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NUMERICAL RANGES OF SUM OF TWO WEIGHTED COMPOSITION OPERATORS ON THE HARDY SPACE H^2

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Abstract. Let φ be an analytic self-map of the open unit disk $\mathbb{D}$ and let ψ be an analytic function on $\mathbb{D}$. The weighted composition operator $C_{\psi,\varphi}$ is the operator on the Hardy space H^2 given by $C_{\psi,\varphi}f = \psi f\circ\varphi$. Under some conditions on φ_1 and φ_2, we try to find a subset of the numerical range of $C_{\psi_1,\varphi_1}+C_{\psi_2,\varphi_2}$ and determine when zero lies in the interior of the numerical range of $C_{\psi_1,\varphi_1}+C_{\psi_2,\varphi_2}$.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. Suppose that S is the subset of $\mathbb{C}$. The boundary points of S is denoted by ∂S. If φ is an analytic self-map of $\mathbb{D}$, then for every analytic function f on $\mathbb{D}$, the composition operator is defined by $C_\varphi(f) = f\circ\varphi$. In this paper, we consider the weighted composition operator $C_{\psi,\varphi}$, where ψ is a bounded analytic function.

The set of functions f analytic on $\mathbb{D}$ with

$$\sup_{0<r<1}\int_0^{2\pi}|f(re^{i\theta})|^2\frac{d\theta}{2\pi}<\infty$$

is called the Hardy space H^2, and $\|f\|$ is the second root of this supremum.

The space $H^\infty(\mathbb{D}) = H^\infty$ is the set of all bounded analytic functions on $\mathbb{D}$ and $\|f\|_\infty = \sup_{|z|<1}|f(z)|$, where $f\in H^\infty$.

For $a,b,c,d\in\mathbb{C}$, a linear fractional transformation is a map of the form $\varphi(z) = \frac{az+b}{cz+d}$, when $ad-bc\neq 0$. We denote the set of these linear fractional transformations that send the open unit disk to itself by $\mathrm{LFT}(\mathbb{D})$.

We know that the automorphisms of the unit disk are the one-to-one analytic maps of the unit disk onto itself. These automorphisms are given by $\varphi(z) = \frac{\lambda(a-z)}{1-\bar{a}z}$, where $|\lambda| = 1$ and $|a|<1$. The set of all automorphisms of $\mathbb{D}$ is denoted by $\mathrm{Aut}(\mathbb{D})$. Let φ be an analytic self-map of $\mathbb{D}$. We say φ has a finite angular derivative $\varphi'(\zeta)$ on the unit circle if there is an η on the circle such that $(\varphi(z)-\eta)/(z-\zeta)$ has a finite nontangential limit as $z\to\zeta$. For $\zeta\in\partial\mathbb{D}$, we denote $\varphi(\zeta) := \lim_{r\to 1}\varphi(r\zeta)$. For an

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analytic self-map φ of $\mathbb{D}$, we say that φ does not make contact with $\partial\mathbb{D}$ if $\{\zeta \in \partial\mathbb{D} : \varphi(\zeta) \in \partial\mathbb{D}\}$ is empty.

A map $\varphi \in \mathrm{LFT}(\mathbb{D})$ is called parabolic if it has a unique fixed point $\zeta \in \partial\mathbb{D}$. Let $T(z) = \frac{\zeta+z}{\zeta-z}$. Put $\phi = T \circ \varphi \circ T^{-1}$. Then ϕ is a linear fractional transformation which takes the right half plane onto itself that fixes only ∞. Therefore, there is a complex number t with $\mathrm{Re}\, t \geqslant 0$ such that $\phi(z) = z + t$. The constant number t is called the translation number of φ. Note that if $\mathrm{Re}\, t = 0$, then $\varphi \in \mathrm{Aut}(\mathbb{D})$. When the translation number t is strictly positive, we call the associated linear fractional self-map of $\mathbb{D}$ a positive parabolic non-automorphism. Among linear fractional transformations with a fixed point $\zeta \in \partial\mathbb{D}$, just for the parabolic ones, we have $\varphi'(\zeta) = 1$. It is easy to see that $\varphi(z) = \frac{(2-t)z+t\zeta}{2+t-t\bar{\zeta}z}$.

Let $\varphi \in \mathrm{LFT}(\mathbb{D})$. Then $C_\varphi^* = T_g C_\sigma T_h^*$, where $\sigma(z) = \frac{\bar{a}z-\bar{c}}{-\bar{b}z+\bar{d}}$ is a self-map of $\mathbb{D}$ into itself, $h(z) = cz + d$, $g(z) = (-\bar{b}z + \bar{d})^{-1}$ and $h, g \in H^\infty$. The maps σ, g and h are called the *Cowen auxiliary functions*. If $\varphi(\zeta) = \eta$ for $\zeta, \eta \in \partial\mathbb{D}$, then $\sigma(\eta) = \zeta$ (see [10]).

Associated with each point $\omega \in \mathbb{D}$ is the function

$$K_\omega(z) := \frac{1}{1-\overline{\omega}z} = \sum_{n=0}^{\infty} \overline{\omega}^n z^n \quad (z \in \mathbb{D}),$$

which is called the reproducing kernel function. Clearly, $K_\omega \in H^2$ for each $\omega \in \mathbb{D}$, and if $f \in H^2$, then $\langle f, K_\omega \rangle = f(\omega)$. It is not hard to see that $\|K_\omega\| = \frac{1}{\sqrt{1-|\omega|^2}}$. Suppose that j is a positive integer. We define $K_\omega^{[j]}$ which is in H^2 by

$$K_\omega^{[j]}(z) = \frac{d^j K(\overline{\omega}z)}{d\overline{\omega}^j}$$

and $\langle f, K_\omega^{[j]} \rangle = f^{(j)}(\omega)$, where $f^{(j)}(\omega)$ is the j^{th} derivative of f at ω.

For each self-map φ of $\mathbb{D}$ and positive integer n, we write $\varphi_1 := \varphi$ and $\varphi_{n+1} := \varphi \circ \varphi_n$. If φ is an analytic self-map of $\mathbb{D}$ that is not an elliptic automorphism, then there is a unique point w in the closed disk $\mathbb{D}$ such that for all $z \in \mathbb{D}$, $\varphi_n(z) \to w$ as $n \to \infty$, this limit point will be referred to as the *Denjoy-Wolff point* of φ. We have

1) $\varphi(w) = w$ and $|\varphi'(w)| < 1$ when $|w| < 1$.

2) $\varphi(w) = w$ and $0 < \varphi'(w) \leqslant 1$ when $|w| = 1$.

The set of all bounded operators and the set of all compact operators from H^2 into itself are denoted by $B(H^2)$ and $B_0(H^2)$, respectively. In this paper, we use $\mathrm{co}(A)$ for the convex hull of a set A which is the intersection of all convex sets that contain A.

The numerical range of a bounded linear operator T on Hilbert space H is denoted by $W(T)$ and is equal to

$$W(T) = \{\langle Tf, f \rangle : f \in H, \|f\| = 1\}.$$

Bourdon and Shapiro in [3] and [4] discussed the numerical ranges of C_φ induced by holomorphic self-maps φ. They investigated in [4] the shape of the numerical range for composition operators when φ is a conformal automorphism. They guessed the numerical range of a finite order elliptic automorphism is not a disk. Gunatillake, Jovovic and Smith in [21] studied numerical ranges of some classes of weighted composition operators on the Hardy space of the unit disk when φ is a rotation of the unit disk. They classified the numerical ranges of isometric weighted composition operators. In [18], Fatehi investigated the numerical ranges of some other weighted composition operators. The spectrum and the essential spectrum have been recently studied for weighted composition operators on H^2.

