NEW WEIGHTED HERMITE–HADAMARD TYPE INEQUALITIES FOR DIFFERENTIABLE STRONGLY CONVEX AND STRONGLY QUASI–CONVEX MAPPINGS

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Abstract. In this paper, new weighted Hermite-Hadamard type inequalities for differentiable strongly convex and strongly quasi-convex mappings are proved. These results strengthen many results proved in earlier works for these classes of functions. Applications of some of our results to statistics are provided.

1. Introduction

The theory of inequalities mainly depends on convex functions:
A function $h : U \subseteq (-\infty, \infty) \to (-\infty, \infty)$, is said to be convex on a convex set U if the inequality

$$h(kv + (1-k)u) \leqslant kh(v) + (1-k)h(u) \tag{1.1}$$

holds for all v, $u \in U$ and $k \in [0,1]$. If (1.1) holds in reversed direction, then h is said to be concave.

The concept of quasi-convex functions generalizes the concept of convex functions.

DEFINITION 1. [32] A function $h : U \subseteq (-\infty, \infty) \to (-\infty, \infty)$ is said quasi-convex on U if

$$h(kv + (1-k)u) \leqslant \max\{h(v), h(u)\}$$

holds for all $v, u \in U$ and $k \in [0,1]$.

There are quasi-convex functions which are not convex functions see for example [32].

It is known that if a function $h : U \subseteq (-\infty, \infty) \to (-\infty, \infty)$ is convex then

$$h\left(\frac{\varepsilon + \epsilon}{2}\right) \leqslant \frac{1}{\epsilon - \varepsilon} \int_{\varepsilon}^{\epsilon} h(v)\, dv \leqslant \frac{h(\varepsilon) + h(\epsilon)}{2} \tag{1.2}$$

for all ε, $\epsilon \in U$ with $\varepsilon < \epsilon$.

The inequality (1.2) is known as Hermite-Hadamard inequality and it holds in reversed direction if the function h is concave on U.

It is also known that if h is continuous, then each of the two sides of (1.2) characterizes the convexity of h (cf. [35]).

Over the past three decades, the definition of convex functions and the inequality (1.2) have been subject to continuous innovative research. The definition of convex functions has been amended in innumerable forms and hence a number of variants of weighted and non weighted forms of the inequality (1.2) have been achieved by many mathematicians, please refer to [7, 8, 11, 19, 20, 25] and [36], and the references mentioned in these papers.

The Hermite-Hadamard double inequality (1.2) was generalized by Fejér [10] by proving that if $\lambda : [\varepsilon, \epsilon] \to [0, \infty)$ is a symmetric density function on $[\varepsilon, \epsilon]$ and a function $h : [\varepsilon, \epsilon] \to (-\infty, \infty)$ is convex then

$$h\left(\frac{\varepsilon + \epsilon}{2}\right) \int_{\varepsilon}^{\epsilon} \lambda(v)\, dv \leqslant \int_{\varepsilon}^{\epsilon} \lambda(v) h(v)\, dv \leqslant \frac{h(\varepsilon) + h(\epsilon)}{2} \int_{\varepsilon}^{\epsilon} \lambda(v)\, dv. \qquad (1.3)$$

A lot of estimates have been obtained by a number of researchers for $\left| h\left(\frac{\varepsilon + \epsilon}{2}\right) - \frac{1}{\epsilon - \varepsilon} \int_{\varepsilon}^{\epsilon} h(v)\, dv \right|$ and $\left| \frac{h(\varepsilon) + h(\epsilon)}{2} - \frac{1}{\epsilon - \varepsilon} \int_{\varepsilon}^{\epsilon} h(v)\, dv \right|$, and their weighted versions with symmetric functions by using different convexity conditions, please cf. [1], [19], [31], [15], [14] and the references cited in these studies.

Strongly convex functions were introduced by Polyak [33]. Strongly convex functions have properties useful in optimization, mathematical economics and other branches of pure and applied mathematics.

DEFINITION 2. [33] Let $U \subset (-\infty, \infty)$ be an interval and r be a positive number. A function $h : U \to (-\infty, \infty)$ is called strongly convex with modulus r if

$$h(kv + (1-k)u) \leqslant kh(v) + (1-k)h(u) - rk(1-k)(v-u)^2$$

for all $v, u \in U$ and all $k \in [0, 1]$.

A counterpart of the Hermite-Hadamard inequality for strongly convex functions is given in the following theorem.

THEOREM 1. [33] *If a function* $h : U \to (-\infty, \infty)$ *is strongly convex with modulus* r *then*

$$h\left(\frac{\varepsilon + \epsilon}{2}\right) + \frac{r}{12}(\varepsilon - \epsilon)^2 \leqslant \frac{1}{\epsilon - \varepsilon} \int_{\varepsilon}^{\epsilon} h(v)\, dv \leqslant \frac{h(\varepsilon) + h(\epsilon)}{2} - \frac{r}{6}(\varepsilon - \epsilon)^2 \qquad (1.4)$$

for all $v, u \in U, v < u$. *Conversely, if* h *is continuous and satisfies the left- or the right-hand side of* (1.4) *for all* $v, u \in U$, $v < u$, *then it is strongly convex with modulus* r.

Since strong convexity is a strengthening of the notion of convexity, some properties of strongly convex functions are just stronger versions of known properties of convex functions

Now we recall the notion of the strongly quasi-convex functions as follows:

DEFINITION 3. [16] A real-valued function h defined on a convex subset C of a real normed space E is uniformly quasi-convex on C if there exists a non-negative function $\delta(s)$, $\delta(0) = 0$, $\delta(s_0) > 0$ for some $s_0 > 0$, such that for all $v, u \in C$ and all $k \in [0, 1]$

$$h(kv + (1-k)u) \leqslant \max\{h(v), h(u)\} - k(1-k)\delta(\|v - u\|)$$

holds. Particularly, if $\delta(s) = rs^2$ with r positive real number, it is said $h(v)$ is a strongly quasi-convex function. For $r = 0$, $h(v)$ is quasi-convex.