Let $\sigma(T)$ and $\sigma_e(T)$ denote for the spectrum of T and the essential spectrum of T for each $T \in B(H^2)$. For the Hardy space H^2, Caughran [5], Cowen [8], Deddens [11], Kamowitz [24] and Nordgren [26] discussed the spectrum of composition operators whose symbols φ are linear fractional self-maps of $\mathbb{D}$. Gunatillake [20] characterized the spectrum of weighted compact composition operators $C_{\psi,\varphi}$ on H^2, when φ has a fixed point in $\mathbb{D}$. Cowen and Ko [9] discovered the spectrum of self-adjoint weighted composition operator $C_{\psi,\varphi}$ on H^2. The spectrum of some normal weighted composition operators $C_{\psi,\varphi}$ were obtained by Bourdon and Narayan (see[2]). Moreover in [14], Fatehi and Haji Shaabani investigated normal, cohyponormal and hyponormal weighted composition operators $C_{\psi,\varphi}$. Gunatillake in [19] found invertible weighted composition operators and investigated their spectrum. The extension of this work has been shown in [23], when φ is an automorphism, but $C_{\psi,\varphi}$ is not necessarily invertible. Recently, other papers have been published about composition operators and weighted composition operators; for instance, we can refer to [12, 13, 14, 15, 16, 17].

In this paper, we put some conditions under φ_1 and φ_2 to study when zero lies in the interior of the numerical range of $C_{\psi_1,\varphi_1} + C_{\psi_2,\varphi_2}$. First, in Proposition 2.2, for analytic self-maps φ_1 and φ_2 which are not identity, we prove that zero belongs to the clouser of $W(C_{\psi_1,\varphi_1} + C_{\psi_2,\varphi_2})$. In Theorems 2.3 and 2.4, for analytic self-maps φ_1 and φ_2 that φ_2 has a fixed point $\zeta \in \partial\mathbb{D}$ and φ_1 makes contact with $\partial\mathbb{D}$, but $\varphi_1(\zeta) \neq \zeta$, we find a subset of the numerical range of $C_{\psi_1,\varphi_1} + C_{\psi_2,\varphi_2}$ and state that zero is an interior point of $W(C_{\psi_1,\varphi_1} + C_{\psi_2,\varphi_2})$. In Theorem 2.5, when φ_1 does not make contact with $\partial\mathbb{D}$ and φ_2 not being a positive parabolic, has a fixed point in $\partial\mathbb{D}$, we show that zero is an interior point of $W(C_{\psi_1,\varphi_1} + C_{\psi_2,\varphi_2})$. In Theorem 2.9, for φ_1 and φ_2 which are parabolic non-automorphisms and not positive parabolic, we investigate $W(C_{\psi_1,\varphi_1} + C_{\psi_2,\varphi_2})$. In Proposition 2.12, we suppose that φ_1 and φ_2, which are not automorphisms, are analytic self maps with the same fixed point in $\mathbb{D}$ and we find some elements in the numerical range of $C_{\psi_1,\varphi_1} + C_{\psi_2,\varphi_2}$.

2. Non-automorphism composition map

First, we state the well-known lemma which will be used frequently in this paper.

LEMMA 2.1. *Suppose that φ is an analytic self-map of $\mathbb{D}$ and $\psi \in H^\infty$. If $C_{\psi,\varphi}$ is bounded on H^2, then $C^*_{\psi,\varphi}(K_w) = \overline{\psi(w)}K_{\varphi(w)}$ for $w \in \mathbb{D}$.*

Proof. For each $f \in H^2$ we have

$$\langle f, C^*_{\psi,\varphi}K_w\rangle = \langle C_{\psi,\varphi}f, K_w\rangle = \psi(w)f(\varphi(w)) = \langle f, \overline{\psi(w)}K_{\varphi(w)}\rangle.$$

Since f is arbitrary, we get the desired result. $\square$

Bourdon et al. in [3] proved that if φ is a holomorphic self-map of $\mathbb{D}$ that is not the identity map, then 0 belongs to the closure of $W(C_\varphi)$. By using the same method which was stated in [3, Theorem 3.1], we show that 0 is an element of the closure of $W(C_{\psi_1,\varphi_1}+C_{\psi_2,\varphi_2})$, when $\psi_1,\psi_2\in H^\infty$ and φ_1,φ_2 are analytic self-maps of $\mathbb{D}$ which are not the identity maps.

PROPOSITION 2.2. *Suppose that φ_1 and φ_2 are analytic self-maps of $\mathbb{D}$ which are not identities and $\psi_1,\psi_2\in H^\infty$. Then 0 belongs to the closure of $W(C_{\psi_1,\varphi_1}+C_{\psi_2,\varphi_2})$.*

Proof. Let $w\in\mathbb{D}$. We know that $\langle (C_{\psi_1,\varphi_1}+C_{\psi_2,\varphi_2})^*\frac{K_w}{\|K_w\|},\frac{K_w}{\|K_w\|}\rangle$ belongs to $W((C_{\psi_1,\varphi_1}+C_{\psi_2,\varphi_2})^*)$. By Lemma 2.1,

$$
\begin{aligned}
(C_{\psi_1,\varphi_1}+C_{\psi_2,\varphi_2})^*\frac{K_w}{\|K_w\|} &= (1-|w|^2)^{\frac{1}{2}}\left(\overline{\psi_1(w)}K_{\varphi_1(w)}+\overline{\psi_2(w)}K_{\varphi_2(w)}\right)\\
&= \overline{\psi_1(w)}\frac{(1-|w|^2)^{\frac{1}{2}}}{1-\overline{\varphi_1(w)}z}+\overline{\psi_2(w)}\frac{(1-|w|^2)^{\frac{1}{2}}}{1-\overline{\varphi_2(w)}z}. \qquad (1)
\end{aligned}
$$

Then

$$
\left\langle (C_{\psi_1,\varphi_1}+C_{\psi_2,\varphi_2})^*\frac{K_w}{\|K_w\|},\frac{K_w}{\|K_w\|}\right\rangle=\overline{\psi_1(w)}\frac{1-|w|^2}{1-\overline{\varphi_1(w)}w}+\overline{\psi_2(w)}\frac{1-|w|^2}{1-\overline{\varphi_2(w)}w}. \qquad (2)
$$

Since φ_1 and φ_2 are not the identity maps, $\{\zeta:\varphi_1(\zeta)=\zeta\}$ and $\{\zeta:\varphi_2(\zeta)=\zeta\}$ are the sets of zero measure. Hence there exists a point $\zeta_0\in\partial\mathbb{D}$ such that $\varphi_1(\zeta_0)\neq\zeta_0$ and $\varphi_2(\zeta_0)\neq\zeta_0$. Then by Equation (2), we have

$$
\lim_{w\to\zeta}\langle (C_{\psi_1,\varphi_1}+C_{\psi_2,\varphi_2})^*\frac{K_w}{\|K_w\|},\frac{K_w}{\|K_w\|}\rangle=0.
$$

It shows that 0 belongs to the closure of $W((C_{\psi_1,\varphi_1}+C_{\psi_2,\varphi_2})^*)$ and so 0 lies in the closure of $W(C_{\psi_1,\varphi_1}+C_{\psi_2,\varphi_2})$. $\square$

Note that for each linear fractional self-map of $\mathbb{D}$, φ, which is not automorphism of $\mathbb{D}$, $\{\zeta:\varphi(\zeta)\in\partial\mathbb{D}\}\cap\partial\mathbb{D}$ has at most one element. We use this observation and the result of Theorem 2.8 achieved by Bourdon in [1] to prove the following theorem.

THEOREM 2.3. *Suppose that φ_1 and φ_2 are two linear fractional self-maps of $\mathbb{D}$ that are not automorphisms of $\mathbb{D}$ and φ_2 is not parabolic. Assume that there are $\zeta_1,\zeta_2,\eta_1\in\partial\mathbb{D}$ such that $\varphi_1(\zeta_1)=\eta_1$ and $\varphi_2(\zeta_2)=\zeta_2$, when $\zeta_1\neq\zeta_2$, $\zeta_2\neq\eta_1$. If $\psi_1,\psi_2\in H^\infty$ and ψ_1 is continuous at ζ_1 and ψ_2 is continuous at ζ_2, then*

$$
\left\{z:|z|<\frac{|\psi_2(\zeta_2)|}{\sqrt{\varphi_2'(\zeta_2)}}\right\}\subseteq W(C_{\psi_1,\varphi_1}+C_{\psi_2,\varphi_2});
$$

in particular, if $\psi_2(\zeta_2) \neq 0$, *then zero is an interior point of* $W(C_{\psi_1,\varphi_1} + C_{\psi_2,\varphi_2})$.

Proof. By [25, Corolary 2.2],

$$\sigma_e(C_{\psi_1,\varphi_1} + C_{\psi_2,\varphi_2}) = \sigma_e(\psi_1(\zeta_1)C_{\varphi_1} + \psi_2(\zeta_2)C_{\varphi_2}).$$

It is not hard to see that the closure of $\varphi_1(\varphi_2(\mathbb{D}))$ and the closure of $\varphi_2(\varphi_1(\mathbb{D}))$ are the subsets of $\mathbb{D}$, so by [10, page 129] $C_{\varphi_1 \circ \varphi_2}$ and $C_{\varphi_2 \circ \varphi_1}$ are compact. By [1, Proposition 3.3], we have

$$\sigma_e(\psi_1(\zeta_1)C_{\varphi_1} + \psi_2(\zeta_2)C_{\varphi_2}) \setminus \{0\} = (\sigma_e(\psi_1(\zeta_1)C_{\varphi_1}) \cup \sigma_e(\psi_2(\zeta_2)C_{\varphi_2})) \setminus \{0\}.$$

Since $\varphi_2(\zeta_2) = \zeta_2$ and φ_2 is not parabolic, we consider two cases (see [1, Theorem 2.8]).

a) Suppose that the Denjoy-Wolff point of φ_2 lies in the open unit disk. By [1, Theorem 2.8(ii)], we get

$$\sigma_e(\psi_2(\zeta_2)C_{\varphi_2}) = \left\{ z : |z| \leqslant \frac{|\psi_2(\zeta_2)|}{\sqrt{\varphi_2'(\zeta_2)}} \right\}.$$

We can see that

$$\left\{ z : |z| < \frac{|\psi_2(\zeta_2)|}{\sqrt{\varphi_2'(\zeta_2)}} \right\} \setminus \{0\} \subseteq \sigma_e(C_{\psi_1,\varphi_1} + C_{\psi_2,\varphi_2}) \setminus \{0\}.$$

By [22, Theorem 1.2-1], the spectrum of an operator is contained in the closure of its numerical range, so $\left\{ z : |z| \leqslant \frac{|\psi_2(\zeta_2)|}{\sqrt{\varphi_2'(\zeta_2)}} \right\} \setminus \{0\}$ is the subset of the closure of $W(C_{\psi_1,\varphi_1} + C_{\psi_2,\varphi_2})$. Since $W(C_{\psi_1,\varphi_1} + C_{\psi_2,\varphi_2})$ is convex, it is easy to see that

$$\left\{ z : |z| < \frac{|\psi_2(\zeta_2)|}{\sqrt{\varphi_2'(\zeta_2)}} \right\} \subseteq W(C_{\psi_1,\varphi_1} + C_{\psi_2,\varphi_2}).$$

It shows that for $\psi_2(\zeta_2) \neq 0$, zero is an interior point of $W(C_{\psi_1,\varphi_1} + C_{\psi_2,\varphi_2})$.

b) Suppose that ζ_2 is the Denjoy-Wolff point of φ_2. By [1, Theorem 2.8(iii)], we get

$$\sigma_e(\psi_2(\zeta_2)C_{\varphi_2}) = \left\{ z : |z| \leqslant \frac{|\psi_2(\zeta_2)|}{\sqrt{\varphi_2'(\zeta_2)}} \right\}.$$

The conclusion follows from the similar idea which was stated in Part(a). □

Let φ_2 be a linear fractional non-automorphism with fixed point $\zeta_2 \in \partial\mathbb{D}$. The map φ_2 must satisfy one of the conditions of [1, Theorem 2.8]. Two cases were investigated in Theorem 2.3 and in the next theorem, we consider the parabolic case and we find a subset of $W(C_{\psi_1,\varphi_1} + C_{\psi_2,\varphi_2})$; we conclude that zero is an interior point of $W(C_{\psi_1,\varphi_1} + C_{\psi_2,\varphi_2})$.

THEOREM 2.4. *Suppose that φ_1 and φ_2 are two linear fractional self-maps of $\mathbb{D}$ that are not automorphisms of $\mathbb{D}$. Assume that there are $\zeta_1, \zeta_2, \eta_1 \in \partial\mathbb{D}$ such that $\varphi_1(\zeta_1) = \eta_1$ and $\varphi_2(\zeta_2) = \zeta_2$ when $\zeta_1 \neq \zeta_2$ and $\zeta_2 \neq \eta_1$. Suppose that φ_2 is parabolic, but not positive parabolic. If $\psi_1, \psi_2 \in H^\infty$ and ψ_1 is continuous at ζ_1 and ψ_2 is continuous at ζ_2 and $\psi_2(\zeta_2) \neq 0$, then co$\left(\{\psi_2(\zeta_2)e^{-\zeta_2\varphi_2''(\zeta_2)\beta} : \beta \geqslant 0\}\right)$ is the subset of the closure of $W(C_{\psi_1,\varphi_1} + C_{\psi_2,\varphi_2})$.*

Proof. By [1, Teorem 2.8(iv)],

$$\sigma_e(\psi_2(\zeta_2)C_{\varphi_2}) = \{\psi_2(\zeta_2)e^{-\zeta_2\varphi_2''(\zeta_2)\beta} : \beta \geqslant 0\} \cup \{0\},$$

(note that since φ_2 is not positive parabolic, $\sigma_e(\psi_2(\zeta_2)C_{\varphi_2})$ is a logarithmic spiral).

By the similar idea stated in the proof of the proceeding theorem, we have co$\left(\{\psi_2(\zeta_2)e^{-\zeta_2\varphi_2''(\zeta_2)\beta} : \beta \geqslant 0\}\right)$ is the subset of the closure of $W(C_{\psi_1,\varphi_1} + C_{\psi_2,\varphi_2})$. Thus, it is easy to see that 0 is an interior point of $W(C_{\psi_1,\varphi_1} + C_{\psi_2,\varphi_2})$. □

In the following theorem, we show that zero is an interior point of $W(C_{\psi_1,\varphi_1} + C_{\psi_2,\varphi_2})$, when φ_1 does not make contact with $\partial\mathbb{D}$ and φ_2 has a fixed point in $\partial\mathbb{D}$.