REMARK 1. If $C = U$ and $E = (-\infty, \infty)$, then the function h is strongly quasi-convex function on U with modulus r if

$$h(kv + (1-k)u) \leqslant \max\{h(v), h(u)\} - k(1-k)(v - u)^2$$

$v, u \in U, v < u$.

REMARK 2. The notion of strongly quasi-convexity strengthens the concept of quasi-convexity.

The following theorem is a counterpart of the Fejér inequalities for strongly convex functions.

THEOREM 2. [34] *Let* $\lambda : [\varepsilon, \epsilon] \to [0, \infty)$ *be a symmetric density function on* $[\varepsilon, \epsilon]$ *and* $h : [\varepsilon, \epsilon] \to (-\infty, \infty)$ *be a strongly convex function with modulus* $r > 0$. *Then*

$$h\left(\frac{\varepsilon + \epsilon}{2}\right) + r\left[\int_\varepsilon^\epsilon v^2 \lambda(v)\, dv - \left(\frac{\varepsilon + \epsilon}{2}\right)^2\right] \leqslant \int_\varepsilon^\epsilon h(v)\lambda(v)\, dv$$

$$\leqslant \frac{h(\varepsilon) + h(\epsilon)}{2} - r\left[\frac{\varepsilon^2 + \epsilon^2}{2} - \int_\varepsilon^\epsilon v^2 \lambda(v)\, dv\right]. \quad (1.5)$$

It can be observed that the inequalities (1.5) are a strengthening of the Fejér inequalities (1.3) if $\lambda : [\varepsilon, \epsilon] \to [0, \infty)$ be a symmetric density function on $[\varepsilon, \epsilon]$.

Gavrea [11] extended the inequality obtained in [1, Theorem 2.2, page 227] to weighted form and generalized the inequalities proved in [14, Theorem 2.8, page 9602] and [15, Theorem 2.8, page 72] in such a way that the weight function $\lambda(v)$ is not necessarily symmetric with respect to the midpoint $\frac{\varepsilon + \epsilon}{2}$.

Inspired by the research towards this direction, the main objectives of this paper are to acquire new weighted Hermite-Hadamard type inequalities for strongly convex and strongly quasi-convex mappings. The results of this paper strengthen the results of Gavrea [11] and hence, in particular, improves the results proved in [1], [19], [31], [15] and [14].

In section 2, we recall some integral identities for a differentiable mapping and a symmetric function with respect to $\frac{\varepsilon + \epsilon}{2}$ defined over an interval $[\varepsilon, \epsilon]$. In section 2, an important inequality for positive linear functional on $C([\varepsilon, \epsilon])$ and a strongly convex

function is proved to obtain some very revivifying results of this manuscript. Section 2 contains some new weighted Hermite-Hadamard type integral inequalities related with the left and right parts of Hermite-Hadamard inequalities (1.3). The results of section 2 provide weighted generalization and improvements of a number of results proved as yet in the field of mathematical inequalities for differentiable strongly convex and strongly quasi-convex functions.

2. Main results

The following notations and results have been used in [11]:
Let $\eta(v) : [\varepsilon, \epsilon] \to [0, \infty)$ be a continuous function with

$$\int_{\varepsilon}^{\epsilon} \eta(v)\, dv = 1$$

and the integral $\int_{\varepsilon}^{\epsilon} v\eta(v)$ is denoted by ε_1, that is

$$\varepsilon_1 = \int_{\varepsilon}^{\epsilon} v\eta(v)\, dv.$$

In case, when $\eta(v) : [\varepsilon, \epsilon] \to [0, \infty)$ is symmetric with respect to $\frac{\varepsilon + \epsilon}{2}$, that is, if

$$\eta(\varepsilon + \epsilon - v) = \eta(v),$$

then the following result holds:

LEMMA 1. [11] *If $\eta(v) : [\varepsilon, \epsilon] \to [0, \infty)$ is symmetric with respect to $\frac{\varepsilon + \epsilon}{2}$. Then*

$$\varepsilon_1 = \frac{\varepsilon + \epsilon}{2}.$$

We need the following result to prove the results of this study.

LEMMA 2. *Let $A : C([\varepsilon, \epsilon]) \to (-\infty, \infty)$ be a positive linear functional on $C([\varepsilon, \epsilon])$ and let e_i be monomials $e_i(v) = v^i$, $v \in [\varepsilon, \epsilon]$, $i \in \mathbb{N}$. Let λ be a strongly convex function on $[\varepsilon, \epsilon]$ with modulus r, then the following inequality holds*

$$A(\lambda(e_1)) \leqslant \frac{A(\epsilon - e_1)\lambda(\varepsilon) + A(e_1 - \varepsilon)\lambda(\epsilon)}{\epsilon - \varepsilon} - rA((\epsilon - e_1)(e_1 - \varepsilon)). \tag{2.1}$$

Proof. By using the convexity of λ on $[\varepsilon, \epsilon]$ and the given equality

$$e_1 = \left(\frac{\epsilon - e_1}{\epsilon - \varepsilon}\right)\varepsilon + \left(\frac{e_1 - \varepsilon}{\epsilon - \varepsilon}\right)\epsilon,$$

we get

$$\lambda(e_1) = \lambda\left(\left(\frac{\epsilon - e_1}{\epsilon - \varepsilon}\right)\varepsilon + \left(\frac{e_1 - \varepsilon}{\epsilon - \varepsilon}\right)\epsilon\right)$$
$$\leqslant \frac{(\epsilon - e_1)\lambda(\varepsilon) + (e_1 - \varepsilon)\lambda(\epsilon)}{\epsilon - \varepsilon} - r(\epsilon - e_1)(e_1 - \varepsilon). \tag{2.2}$$

Since A is a positive linear functional, we get the inequality (2.1) by applying A on both sides of (2.2). $\square$

The following theorem strengthens the result given by Gavrea in [11].

THEOREM 3. *Let $h : U \subseteq (-\infty, \infty) \to (-\infty, \infty)$ be a differentiable mapping on U° and $h' \in L([\varepsilon, \epsilon])$, where $[\varepsilon, \epsilon] \subseteq U^{\circ}$. If $\eta : [\varepsilon, \epsilon] \to [0, \infty)$ is a continuous mapping and $|h'|$ is strongly convex function on $[\varepsilon, \epsilon]$ with modulus r, then the inequality holds*

$$\left| \int_{\varepsilon}^{\epsilon} h(v)\,\eta(v)\,dv - h(\varepsilon_1) \right|$$

$$\leqslant \frac{\chi_1(\varepsilon, \varepsilon_1, \epsilon)\,|h'(\varepsilon)| + \chi_2(\varepsilon, \varepsilon_1, \epsilon)\,|h'(\epsilon)|}{\epsilon - \varepsilon} - r\chi_3(\varepsilon, \varepsilon_1, \epsilon), \quad (2.3)$$

where

$$\chi_1(\varepsilon, \varepsilon_1, \epsilon) = \frac{1}{2}\int_{\varepsilon}^{\varepsilon_1} (\varepsilon_1 - v)(\epsilon - v)\,\eta(v)\,dv - \frac{1}{2}\int_{\varepsilon_1}^{\epsilon} (\varepsilon_1 - v)(\epsilon - v)\,\eta(v)\,dv,$$

$$\chi_2(\varepsilon, \varepsilon_1, \epsilon) = \frac{1}{2}\int_{\varepsilon_1}^{\epsilon} (\varepsilon_1 - v)(v - \varepsilon)\,\eta(v)\,dv - \frac{1}{2}\int_{\varepsilon}^{\varepsilon_1} (\varepsilon_1 - v)(v - \varepsilon)\,\eta(v)\,dv$$

and

$$\chi_3(\varepsilon, \varepsilon_1, \epsilon) = \frac{1}{6}\int_{\varepsilon}^{\varepsilon_1} (v - \varepsilon_1)\left[2\left(v^2 + \varepsilon_1 v + \varepsilon_1^2\right) - 3(\varepsilon + \epsilon)(v - \varepsilon_1) + 6\varepsilon\,\epsilon\right]\eta(v)\,dv$$

$$-\frac{1}{6}\int_{\varepsilon_1}^{\epsilon} (v - \varepsilon_1)\left[2\left(v^2 + \varepsilon_1 v + \varepsilon_1^2\right) - 3(\varepsilon + \epsilon)(v - \varepsilon_1) + 6\varepsilon\,\epsilon\right]\eta(v)\,dv.$$

Proof. We can write

$$h(v) - h(\varepsilon_1) = \int_{\varepsilon}^{\epsilon} [\sigma(v - s) - \sigma(\varepsilon_1 - s)]\,h'(s)\,ds, \quad (2.4)$$

where $\sigma(\cdot)$ is the Heavyside function defined by

$$\sigma(x) = \begin{cases} 0, & x < 0 \\ 1 & x > 0. \end{cases}$$

Multiplying both sides of (2.4) by $\eta(v)$, integrating over the interval $[\varepsilon, \epsilon]$ with respect to v and using the fact that $\int_{\varepsilon}^{\epsilon} \eta(v)\,dv = 1$, we obtain

$$\int_{\varepsilon}^{\epsilon} h(v)\,\eta(v)\,dv - h(\varepsilon_1) = \int_{\varepsilon}^{\epsilon}\left[\int_{\varepsilon}^{\epsilon}\sigma(v - s)\,\eta(v)\,dv - \sigma(\varepsilon_1 - s)\right]h'(s)\,ds$$

$$= \int_{\varepsilon}^{\epsilon}\left(\int_{s}^{\epsilon}\eta(v)\,dv - \sigma(\varepsilon_1 - s)\right)h'(s)\,ds. \quad (2.5)$$

Using the definition of the Heavyside function, we get that

$$\int_{\varepsilon}^{\epsilon} \sigma(v-s)\eta(v)\,dv = \int_{\varepsilon}^{s} \sigma(v-s)\eta(v)\,dv + \int_{s}^{\epsilon} \sigma(v-s)\eta(v)\,dv$$

$$= \int_{s}^{\epsilon} \eta(v)\,dv. \tag{2.6}$$

Applying (2.6) in (2.5), we have

$$\int_{\varepsilon}^{\epsilon} h(v)\eta(v)\,dv - h(\varepsilon_1) = \int_{\varepsilon}^{\epsilon}\left(\int_{s}^{\epsilon}\eta(v)\,dv - \sigma(\varepsilon_1 - s)\right)h'(s)\,ds. \tag{2.7}$$

Taking absolute value on both sides of (2.7) and applying Lemma 2, we have

$$\left|\int_{\varepsilon}^{\epsilon} h(v)\eta(v)\,dv - h(\varepsilon_1)\right| \leqslant \int_{\varepsilon}^{\epsilon}\left|\int_{s}^{\epsilon}\eta(v)\,dv - \sigma(\varepsilon_1 - s)\right||h'(s)|\,ds$$

$$\leqslant \frac{|h'(\varepsilon)|}{\epsilon - \varepsilon}\int_{\varepsilon}^{\epsilon}\left|\int_{s}^{\epsilon}\eta(v)\,dv - \sigma(\varepsilon_1 - s)\right|(\epsilon - s)\,ds$$

$$+ \frac{|h'(\epsilon)|}{\epsilon - \varepsilon}\int_{\varepsilon}^{\epsilon}\left|\int_{s}^{\epsilon}\eta(v)\,dv - \sigma(\varepsilon_1 - s)\right|(s - \varepsilon)\,ds$$

$$- r\int_{\varepsilon}^{\epsilon}\left|\int_{s}^{\epsilon}\eta(v)\,dv - \sigma(\varepsilon_1 - s)\right|(s - \varepsilon)(\epsilon - s)\,ds. \tag{2.8}$$