THEOREM 2.5. *Let $\psi_1, \psi_2 \in H^\infty$ and φ_1, φ_2 be linear fractional non-automorphisms. Suppose that φ_1 does not make contact with $\partial\mathbb{D}$ and φ_2 has a fixed point $\zeta_2 \in \partial\mathbb{D}$. Assume that ψ_2 is continuous at ζ_2, $\psi_2(\zeta_2) \neq 0$ and φ_2 is not positive parabolic. Then zero is an interior point of $W(C_{\psi_1,\varphi_1} + C_{\psi_2,\varphi_2})$.*

Proof. By [10, p. 129], C_{ψ_1,φ_1} is compact. One can easily see that

$$\sigma_e(C_{\psi_1,\varphi_1} + C_{\psi_2,\varphi_2}) = \sigma_e(C_{\psi_2,\varphi_2}) = \psi_2(\zeta_2)\sigma_e(C_{\varphi_2}).$$

We infer from [22, Teorem 1.2-1] that co$(\psi_2(\zeta_2)\sigma_e(C_{\varphi_2}))$ is the subset of the closure of $W(C_{\psi_1,\varphi_1} + C_{\psi_2,\varphi_2})$. By [1, Theorem 2.8] and the fact that $W(C_{\psi_1,\varphi_1} + C_{\psi_2,\varphi_2})$ is convex, the result follows. □

Now we give examples for Theorems 2.3, 2.4 and 2.5.

EXAMPLE 2.6. (a) Let $\varphi_1(z) = \frac{iz}{2+z}$ and $\varphi_2(z) = \frac{1+z}{2}$. We have $\varphi_1(-1) = -i$ and $\varphi_2(1) = 1$. It is obvious that φ_2 is not parabolic. Put $\psi_1(z) = \frac{z^2}{z+4}$ and $\psi_2(z) = (2z+1)e^z$. Obviously, ψ_1 is continuous at -1 and ψ_2 is continuous at 1. By Theorem 2.3, we observe that $\left\{z : |z| < 3\sqrt{2}e\right\} \subseteq W(C_{\psi_1,\varphi_1} + C_{\psi_2,\varphi_2})$.

(b) Let $\varphi_1(z) = \frac{-z}{2+z}$ and $\varphi_2(z) = \frac{(1-i)z+(i-1)}{(3+i)-(1-i)z}$. We have $\varphi_1(-1) = 1$ and $\varphi_2(i) = i$. It is clear that φ_2 is parabolic, but is not positive parabolic. Put $\psi_1(z) = e^{-iz}$ and $\psi_2(z) = iz^2$. By Theorem 2.4, co$\left(\{-ie^{-(i+1)\beta} : \beta \geqslant 0\}\right)$ is the subset of the closure of $W(C_{\psi_1,\varphi_1} + C_{\psi_2,\varphi_2})$.

(c) Let $\varphi_1(z) = \frac{1+iz}{4}$, $\varphi_2(z) = \frac{(1+i)z+(1-i)}{(3-i)+(-1+i)z}$ and $\varphi_3(z) = \frac{z}{2-z}$ be three linear fractional non-automorphisms. Clearly, φ_2 and φ_3 are not positive parabolic. Furthermore, we can see that φ_1 does not make contant with $\partial\mathbb{D}$ and $\varphi_2(1) = \varphi_3(1) = 1$.

Put $\psi_1(z) = \frac{1}{1-\frac{z}{2}}$ and $\psi_2(z) = 1 + \frac{z^2}{2}$. By Theorem 2.5, zero is an interior point of $W(C_{\psi_1,\varphi_1} + C_{\psi_2,\varphi_2})$ and $W(C_{\psi_1,\varphi_1} + C_{\psi_2,\varphi_3})$.

For each $z \in \mathbb{C}$, we can write $z = |z|e^{i\theta}$ when $-\pi < \theta \leqslant \pi$. We denote $\arg(z) = \theta$ and we use it in the proof of the following proposition. In Proposition 2.7, we determine the spectrum of linear combination of two composition operators which their composition maps are parabolic non-automorphism self-maps of $\mathbb{D}$ with the same fixed point. In the proof of the next result, we utilize the idea used in [8, Theorem 6.1] and [8, Corollary 6.2].

PROPOSITION 2.7. *Let φ_1 and φ_2 be two parabolic non-automorphism self-maps of $\mathbb{D}$ and $\zeta \in \partial\mathbb{D}$ is a fixed point of φ_1 and φ_2. Suppose that α_1 and α_2 are two arbitrary complex numbers and t_1 and t_2 are the translation numbers of φ_1 and φ_2, respectively. Then*

$$\sigma(\alpha_1 C_{\varphi_1} + \alpha_2 C_{\varphi_2}) = \sigma_e(\alpha_1 C_{\varphi_1} + \alpha_2 C_{\varphi_2}) = \{\alpha_1 e^{-\beta t_1} + \alpha_2 e^{-\beta t_2} : \beta \geqslant 0\} \cup \{0\}.$$

Proof. Define $\tilde{\varphi}_1(z) = \overline{\zeta}(\varphi_1(\zeta z))$ and $\tilde{\varphi}_2(z) = \overline{\zeta}(\varphi_2(\zeta z))$. It is easy to see that 1 is the fixed point of two parabolic functions $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$, because

$$\tilde{\varphi}_i(1) = \overline{\zeta}(\varphi_i(\zeta)) = \overline{\zeta}\zeta = |\zeta|^2 = 1$$

for $i = 1,2$. It shows that $C_{\tilde{\varphi}_i} = C_{\zeta z} C_{\varphi_i} C^*_{\zeta z}$. Then C_{φ_i} is unitary equivalent to $C_{\tilde{\varphi}_i}$. Hence without lose of generality, we can assume that 1 is the fixed point of two parabolic non-automorphism self-maps φ_1 and φ_2. Put $\tau(G) = \{t : |\arg(t)| < \frac{\pi}{2}\}$ and $\varphi_t(z) = \sigma^{-1}(\sigma(z) + t)$, when $\sigma(z) = \frac{1+z}{1-z}$ and $t \in \tau(G)$. As stated in the introduction, φ_t is parabolic non-automorphism. Let U be the norm closed algebra of operators generated by $\{C_{\varphi_t} : t \in \tau(G)\} \cup \{I\}$. If Σ is the set of all nonzero multiplicative linear functionals on U, then by the proof of [8, Theorem 6.1], for each $t \in \tau(G)$ and $\Lambda \in \Sigma$, we have $\Lambda(C_{\varphi_t}) = 0$ or $\Lambda(C_{\varphi_t}) = e^{-\beta t}$, for some $\beta \geqslant 0$. By [6, Theorem 8.6, P. 219],

$$\begin{aligned}\sigma(\alpha_1 C_{\varphi_1} + \alpha_2 C_{\varphi_2}) &\subseteq \{\Lambda(\alpha_1 C_{\varphi_1} + \alpha_2 C_{\varphi_2}) : \Lambda \in \Sigma\} \\ &= \{\alpha_1 \Lambda(C_{\varphi_1}) + \alpha_2 \Lambda(C_{\varphi_2}) : \Lambda \in \Sigma\} \\ &= \{\alpha_1 e^{-\beta t_1} + \alpha_2 e^{-\beta t_2} : \beta \geqslant 0\} \cup \{0\}. \qquad (3)\end{aligned}$$

By [8, Corolary 6.2], $e^{-\beta t}$ is an eigenvalue corresponding to the eigenvector $f(z) = \exp(-\beta\sigma(z))$ for C_{φ_t}. Hence $(\alpha_1 C_{\varphi_1} + \alpha_2 C_{\varphi_2})(f) = (\alpha_1 e^{-\beta t_1} + \alpha_2 e^{-\beta t_2})(f)$. Thus $\alpha_1 e^{-\beta t_1} + \alpha_2 e^{-\beta t_2} \in \sigma(\alpha_1 C_{\varphi_1} + \alpha_2 C_{\varphi_2})$. Since φ_1 and φ_2 are parabolic non-automorphisms, $\mathrm{Re}t_1 > 0$ and $\mathrm{Re}t_2 > 0$. It is clear that 0 is a limit point of $\alpha_1 e^{-\beta t_1} + \alpha_2 e^{-\beta t_2}$. Since spectrum is compact we have

$$\{\alpha_1 e^{-\beta t_1} + \alpha_2 e^{-\beta t_2}\} \cup \{0\} \subseteq \sigma(\alpha_1 C_{\varphi_1} + \alpha_2 C_{\varphi_2}). \qquad (4)$$