We observe that

$$\int_{\varepsilon}^{\epsilon}\left|\int_{s}^{\epsilon}\eta(v)\,dv - \sigma(\varepsilon_1 - s)\right|(\epsilon - s)\,ds$$

$$= \int_{\varepsilon}^{\varepsilon_1}\left|\int_{s}^{\epsilon}\eta(v)\,dv - \int_{\varepsilon}^{\epsilon}\eta(v)\,dv\right|(\epsilon - s)\,ds + \int_{\varepsilon_1}^{\epsilon}\left(\int_{s}^{\epsilon}\eta(v)\,dv\right)(\epsilon - s)\,ds$$

$$= \int_{\varepsilon}^{\varepsilon_1}\left(\int_{\varepsilon}^{s}\eta(v)\,dv\right)(\epsilon - s)\,ds + \int_{\varepsilon_1}^{\epsilon}\left(\int_{s}^{\epsilon}\eta(v)\,dv\right)(\epsilon - s)\,ds$$

$$= \frac{(\epsilon - \varepsilon_1)^2}{2}\left[\int_{\varepsilon_1}^{\epsilon}\eta(v)\,dv - \int_{\varepsilon}^{\varepsilon_1}\eta(v)\,dv\right] + \frac{1}{2}\int_{\varepsilon}^{\varepsilon_1}(\epsilon - v)^2\eta(v)\,dv$$

$$- \frac{1}{2}\int_{\varepsilon_1}^{\epsilon}(\epsilon - v)^2\eta(v)\,dv = \frac{1}{2}\int_{\varepsilon}^{\varepsilon_1}(\varepsilon_1 - v)^2\eta(v)\,dv$$

$$- \frac{1}{2}\int_{\varepsilon_1}^{\epsilon}(\varepsilon_1 - v)^2\eta(v)\,dv + \frac{(\epsilon - \varepsilon_1)}{2}\int_{\varepsilon}^{\varepsilon_1}(\varepsilon_1 - v)\eta(v)\,dv$$

$$- \frac{(\epsilon - \varepsilon_1)}{2}\int_{\varepsilon_1}^{\epsilon}(\varepsilon_1 - v)\eta(v)\,dv$$

$$= \chi_1(\varepsilon, \varepsilon_1, \epsilon). \tag{2.9}$$

In a similar way

$$\int_{\varepsilon}^{\epsilon} \left| \int_{s}^{\epsilon} \eta(v)\, dv - \sigma(\varepsilon_1 - s) \right| (s - \varepsilon)\, ds$$

$$= \frac{1}{2} \int_{\varepsilon_1}^{\epsilon} (v - \varepsilon_1)^2 \eta(v)\, dv - \frac{1}{2} \int_{\varepsilon}^{\varepsilon_1} (v - \varepsilon_1)^2 \eta(v)\, dv$$

$$+ \frac{(\varepsilon_1 - \varepsilon)}{2} \int_{\varepsilon_1}^{\epsilon} (\varepsilon_1 - v) \eta(v)\, dv - \frac{(\varepsilon_1 - \varepsilon)}{2} \int_{\varepsilon}^{\varepsilon_1} (\varepsilon_1 - v) \eta(v)\, dv$$

$$= \chi_2(\varepsilon, \varepsilon_1, \epsilon) \tag{2.10}$$

and

$$\int_{\varepsilon}^{\epsilon} \left| \int_{s}^{\epsilon} \eta(v)\, dv - \sigma(\varepsilon_1 - s) \right| (s - \varepsilon)(\epsilon - s)\, ds$$

$$= (\epsilon - \varepsilon_1) \int_{\varepsilon}^{\epsilon} \left| \int_{s}^{\epsilon} \eta(v)\, dv - \sigma(\varepsilon_1 - s) \right| (s - \varepsilon_1)\, ds - \int_{\varepsilon}^{\epsilon} \left| \int_{s}^{\epsilon} \eta(v)\, dv - \sigma(\varepsilon_1 - s) \right|$$

$$\times (s - \varepsilon_1)^2\, ds - (\varepsilon_1 - \varepsilon) \int_{\varepsilon}^{\epsilon} \left| \int_{s}^{\epsilon} \eta(v)\, dv - \sigma(\varepsilon_1 - s) \right| (s - \varepsilon_1)\, ds$$

$$+ (\epsilon - \varepsilon_1)(\varepsilon_1 - \varepsilon) \int_{\varepsilon}^{\epsilon} \left| \int_{s}^{\epsilon} \eta(v)\, dv - \sigma(\varepsilon_1 - s) \right| ds$$

$$= \frac{(\epsilon - \varepsilon_1)}{2} \left[\int_{\varepsilon_1}^{\epsilon} (v - \varepsilon_1)^2 \eta(v)\, dv - \int_{\varepsilon}^{\varepsilon_1} (v - \varepsilon_1)^2 \eta(v)\, dv \right]$$

$$- \frac{(\varepsilon_1 - \varepsilon)}{2} \left[\int_{\varepsilon_1}^{\epsilon} (v - \varepsilon_1)^2 \eta(v)\, dv - \int_{\varepsilon}^{\varepsilon_1} (v - \varepsilon_1)^2 \eta(v)\, dv \right]$$

$$- \frac{1}{3} \left[\int_{\varepsilon_1}^{\epsilon} (v - \varepsilon_1)^3 \eta(v)\, dv - \int_{\varepsilon}^{\varepsilon_1} (v - \varepsilon_1)^3 \eta(v)\, dv \right] + (\epsilon - \varepsilon_1)(\varepsilon_1 - \varepsilon)$$

$$\times \left[\int_{\varepsilon_1}^{\epsilon} (v - \varepsilon_1) \eta(v)\, dv - \int_{\varepsilon}^{\varepsilon_1} (v - \varepsilon_1) \eta(v)\, dv \right] = \chi_2(\varepsilon, \varepsilon_1, \epsilon) \tag{2.11}$$

This concludes the proof of the theorem by applying (2.9)–(2.11) in (2.8). □

COROLLARY 1. *Suppose that the assumptions of Theorem* 3 *are satisfied and that* $\eta(v)$ *is symmetric with respect to* $\frac{\varepsilon + \epsilon}{2}$ *on* $[\varepsilon, \epsilon]$. *Then the following inequality holds:*

$$\left| \int_{\varepsilon}^{\epsilon} h(v) \eta(v)\, dv - h\left(\frac{\varepsilon + \epsilon}{2} \right) \right| \le \frac{[|h'(\varepsilon)| + |h'(\epsilon)|]}{\epsilon - \varepsilon} \chi_1(\varepsilon, \epsilon) - r \chi_3(\varepsilon, \epsilon), \tag{2.12}$$

where

$$\chi_1(\varepsilon, \epsilon) = \frac{1}{2} \int_{\varepsilon}^{\frac{\varepsilon + \epsilon}{2}} \left(\frac{\varepsilon + \epsilon}{2} - v \right) (\epsilon - v) \eta(v)\, dv$$

$$- \frac{1}{2} \int_{\frac{\varepsilon + \epsilon}{2}}^{\epsilon} \left(\frac{\varepsilon + \epsilon}{2} - v \right) (\epsilon - v) \eta(v)\, dv$$

and

$$\chi_3\left(\varepsilon,\epsilon\right) = \frac{1}{6}\int_{\varepsilon}^{\frac{\varepsilon+\epsilon}{2}}\left(v - \frac{\varepsilon+\epsilon}{2}\right)\left[8\varepsilon\,\epsilon + (\varepsilon+\epsilon)^2 + (v-\varepsilon)^2 + (\epsilon-v)^2\right]\eta\left(v\right)dv$$
$$-\frac{1}{6}\int_{\frac{\varepsilon+\epsilon}{2}}^{\epsilon}\left(v - \frac{\varepsilon+\epsilon}{2}\right)\left[8\varepsilon\,\epsilon + (\varepsilon+\epsilon)^2 + (v-\varepsilon)^2 + (\epsilon-v)^2\right]\eta\left(v\right)dv.$$

Proof. Since the function $\eta\left(v\right)$ is symmetric with respect to $\frac{\varepsilon+\epsilon}{2}$ on $[\varepsilon,\epsilon]$ so $\varepsilon_1 = \frac{\varepsilon+\epsilon}{2}$ and the function $\eta\left(v\right)$ is symmetric with respect $\frac{\varepsilon+\epsilon}{2}$ on $[\varepsilon,\epsilon]$. This fact gives

$$\chi_1\left(\varepsilon,\varepsilon_1,\epsilon\right) = \chi_1\left(\varepsilon,\frac{\varepsilon+\epsilon}{2},\epsilon\right) = \chi_2\left(\varepsilon,\frac{\varepsilon+\epsilon}{2},\epsilon\right) = \chi_2\left(\varepsilon,\varepsilon_1,\epsilon\right)$$

Thus

$$\chi_1\left(\varepsilon,\frac{\varepsilon+\epsilon}{2},\epsilon\right) = \chi_2\left(\varepsilon,\frac{\varepsilon+\epsilon}{2},\epsilon\right)$$
$$= \frac{1}{2}\int_{\varepsilon}^{\frac{\varepsilon+\epsilon}{2}}\left(\frac{\varepsilon+\epsilon}{2}-v\right)(\epsilon-v)\,\eta\left(v\right)dv$$
$$-\frac{1}{2}\int_{\frac{\varepsilon+\epsilon}{2}}^{\epsilon}\left(\frac{\varepsilon+\epsilon}{2}-v\right)(\epsilon-v)\,\eta\left(v\right)dv = \chi_1\left(\varepsilon,\epsilon\right). \quad (2.13)$$

Moreover,

$$\chi_3\left(\varepsilon,\varepsilon_1,\epsilon\right) = \chi_3\left(\varepsilon,\frac{\varepsilon+\epsilon}{2},\epsilon\right) = \chi_3\left(\varepsilon,\epsilon\right)$$
$$= \frac{1}{6}\int_{\varepsilon}^{\frac{\varepsilon+\epsilon}{2}}\left(v - \frac{\varepsilon+\epsilon}{2}\right)\left[8\varepsilon\,\epsilon + (\varepsilon+\epsilon)^2 + (v-\varepsilon)^2 + (\epsilon-v)^2\right]\eta\left(v\right)dv$$
$$-\frac{1}{6}\int_{\frac{\varepsilon+\epsilon}{2}}^{\epsilon}\left(v - \frac{\varepsilon+\epsilon}{2}\right)\left[8\varepsilon\,\epsilon + (\varepsilon+\epsilon)^2 + (v-\varepsilon)^2 + (\epsilon-v)^2\right]\eta\left(v\right)dv. \quad \square$$

COROLLARY 2. *If we take* $\eta\left(v\right) = \frac{\lambda\left(v\right)}{\int_{\varepsilon}^{\epsilon}\lambda\left(v\right)dv}$ *in* (2.3) *and* $\lambda\left(v\right)$ *is symmetric with respect to* $\frac{\varepsilon+\epsilon}{2}$. *Then the following inequality holds*

$$\left|\int_{\varepsilon}^{\epsilon}h\left(v\right)\lambda\left(v\right)dv - h\left(\frac{\varepsilon+\epsilon}{2}\right)\int_{\varepsilon}^{\epsilon}\lambda\left(v\right)dv\right|$$
$$\leqslant \frac{\left[|h'\left(\varepsilon\right)| + |h'\left(\epsilon\right)|\right]}{\epsilon-\varepsilon}\chi_4\left(\varepsilon,\epsilon\right) - r\chi_5\left(\varepsilon,\epsilon\right), \quad (2.14)$$

where

$$\chi_4\left(\varepsilon,\epsilon\right) = \frac{1}{2}\int_{\varepsilon}^{\frac{\varepsilon+\epsilon}{2}}\left(\frac{\varepsilon+\epsilon}{2}-v\right)(\epsilon-v)\,\lambda\left(v\right)dv$$
$$-\frac{1}{2}\int_{\frac{\varepsilon+\epsilon}{2}}^{\epsilon}\left(\frac{\varepsilon+\epsilon}{2}-v\right)(\epsilon-v)\,\lambda\left(v\right)dv$$

and

$$\chi_5(\varepsilon,\epsilon) = \frac{1}{6}\int_\varepsilon^{\frac{\varepsilon+\epsilon}{2}} \left(v - \frac{\varepsilon+\epsilon}{2}\right) \left[8\varepsilon\,\epsilon + (\varepsilon+\epsilon)^2 + (v-\varepsilon)^2 + (\epsilon-v)^2\right]\lambda(v)\,dv$$

$$-\frac{1}{6}\int_{\frac{\varepsilon+\epsilon}{2}}^\epsilon \left(v - \frac{\varepsilon+\epsilon}{2}\right) \left[8\varepsilon\,\epsilon + (\varepsilon+\epsilon)^2 + (v-\varepsilon)^2 + (\epsilon-v)^2\right]\lambda(v)\,dv.$$