From Equations (3) and (4), $\sigma(\alpha_1 C_{\varphi_1} + \alpha_2 C_{\varphi_2}) = \{\alpha_1 e^{-\beta t_1} + \alpha_2 e^{-\beta t_2} : \beta \geqslant 0\} \cup \{0\}$. Since every point of $\sigma(\alpha_1 C_{\varphi_1} + \alpha_2 C_{\varphi_2})$ is a boundary point of spectrum and none is isolated, we conclude that $\sigma(\alpha_1 C_{\varphi_1} + \alpha_2 C_{\varphi_2}) = \sigma_e(\alpha_1 C_{\varphi_1} + \alpha_2 C_{\varphi_2})$ (see [7, Theorem 37.8]). $\square$

Now we use Proposition 2.7 to find the numerical range of $\alpha_1 C_{\varphi_1} + \alpha_2 C_{\varphi_2}$, when φ_1 and φ_2 are two parabolic non-automorphism self-maps of $\mathbb{D}$ which are not positive.

PROPOSITION 2.8. *Let φ_1 and φ_2 be two parabolic non-automorphisms which are not positive parabolic, and $\zeta \in \partial\mathbb{D}$ be the fixed point of φ_1 and φ_2. Suppose that α_1 and α_2 are two arbitrary complex numbers with the same arguments. Assume that $t_1 = t_{1,1} + it_{1,2}$ and $t_2 = t_{2,1} + it_{2,2}$ are translation numbers of φ_1 and φ_2, respectively.*

a) If $t_{1,2} = t_{2,2}$, then zero is an interior point of $W(\alpha_1 C_{\varphi_1} + \alpha_2 C_{\varphi_2})$.

b) If $t_{1,2} \neq \frac{k}{2}(t_{1,2} - t_{2,2})$ for each positive integer k, then zero is an interior point of $W(\alpha_1 C_{\varphi_1} + \alpha_2 C_{\varphi_2})$.

c) If $t_{1,2} = \frac{k}{2}(t_{1,2} - t_{2,2})$ for some positive odd integer k, then 0 belongs to $W(\alpha_1 C_{\varphi_1} + \alpha_2 C_{\varphi_2})$.

Proof. For each $\beta \geqslant 0$, $e^{-\beta t_1} = e^{-\beta t_{1,1}} e^{-i\beta t_{1,2}}$ and $e^{-\beta t_2} = e^{-\beta t_{2,1}} e^{-i\beta t_{2,2}}$. Note that since φ_1 and φ_2 are not positive parabolic, $t_{1,2} \neq 0$ and $t_{2,2} \neq 0$.

a) Let $t_{1,2} = t_{2,2} = t$. We have

$$\alpha_1 e^{-\beta t_1} + \alpha_2 e^{-\beta t_2} = (\alpha_1 e^{-\beta t_{1,1}} + \alpha_2 e^{-\beta t_{2,1}}) e^{-i\beta t}.$$

Since α_1 and α_2 have same arguments, for each $\beta \geqslant 0$, $\alpha_1 e^{-\beta t_1}$ and $\alpha_2 e^{-\beta t_2}$ have the same arguments too. It is easy to see that by choosing appropriate $\beta \geqslant 0$, we can find four points of the set $\{\alpha_1 e^{-\beta t_1} + \alpha_2 e^{-\beta t_2} : \beta \geqslant 0\}$ in each quarter. By Proposition 2.7 and the fact that the spectrum of an operator is contained in the closure of its numerical range, and the convexity of the numerical range, we find that the interior of polygonal created by joining these four points is contained in the numerical range. Therefore, zero is an interior piont of $W(\alpha_1 C_{\varphi_1} + \alpha_2 C_{\varphi_2})$.

b) Suppose that $t_{1,2} \neq \frac{k}{2}(t_{1,2} - t_{2,2})$ for each positive integer k. Without loss of generality, assume that $t_{1,2} > t_{2,2}$. Put $\gamma = t_{1,2} - t_{2,2} > 0$. There is $\beta_0 > 0$ such that $\beta_0 \gamma = 2\pi$, so

$$\beta_0 t_{1,2} = \beta_0(\gamma + t_{2,2}) = 2\pi + \beta_0 t_{2,2}.$$

It states that for each positive integer n, $\alpha_1 e^{-\beta_0 t_1 n}$ and $\alpha_2 e^{-\beta_0 t_2 n}$ have the same arguments. We have $e^{-i\beta_0 t_{1,2}} \neq 1$ and $e^{-i\beta_0 t_{1,2}} \neq -1$ because φ_1 and φ_2 are not positive parabolic and $t_{1,2} \neq \frac{k}{2}(t_{1,2} - t_{2,2})$ for each positive integer k. Therefore, $e^{-i\beta_0 t_{1,2}}$ must be either a primitive root of 1 with the order $n > 2$ or must not be a root of 1. First suppose that $e^{-i\beta_0 t_{1,2}}$ is a primitive root of 1 with the order $n > 2$. We can see that 1, $e^{-it_{1,2}\beta_0}$, $e^{-2it_{1,2}\beta_0}$, $\ldots$, $e^{-(n-1)it_{1,2}\beta_0}$ are the nth root of 1. Let P be the polygonal region whose vertices are $\alpha_1 + \alpha_2$, $\alpha_1 e^{-i\beta_0 t_1} + \alpha_2 e^{-i\beta_0 t_2}$, $\ldots$, $\alpha_1 e^{-i(n-1)\beta_0 t_1} + \alpha_2 e^{-i(n-1)\beta_0 t_2}$. Since $n > 2$, it is not hard to see that non of the sides of P contains zero and so 0 belongs to the interior of the polygonal region P. Thus, 0 is an interior point of $W(\alpha_1 C_{\varphi_1} + \alpha_2 C_{\varphi_2})$. Now suppose that $e^{-i\beta_0 t_{1,2}}$ is not a root of 1. Then $\{e^{-in\beta_0 t_{1,2}} : n \text{ is a non-negative integer}\}$ is dense in $\partial\mathbb{D}$. One can easily see that we can find n_1, n_2, n_3, n_4 such that $\alpha_1 e^{-in_1\beta_0 t_1} + \alpha_2 e^{-in_1\beta_0 t_2}$, $\alpha_1 e^{-in_2\beta_0 t_1} +$

$\alpha_2 e^{-in_2\beta_0 t_2}$, $\alpha_1 e^{-in_3\beta_0 t_1} + \alpha_2 e^{-in_3\beta_0 t_2}$, $\alpha_1 e^{-in_4\beta_0 t_1} + \alpha_2 e^{-in_4\beta_0 t_2}$ lie in the quadrants I, II, III, IV, respectively. It is not hard to see that 0 is contained in the interior of polygonal region P whose vertices are $\alpha_1 e^{-in_1\beta_0 t_1} + \alpha_2 e^{-in_1\beta_0 t_2}$, $\alpha_1 e^{-in_2\beta_0 t_1} + \alpha_2 e^{-in_2\beta_0 t_2}$, $\alpha_1 e^{-in_3\beta_0 t_1} + \alpha_2 e^{-in_3\beta_0 t_2}$, $\alpha_1 e^{-in_4\beta_0 t_1} + \alpha_2 e^{-in_4\beta_0 t_2}$. Then 0 is an interior point of $W(\alpha_1 C_{\varphi_1} + \alpha_2 C_{\varphi_2})$.

c) Let $t_{1,2} = \frac{k}{2}(t_{1,2} - t_{2,2})$ for some positive odd integer k. Without loss of generality, assume that $t_{1,2} > t_{2,2}$. Put $\gamma = t_{1,2} - t_{2,2} > 0$. There is $\beta_0 > 0$ such that $\beta_0\gamma = 2\pi$ (note that in case that $t_{2,2} > t_{1,2}$, we can find $\beta_0 > 0$ such that $\beta_0\gamma = -2\pi$), so

$$\beta_0 t_{1,2} = \beta_0(\gamma + t_{2,2}) = 2\pi + \beta_0 t_{2,2}.$$

Since k is a positive odd integer, we have $e^{i\beta_0 t_{1,2}} = e^{ik\pi} = -1$ and $e^{i\beta_0 t_{2,2}} = e^{i(\beta_0 t_{1,2} - 2\pi)} = -1$. Thus $\alpha_1 e^{-\beta_0 t_1} + \alpha_2 e^{-\beta_0 t_2} = (\alpha_1 e^{-\beta_0 t_{1,1}} + \alpha_2 e^{-\beta_0 t_{2,1}})e^{i\pi}$. We know that $e^{-\beta_0 t_{1,1}}$ and $e^{-\beta_0 t_{2,1}}$ are positive real numbers; so, one can easily see that $\arg(\alpha_1 e^{-\beta_0 t_1} + \alpha_2 e^{-\beta_0 t_2}) = \arg(\alpha_1 e^{-\beta_0 t_{1,1}} + \alpha_2 e^{-\beta_0 t_{2,1}}) + \pi = \arg(\alpha_1 + \alpha_2) + \pi$. By Proposition 2.7, $\alpha_1 + \alpha_2$ and $\alpha_1 e^{-\beta_0 t_1} + \alpha_2 e^{-\beta_0 t_2}$ belong to $\sigma(\alpha_1 C_{\varphi_1} + \alpha_2 C_{\varphi_2})$. Since numerical range is convex, the closed line segment with endpoints $\alpha_1 e^{-\beta_0 t_1} + \alpha_2 e^{-\beta_0 t_2}$ and $\alpha_1 + \alpha_2$ is contained in the closure of $W(\alpha_1 C_{\varphi_1} + \alpha_2 C_{\varphi_2})$. Therefore, the open line segment with endpoints $\alpha_1 e^{-\beta_0 t_1} + \alpha_2 e^{-\beta_0 t_2}$ and $\alpha_1 + \alpha_2$ is contained in $W(\alpha_1 C_{\varphi_1} + \alpha_2 C_{\varphi_2})$ and so zero belongs to the numerical range. □

Note that in Proposition 2.8, existing 0 in $W(\alpha_1 C_{\varphi_1} + \alpha_2 C_{\varphi_2})$ was investigated for all parabolic non-automorphisms φ_1 and φ_2 which are not positive except for the case that $t_{1,2} = \frac{k}{2}(t_{1,2} - t_{2,2})$ for some even integer k.

THEOREM 2.9. *Let φ_1 and φ_2 be two parabolic non-automorphisms which are not positive parabolic, and $\zeta \in \partial\mathbb{D}$ be the fixed point of φ_1 and φ_2. Suppose that $\psi_1, \psi_2 \in H^\infty$ and $\psi_1(\zeta)$ and $\psi_2(\zeta)$ have the same arguments . Assume that $t_1 = t_{1,1} + it_{1,2}$ and $t_2 = t_{2,1} + it_{2,2}$ are translation numbers of φ_1 and φ_2, respectively.*

a) If $t_{1,2} = t_{2,2}$, then zero is an interior point of $W(C_{\psi_1,\varphi_1} + C_{\psi_2,\varphi_2})$.

b) If $t_{1,2} \neq \frac{k}{2}(t_{1,2} - t_{2,2})$ for each positive integer k, then zero is an interior point of $W(C_{\psi_1,\varphi_1} + C_{\psi_2,\varphi_2})$.

c) If $t_{1,2} = \frac{k}{2}(t_{1,2} - t_{2,2})$ for some positive odd integer k, then 0 belongs to $W(C_{\psi_1,\varphi_1} + C_{\psi_2,\varphi_2})$.

Proof. By [25, Corolary 2.2],

$$\sigma_e(C_{\psi_1,\varphi_1} + C_{\psi_2,\varphi_2}) = \sigma_e(\psi_1(\zeta)C_{\varphi_1} + \psi_2(\zeta)C_{\varphi_2}).$$

By Proposition 2.7, we have

$$\sigma_e(\psi_1(\zeta)C_{\varphi_1} + \psi_2(\zeta)C_{\varphi_2}) = \{\psi_1(\zeta)e^{-\beta t_1} + \psi_2(\zeta)e^{-\beta t_2} : \beta \geqslant 0\} \cup \{0\}.$$

Now by Proposition 2.8, the results follow. □

In the following example, three illustrations for Theorem 2.9 are given.

EXAMPLE 2.10. (a) Consider $\varphi_1(z) = \frac{(1-i)z+i-1}{(i-1)z+3+i}$ and $\varphi_2(z) = \frac{-iz+2i-1}{(2i-1)z+4+i}$. It is obvious that i is the fixed point of φ_1 and φ_2. We have $\varphi_1'(i) = \varphi_2'(i) = 1$, so φ_1 and

φ_2 are parabolic. Now put $T(z)=\frac{i+z}{i-z}$, so $T^{-1}(z)=\frac{iz-i}{1+z}$. We have $T(\varphi_1(T^{-1}(z)))=z+1+i$ and $T(\varphi_2(T^{-1}(z)))=z+2+i$. We obtain that $t_1=1+i$ and $t_2=2+i$ are translation numbers of φ_1 and φ_2, respectively. Put $\psi_1(z)=e^{iz}$ and $\psi_2(z)=e^{2iz}$. We can see that $\psi_1(i)$ and $\psi_2(i)$ have the same arguments. Since the imaginary part of t_1 and t_2 are equal, by Part (a) of Theorem 2.9, zero is an interior point of $W(C_{\psi_1,\varphi_1}+C_{\psi_2,\varphi_2})$.

(b) Let $\varphi_1(z)=\frac{-5iz+2+5i}{(-2-5i)z+4+5i}$ and $\varphi_2(z)=\frac{(-1-2i)z+3+2i}{(-3-2i)z+5+2i}$. It is not hard to see that 1 is the fixed point of φ_1 and φ_2. Since $\varphi_1'(1)=\varphi_2'(1)=1$, φ_1 and φ_2 are parabolic. Let $T(z)=\frac{1+z}{1-z}$, so $T^{-1}(z)=\frac{z-1}{1+z}$. We have $T(\varphi_1(T^{-1}(z)))=z+2+5i$ and $T(\varphi_2(T^{-1}(z)))=z+3+2i$. Therefore, $t_1=2+5i$ and $t_2=3+2i$ are translation numbers of φ_1 and φ_2, respectively. Put $\psi_1(z)=e^{z+\frac{\pi}{2}i}$ and $\psi_2(z)=e^{2z+\frac{\pi}{2}i}$. Obviously, $\psi_1(1)$ and $\psi_2(1)$ have the same arguments. Since $t_{1,2}\neq\frac{k}{2}(t_{1,2}-t_{2,2})$ for each positive integer k, by Part (b) of Theorem 2.9, zero is an interior point of $W(C_{\psi_1,\varphi_1}+C_{\psi_2,\varphi_2})$.