THEOREM 4. *Let $h : U \subseteq (-\infty,\infty) \to (-\infty,\infty)$ be a differentiable mapping on U° and $h' \in L([\varepsilon,\epsilon])$, where $[\varepsilon,\epsilon] \subseteq U^\circ$. If $\eta : [\varepsilon,\epsilon] \to [0,\infty)$ is a continuous mapping and $|h'|^q$ is strongly convex function on $[\varepsilon,\epsilon]$ with modulus r for $q \geqslant 1$, then the following inequality holds*

$$\left|\int_\varepsilon^\epsilon h(v)\,\eta(v)\,dv - h(\varepsilon_1)\right|$$

$$\leqslant \left(\int_{\varepsilon_1}^\epsilon (v-\varepsilon_1)\,\eta(v)\,dv - \int_\varepsilon^{\varepsilon_1} (v-\varepsilon_1)\,\eta(v)\,dv\right)^{1-\frac{1}{q}}$$

$$\times \left(\frac{\chi_1(\varepsilon,\varepsilon_1,\epsilon)\,|h'(\varepsilon)|^q + \chi_2(\varepsilon,\varepsilon_1,\epsilon)\,|h'(\epsilon)|^q}{\epsilon-\varepsilon} - r\chi_3(\varepsilon,\varepsilon_1,\epsilon)\right)^{\frac{1}{q}}, \quad (2.15)$$

where $\chi_1(\varepsilon,\varepsilon_1,\epsilon)$, $\chi_2(\varepsilon,\varepsilon_1,\epsilon)$ and $\chi_3(\varepsilon,\varepsilon_1,\epsilon)$ are as defined in Theorem 3.

Proof. Application of Hölder inequality in (2.7) yields that

$$\left|\int_\varepsilon^\epsilon h(v)\,\eta(v)\,dv - h(\varepsilon_1)\right| \leqslant \int_\varepsilon^\epsilon \left|\int_s^\epsilon \eta(v)\,dv - \sigma(\varepsilon_1-s)\right| |h'(s)|^q\,ds$$

$$\leqslant \left(\int_\varepsilon^\epsilon \left|\int_s^\epsilon \eta(v)\,dv - \sigma(\varepsilon_1-s)\right| ds\right)^{1-\frac{1}{q}}$$

$$\times \left(\int_\varepsilon^\epsilon \left|\int_s^\epsilon \eta(v)\,dv - \sigma(\varepsilon_1-s)\right| |h'(s)|^q\,ds\right)^{\frac{1}{q}}. \quad (2.16)$$

Applying Lemma 2, we have

$$\int_\varepsilon^\epsilon \left|\int_s^\epsilon \eta(v)\,dv - \sigma(\varepsilon_1-s)\right| |h'(s)|^q\,ds$$

$$\leqslant \chi_1(\varepsilon,\varepsilon_1,\epsilon)\,|h'(\varepsilon)|^q + \chi_2(\varepsilon,\varepsilon_1,\epsilon)\,|h'(\epsilon)|^q$$

$$-r\int_\varepsilon^\epsilon \left|\int_s^\epsilon \eta(v)\,dv - \sigma(\varepsilon_1-s)\right| (s-\varepsilon)(\epsilon-s)\,ds. \quad (2.17)$$

On the other hand, we have

$$\int_\varepsilon^\epsilon \left|\int_s^\epsilon \eta(v)\,dv - \sigma(\varepsilon_1-s)\right| ds$$

$$= \int_\varepsilon^{\varepsilon_1} \left(\int_\varepsilon^s \eta(v)\,dv\right) ds + \int_{\varepsilon_1}^\epsilon \left(\int_s^\epsilon \eta(v)\,dv\right) ds$$

$$= \int_{\varepsilon_1}^\epsilon (v-\varepsilon_1)\,\eta(v)\,dv - \int_\varepsilon^{\varepsilon_1} (v-\varepsilon_1)\,\eta(v)\,dv. \quad (2.18)$$

A combination of (2.16)–(2.18) gives (2.15). This completes the proof of the theorem. □

COROLLARY 3. *If $\eta(v)$ is symmetric with respect to $\frac{\varepsilon+\epsilon}{2}$ on $[\varepsilon,\epsilon]$, then from (2.15), we obtain the following inequality*

$$\left| \int_\varepsilon^\epsilon h(v)\eta(v)\,dv - h\left(\frac{\varepsilon+\epsilon}{2}\right)\right| \leqslant \left(2\int_{\frac{\varepsilon+\epsilon}{2}}^\epsilon \left(v - \frac{\varepsilon+\epsilon}{2}\right)\eta(v)\,dv\right)^{1-\frac{1}{q}}$$
$$\times \left(\chi_1(\varepsilon,\epsilon)\left[\frac{|h'(\varepsilon)|^q + |h'(\epsilon)|^q}{\epsilon - \varepsilon}\right] - r\chi_3(\varepsilon,\epsilon)\right)^{\frac{1}{q}}, \quad (2.19)$$

where $\chi_1(\varepsilon,\epsilon)$ and $\chi_3(\varepsilon,\epsilon)$ as defined in Corollary 1.

COROLLARY 4. *If $\eta(v) = \frac{\lambda(v)}{\int_\varepsilon^\epsilon \lambda(v)\,dv}$ and $\lambda(v)$ is symmetric with respect to $\frac{\varepsilon+\epsilon}{2}$ on $[\varepsilon,\epsilon]$, then the following inequality holds:*

$$\left| \int_\varepsilon^\epsilon h(v)\lambda(v)\,dv - h\left(\frac{\varepsilon+\epsilon}{2}\right)\int_\varepsilon^\epsilon \lambda(v)\,dv\right| \leqslant \left(2\int_{\frac{\varepsilon+\epsilon}{2}}^\epsilon \left(v - \frac{\varepsilon+\epsilon}{2}\right)\lambda(v)\,dv\right)^{1-\frac{1}{q}}$$
$$\times \left(\chi_4(\varepsilon,\epsilon)\left[\frac{|h'(\varepsilon)|^q + |h'(\epsilon)|^q}{\epsilon - \varepsilon}\right] - r\chi_5(\varepsilon,\epsilon)\right)^{\frac{1}{q}}, \quad (2.20)$$

where $\chi_{12}(\varepsilon,\epsilon)$ is as defined in Corollary 2.

For our next results, we use the following notations.

$$\varphi(\eta,h) := \frac{h(\varepsilon)\int_\varepsilon^\epsilon (\epsilon - v)\eta(v)\,dv + h(\epsilon)\int_\varepsilon^\epsilon (v - \varepsilon)\eta(v)\,dv}{\epsilon - \varepsilon} - \int_\varepsilon^\epsilon \eta(v)h(v)\,dv. \quad (2.21)$$

It is clear from (2.21) that

$$\varphi\left(\frac{1}{\epsilon - \varepsilon}, h\right) := \frac{h(\varepsilon) + h(\epsilon)}{2} - \int_\varepsilon^\epsilon h(v)\eta(v)\,dv. \quad (2.22)$$

The next result gives upper bound of $|\varphi(\eta,h)|$ when the function $h(v)$ is strongly quasi-convex function on $[\varepsilon,\epsilon]$ with modulus r.

To prove our next results, we need the following result.

LEMMA 3. [11] *Let $h : U \subseteq (-\infty,\infty) \to (-\infty,\infty)$ be a differentiable mapping on U° and $h' \in L([\varepsilon,\epsilon])$, where $[\varepsilon,\epsilon] \subseteq U^\circ$. Let $\eta : [\varepsilon,\epsilon] \to [0,\infty)$ be a continuous non-negative mapping, then*

$$\frac{h(\varepsilon)\int_\varepsilon^\epsilon (\epsilon - v)\eta(v)\,dv + h(\epsilon)\int_\varepsilon^\epsilon (v - \varepsilon)\eta(v)\,dv}{\epsilon - \varepsilon} - \int_\varepsilon^\epsilon h(v)\eta(v)\,dv$$
$$= \frac{\varepsilon_1 - \varepsilon}{\epsilon - \varepsilon}\int_0^1 H(\eta,\varepsilon,\varepsilon_1;u)h'(u\varepsilon_1 + (1-u)\varepsilon)\,du$$
$$+ \frac{\epsilon - \varepsilon_1}{\epsilon - \varepsilon}\int_0^1 H(\eta,\varepsilon_1,\epsilon;u)h'((1-u)\varepsilon_1 + u\epsilon)\,du, \quad (2.23)$$

where

$$H(\eta,\alpha,\beta;u) = \int_{\varepsilon}^{(1-u)\alpha+u\beta}(v-\varepsilon)\,\eta\,(v)\,dv - \int_{(1-u)\alpha+u\beta}^{\epsilon}(\epsilon-v)\,\eta\,(v)\,dv,$$

$\alpha,\beta \in [\varepsilon,\epsilon].$

THEOREM 5. *Let* $h : U \subseteq (-\infty,\infty) \to (-\infty,\infty)$ *be a differentiable mapping on* U° *and* $h' \in L([\varepsilon,\epsilon])$, *where* $[\varepsilon,\epsilon] \subseteq U^{\circ}$. *If* $\eta : [\varepsilon,\epsilon] \to [0,\infty)$ *be a continuous mapping and* $|h'|$ *is strongly quasi-convex function on* $[\varepsilon,\epsilon]$ *with modulus* r, *then the following inequality holds*

$$\begin{aligned}
|\varphi(\eta,h)| \leqslant{} & \left(\frac{\varepsilon_1-\varepsilon}{\epsilon-\varepsilon}\right)\left(\sup\left\{|h'(\varepsilon)|,|h'(\varepsilon_1)|\right\}\right)\int_0^1|H(\eta,\varepsilon,\varepsilon_1;u)|\,du \\
& + \left(\frac{\epsilon-\varepsilon_1}{\epsilon-\varepsilon}\right)\left(\sup\left\{|h'(\varepsilon_1)|,|h'(\epsilon)|\right\}\right)\int_0^1|H(\eta,\varepsilon_1,\epsilon;u)|\,du \\
& -r\frac{(\varepsilon_1-\varepsilon)^3}{\epsilon-\varepsilon}\int_0^1 u\,(1-u)\,|H(\eta,\varepsilon,\varepsilon_1;u)|\,du \\
& -r\frac{(\epsilon-\varepsilon_1)^3}{\epsilon-\varepsilon}\int_0^1 u\,(1-u)\,|H(\eta,\varepsilon_1,\epsilon;u)|\,du,
\end{aligned} \tag{2.24}$$

where $H(\eta,\alpha,\beta;u)$ *is defined as in Lemma* 3.