(c) Let $\varphi_1(z)=\frac{(1-10i)z+i-10}{(i-10)z+3+10i}$ and $\varphi_2(z)=\frac{-6iz+2i-6}{(2i-6)z+4+6i}$. We can easily see that i is the fixed point of φ_1 and φ_2. We have $\varphi_1'(i)=\varphi_2'(i)=1$, so φ_1 and φ_2 are parabolic. Now put $T(z)=\frac{i+z}{i-z}$, so $T^{-1}(z)=\frac{iz-i}{1+z}$. We know that $T(\varphi_1(T^{-1}(z)))=z+1+10i$ and $T(\varphi_2(T^{-1}(z)))=z+2+6i$. We have $t_1=1+10i$ and $t_2=2+6i$ are translation numbers of φ_1 and φ_2, respectively. Put $\psi_1(z)=z^2$ and $\psi_2(z)=3z^2$, thus $\psi_1(i)$ and $\psi_2(i)$ have the same arguments. We have for $k=5$, $t_{1,2}=\frac{k}{2}(t_{1,2}-t_{2,2})$, so by Part (c) of Theorem 2.9, zero belongs to $W(C_{\psi_1,\varphi_1}+C_{\psi_2,\varphi_2})$.

In the following lemma, we obtain that $(C_{\psi_1,\varphi_1}+C_{\psi_2,\varphi_2})^*\left(\frac{K_0^{[n]}}{\|K_0^{[n]}\|}\right)$ which will be used to prove Proposition 2.12.

LEMMA 2.11. *Let n be a positive integer. Then*

$$(C_{\psi_1,\varphi_1}+C_{\psi_2,\varphi_2})^*\left(\frac{K_0^{[n]}}{\|K_0^{[n]}\|}\right)=\sum_{j=0}^{n-1}\overline{\alpha_{1,j}(0)+\alpha_{2,j}(0)}\frac{K_0^{[j]}}{n!}$$
$$+\overline{\psi_1(0)(\varphi_1'(0))^n+\psi_2(0)(\varphi_2'(0))^n}\frac{K_0^{[n]}}{n!},$$

where for $i=1,2$, $\alpha_{i,j}$'s are maps that contain various of products of derivatives of ψ_i and φ_i.

Proof. Let f be an arbitrary function on H^2. Then

$$\begin{aligned}
&\langle f,(C_{\psi_1,\varphi_1}+C_{\psi_2,\varphi_2})^*K_0^{[n]}\rangle\\
&=\langle (C_{\psi_1,\varphi_1}+C_{\psi_2,\varphi_2})f,K_0^{[n]}\rangle\\
&=\langle \psi_1 f\circ\varphi_1+\psi_2 f\circ\varphi_2,K_0^{[n]}\rangle\\
&=(\psi_1 f\circ\varphi_1+\psi_2 f\circ\varphi_2)^{(n)}(0)\\
&=\sum_{j=0}^{n-1}\alpha_{1,j}(0)f^{(j)}(0)+(\varphi_1'(0))^n f^{(n)}(0)\psi_1(0)+\sum_{j=0}^{n-1}\alpha_{2,j}(0)f^{(j)}(0)\\
&\quad+(\varphi_2'(0))^n f^{(n)}(0)\psi_2(0)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{n-1} \alpha_{1,j}(0)\langle f, K_0^{[j]}\rangle + (\varphi_1'(0))^n \psi_1(0)\langle f, K_0^{[n]}\rangle \\
&\quad + \sum_{j=0}^{n-1} \alpha_{2,j}(0)\langle f, K_0^{[j]}\rangle + (\varphi_2'(0))^n \psi_2(0)\langle f, K_0^{[n]}\rangle \\
&= \left\langle f, \sum_{j=0}^{n-1} \overline{\alpha_{1,j}(0) + \alpha_{2,j}(0)} K_0^{[j]} + \overline{\psi_1(0)(\varphi_1'(0))^n + \psi_2(0)(\varphi_2'(0))^n} K_0^{[n]} \right\rangle.
\end{aligned}
$$

Since f is arbitrary and $\|K_0^{[n]}\| = n!$, we get

$$
\begin{aligned}
(C_{\psi_1,\varphi_1} + C_{\psi_2,\varphi_2})^* \left(\frac{K_0^{[n]}}{\|K_0^{[n]}\|} \right) &= \sum_{j=0}^{n-1} \overline{\alpha_{1,j}(0) + \alpha_{2,j}(0)} \frac{K_0^{[j]}}{n!} \\
&\quad + \overline{\psi_1(0)(\varphi_1'(0))^n + \psi_2(0)(\varphi_2'(0))^n} \frac{K_0^{[n]}}{n!}. \quad \square
\end{aligned}
$$

We know that the numerical range of the compression of operator T is contained in $W(T)$. This fact is used in following proposition to find some elements of $W(C_{\psi_1,\varphi_1} + C_{\psi_2,\varphi_2})$, where φ_1 and φ_2 are two non-automorphism self-maps of $\mathbb{D}$ with the same fixed point in $\mathbb{D}$.

PROPOSITION 2.12. *Suppose that φ_1 and φ_2 are two self-maps of $\mathbb{D}$ that are not automorphisms and $\psi_1, \psi_2 \in H^\infty$. Let $p \in \mathbb{D}$ be the fixed point of φ_1 and φ_2. Then for each non-negative integer n,*

$$
\psi_1(p)(\varphi_1'(p))^n + \psi_2(p)(\varphi_2'(p))^n \in W(C_{\psi_1,\varphi_1} + C_{\psi_2,\varphi_2}).
$$

Proof. Let $\varphi_1(p) = \varphi_2(p) = p$. Suppose that $\psi_p := \frac{K_p}{\|K_p\|}$ and $\alpha_p(z) := \frac{p-z}{1-\overline{p}z}$. By [2, Theorem 6], C_{ψ_p,α_p} is unitary, that is, $C^*_{\psi_p,\alpha_p} C_{\psi_p,\alpha_p} = C_{\psi_p,\alpha_p} C^*_{\psi_p,\alpha_p} = I$. Let σ, g, h be the Cowen auxiliary functions for α_p. Since $T_h^* T_{\psi_p}^* = (T_{\psi_p h})^* = \frac{1}{\|K_p\|}$, $\sigma = \alpha_p$ and $C^*_{\alpha_p} = T_g C_{\alpha_p} T_h^*$, we have

$$
\begin{aligned}
& C^*_{\psi_p,\alpha_p}(C_{\psi_1,\varphi_1} + C_{\psi_2,\varphi_2})C_{\psi_p,\alpha_p} \\
&= C^*_{\psi_p,\alpha_p} C_{\psi_1,\varphi_1} C_{\psi_p,\alpha_p} + C^*_{\psi_p,\alpha_p} C_{\psi_2,\varphi_2} C_{\psi_p,\alpha_p} \\
&= C^*_{\alpha_p} T^*_{\psi_p} T_{\psi_1} C_{\varphi_1} T_{\psi_p} C_{\alpha_p} + C^*_{\alpha_p} T^*_{\psi_p} T_{\psi_2} C_{\varphi_2} T_{\psi_p} C_{\alpha_p} \\
&= \frac{1}{\|K_p\|} T_g C_{\alpha_p} T_{\psi_1} C_{\varphi_1} T_{\psi_p} C_{\alpha_p} + \frac{1}{\|K_p\|} T_g C_{\alpha_p} T_{\psi_2} C_{\varphi_2} T_{\psi_p} C_{\alpha_p} \\
&= \frac{1}{\|K_p\|} T_g T_{\psi_1 \circ \alpha_p} T_{\psi_p \circ \varphi_1 \circ \alpha_p} C_{\alpha_p} C_{\varphi_1} C_{\alpha_p} + \frac{1}{\|K_p\|} T_g T_{\psi_2 \circ \alpha_p} T_{\psi_p \circ \varphi_2 \circ \alpha_p} C_{\alpha_p} C_{\varphi_2} C_{\alpha_p} \\
&= \frac{1}{\|K_p\|} T_{g.\psi_1 \circ \alpha_p.\psi_p \circ \varphi_1 \circ \alpha_p} C_{\alpha_p \circ \varphi_1 \circ \alpha_p} + \frac{1}{\|K_p\|} T_{g.\psi_2 \circ \alpha_p.\psi_p \circ \varphi_2 \circ \alpha_p} C_{\alpha_p \circ \varphi_2 \circ \alpha_p}.
\end{aligned}
$$

We put $\tilde{\psi}_1 = \frac{1}{\|K_p\|} g.\psi_1 \circ \alpha_p.\psi_p \circ \varphi_1 \circ \alpha_p, \tilde{\varphi}_1 = \alpha_p \circ \varphi_1 \circ \alpha_p, \tilde{\psi}_2 = \frac{1}{\|K_p\|} g.\psi_2 \circ \alpha_p.\psi_p \circ \varphi_2 \circ \alpha_p, \tilde{\varphi}_2 = \alpha_p \circ \varphi_2 \circ \alpha_p$, so we have $\tilde{\varphi}_1(0) = \tilde{\varphi}_2(0) = 0$. Let $Q = \text{span}\{e_1, e_2\}$, where $e_1 = \frac{K_0}{\|K_0\|}$ and $e_2 = \frac{K_0^{[n]}}{\|K_0^{[n]}\|}$ for some $n \geqslant 1$. From Lemma 2.1, we obtain that

$$(C_{\tilde{\psi}_1,\tilde{\varphi}_1} + C_{\tilde{\psi}_2,\tilde{\varphi}_2})^*(1) = \overline{\tilde{\psi}_1(0)} + \overline{\tilde{\psi}_2(0)}.$$

By Lemma 2.11,

$$(C_{\tilde{\psi}_1,\tilde{\varphi}_1} + C_{\tilde{\psi}_2,\tilde{\varphi}_2})^* \left(\frac{K_0^{[n]}}{\|K_0^{[n]}\|} \right) = \sum_{j=0}^{n-1} \left(\overline{\alpha_{1,j}(0) + \alpha_{2,j}(0)} \right) \frac{K_0^{[j]}}{n!} + \left(\overline{\tilde{\psi}_1(0)(\tilde{\varphi}_1'(0))^n + \tilde{\psi}_2(0)(\tilde{\varphi}_2'(0))^n} \right) \frac{K_0^{[n]}}{n!}.$$

Considering the compression of $(C_{\tilde{\psi}_1,\tilde{\varphi}_1} + C_{\tilde{\psi}_2,\tilde{\varphi}_2})^*$ to Q, its matrix representation is as follow

$$\begin{bmatrix} \overline{\tilde{\psi}_1(0) + \tilde{\psi}_2(0)} & \frac{\overline{\tilde{\psi}_1{}^{(n)}(0) + \tilde{\psi}_2{}^{(n)}(0)}}{n!} \\ 0 & \overline{\tilde{\psi}_1(0)(\tilde{\varphi}_1'(0))^n + \tilde{\psi}_2(0)(\tilde{\varphi}_2'(0))^n} \end{bmatrix}.$$

We have $\tilde{\psi}_1(0) = \psi_1(p)$, $\tilde{\psi}_2(0) = \psi_2(p)$ and $\tilde{\varphi}_1'(0) = \varphi_1'(p)$, $\tilde{\varphi}_2'(0) = \varphi_2'(p)$. Therefore, the numerical range of the compression of $(C_{\psi_1,\varphi_1} + C_{\psi_2,\varphi_2})^*$ is the ellipse with foci at $\overline{\psi_1(p) + \psi_2(p)}$ and $\overline{\psi_1(p)(\varphi_1'(p))^n + \psi_2(p)(\varphi_2'(p))^n}$ and minor axis

$$\left| \frac{\overline{\tilde{\psi}_1{}^{(n)}(0) + \tilde{\psi}_2{}^{(n)}(0)}}{n!} \right|$$

and major axis

$$\sqrt{\left| (\overline{\psi_1(p) + \psi_2(p)}) - (\overline{\psi_1(p)(\varphi_1'(p))^n + \psi_2(p)(\varphi_2'(p))^n}) \right|^2 + \left| \frac{\overline{\tilde{\psi}_1{}^{(n)}(0) + \tilde{\psi}_2{}^{(n)}(0)}}{n!} \right|^2}.$$

Thus for each n, $\psi_1(p)(\varphi_1'(p))^n + \psi_2(p)(\varphi_2'(p))^n \in W(C_{\psi_1,\varphi_1} + C_{\psi_2,\varphi_2})$. □

Following example gives two illustrations of Proposition 2.12 in which 0 belongs to the interior of the numerical range.

EXAMPLE 2.13. (a) Let $\varphi_1(z) = \frac{9-9z}{15-12z}$, $\varphi_2(z) = \frac{7-8z}{11-10z}$, $\psi_1(z) = e^z$ and $\psi_2(z) = 3z^2$. It is easy to see that $\frac{1}{2}$ is the fixed point of φ_1 and φ_2. Moreover, $\varphi_1'(\frac{1}{2}) = -\frac{1}{3}$ and $\varphi_2'(\frac{1}{2}) = -\frac{1}{2}$. By Proposition 2.12, $\psi_1(\frac{1}{2}) + \psi_2(\frac{1}{2})$ and $\psi_1(\frac{1}{2})\varphi_1'(\frac{1}{2}) + \psi_2(\frac{1}{2})\varphi_2'(\frac{1}{2})$ belong to $W(C_{\psi_1,\varphi_1} + C_{\psi_2,\varphi_2})$. It says that $\lambda_1 = e^{\frac{1}{2}} + \frac{3}{4}$ and $\lambda_2 = -\frac{1}{3}e^{\frac{1}{2}} - \frac{3}{8}$ lie in $W(C_{\psi_1,\varphi_1} + C_{\psi_2,\varphi_2})$. Since by the proof of Proposition 2.12, $W(C_{\psi_1,\varphi_1} + C_{\psi_2,\varphi_2})$ contains the ellips with foci λ_1 and λ_2 and the length of minor axis is positive, it is easy to see that 0 belongs to the interior of $W(C_{\psi_1,\varphi_1} + C_{\psi_2,\varphi_2})$.

(b) Suppose that $\varphi_1(z) = \frac{8}{40-48z}$ and $\varphi_2(z) = \frac{46-60z+22z^2}{114-100z+42z^2}$. It is easy to see that $\frac{1}{3}$ is the fixed point of φ_1 and φ_2. Furthermore, $\varphi_1'(\frac{1}{3}) = \frac{2}{3}$ and $\varphi_2'(\frac{1}{3}) = -\frac{1}{4}$. Let

$\psi_1(z) = \frac{z^2}{5}$ and $\psi_2(z) = 36z^2$. Again by Proposition 2.12, $\lambda_1 = \psi_1(\frac{1}{3}) + \psi_2(\frac{1}{3}) = \frac{1}{45} + 4$ and $\lambda_2 = \psi_1(\frac{1}{3})\varphi_1'(\frac{1}{3}) + \psi_2(\frac{1}{3})\varphi_2'(\frac{1}{3}) = -1 + \frac{2}{135}$ belong to $W(C_{\psi_1,\varphi_1} + C_{\psi_2,\varphi_2})$. Since again by the proof of Proposition 2.12, $W(C_{\psi_1,\varphi_1} + C_{\psi_2,\varphi_2})$ contains the ellipse with foci λ_1 and λ_2 and the length of minor axis is positive, it is easy to see that 0 belongs to the interior $W(C_{\psi_1,\varphi_1} + C_{\psi_2,\varphi_2})$.

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