Proof. Since $|h'|$ is strongly quasi-convex function on $[\varepsilon,\epsilon]$ with modulus r, we have

$$\left|h'((1-u)\varepsilon+u\varepsilon_1)\right| \leqslant \sup\left\{|h'(\varepsilon)|,|h'(\varepsilon_1)|\right\} - ru\,(1-u)\,(\varepsilon-\varepsilon_1)^2$$

and

$$\left|h'((1-u)\varepsilon_1+u\,\epsilon)\right| \leqslant \sup\left\{|h'(\varepsilon_1)|,|h'(\epsilon)|\right\} - ru\,(1-u)\,(\varepsilon_1-\epsilon)^2$$

for all $u \in [0,1]$. Hence the inequality (2.24) follows from (2.23). $\square$

THEOREM 6. *Let* $h : U \subseteq (-\infty,\infty) \to (-\infty,\infty)$ *be a differentiable mapping on* U° *and* $h' \in L([\varepsilon,\epsilon])$, *where* $[\varepsilon,\epsilon] \subseteq U^{\circ}$. *If* $\eta : [\varepsilon,\epsilon] \to [0,\infty)$ *be a continuous mapping and* $|h'|$ *is strongly quasi-convex function on* $[\varepsilon,\epsilon]$ *with modulus* r, *then the following inequality holds*

$$\begin{aligned}
|\varphi(\eta,h)| \leqslant{} & \left(\int_{\varepsilon}^{\frac{\varepsilon+\epsilon}{2}}(v-\varepsilon)\,ds\right)\left[\sup\left\{\left|h'\left(\frac{\varepsilon+\epsilon}{2}\right)\right|,|h'(\epsilon)|\right\}\right. \\
& \left. + \sup\left\{|h'(\varepsilon)|,\left|h'\left(\frac{\varepsilon+\epsilon}{2}\right)\right|\right\}\right] \\
& -\frac{2r}{3}\int_{\varepsilon}^{\frac{\varepsilon+\epsilon}{2}}(v-\varepsilon)^2\left(v-\frac{\varepsilon+3\epsilon}{4}\right)\eta\,(v)\,dv.
\end{aligned} \tag{2.25}$$

Proof. The symmetry of $\eta(v)$ with respect to $\frac{\varepsilon+\epsilon}{2}$ on $[\varepsilon,\epsilon]$ gives

$$\varphi(\eta,h) = \frac{h(\varepsilon)+h(\epsilon)}{2} - \int_\varepsilon^\epsilon h(v)\eta(v)\,dv.$$

We also observe that

$$\frac{\varepsilon_1-\varepsilon}{\epsilon-\varepsilon}\int_0^1 |H(\eta,\varepsilon,\varepsilon_1;u)|\,du$$

$$= \frac{1}{\epsilon-\varepsilon}\int_\varepsilon^{\varepsilon_1}\left|\int_\varepsilon^s (v-\varepsilon)\eta(v)\,dv - \int_s^\epsilon (\epsilon-v)\eta(v)\,dv\right|ds$$

and

$$\frac{\epsilon-\varepsilon_1}{\epsilon-\varepsilon}\int_0^1 |H(\eta,\varepsilon_1,\epsilon;u)|\,du$$

$$= \frac{1}{\epsilon-\varepsilon}\int_{\varepsilon_1}^\epsilon\left|\int_\varepsilon^s (v-\varepsilon)\eta(v)\,dv - \int_s^\epsilon (\epsilon-v)\eta(v)\,dv\right|ds$$

Consider the function $p:[\varepsilon,\epsilon]\to(-\infty,\infty)$ defined by

$$p(s) = \int_\varepsilon^s (v-\varepsilon)\eta(v)\,dv - \int_s^\epsilon (\epsilon-v)\eta(v)\,dv.$$

Then

$$p'(s) = (\epsilon-\varepsilon)\eta(s) > 0, s\in[\varepsilon,\epsilon]$$

This shows that $p(s)$ is an increasing function on $[\varepsilon,\epsilon]$ and

$$p(\varepsilon_1) = 0.$$

Now it is easy to see that

$$\left(\frac{\varepsilon_1-\varepsilon}{\epsilon-\varepsilon}\right)\int_0^1 |H(\eta,\varepsilon,\varepsilon_1;u)|\,du = \int_\varepsilon^{\varepsilon_1} (v-\varepsilon)\eta(v)\,dv$$

and

$$\left(\frac{\epsilon-\varepsilon_1}{\epsilon-\varepsilon}\right)\int_0^1 |H(\eta,\varepsilon_1,\epsilon;u)|\,du = \int_\varepsilon^{\varepsilon_1} (v-\varepsilon)\eta(v)\,dv.$$

Finally,

$$\frac{(\varepsilon_1-\varepsilon)^3}{\epsilon-\varepsilon}\int_0^1 u(1-u)|H(\eta,\varepsilon,\varepsilon_1;u)|\,du$$

$$= \frac{1}{\epsilon-\varepsilon}\int_\varepsilon^{\varepsilon_1} (v-\varepsilon)(\varepsilon_1-v)p(v)\,dv$$

$$= \frac{1}{6}\int_\varepsilon^{\varepsilon_1} (\varepsilon-s)^2(\varepsilon+2s-3\varepsilon_1)\eta(s)\,ds$$

$$= \frac{1}{3}\int_\varepsilon^{\frac{\varepsilon+\epsilon}{2}} (v-\varepsilon)^2\left(v-\frac{\varepsilon+3\epsilon}{4}\right)\eta(v)\,dv.$$

and

$$\frac{(\epsilon - \varepsilon_1)^3}{\epsilon - \varepsilon} \int_0^1 u(1-u)|H(\eta, \varepsilon_1, \epsilon; u)|\, du$$

$$= -\frac{1}{\epsilon - \varepsilon} \int_{\varepsilon_1}^{\epsilon} (\epsilon - v)(v - \varepsilon_1) p(v)\, dv$$

$$= -\frac{1}{6} \int_{\varepsilon_1}^{\epsilon} (\epsilon - s)^2 (\epsilon + 2s - 3\varepsilon_1)\eta(s)\, ds$$

$$= \frac{1}{3} \int_{\varepsilon}^{\frac{\varepsilon + \epsilon}{2}} (v - \varepsilon)^2 \left(v - \frac{\varepsilon + 3\epsilon}{4}\right)\eta(v)\, dv.$$

Hence the inequality (2.25) follows from the inequality.   □

3. Applications

Let $\eta : [\varepsilon, \epsilon] \to [0, \infty)$ be the probability density function of a continuous random variable X with $0 < \varepsilon < \epsilon$. The rth moment of X is defined as

$$E_r(X) = \int_{\varepsilon}^{\epsilon} v^r \eta(v) < \infty. \tag{3.1}$$

THEOREM 7. *For $r \geqslant 2$, the following inequality holds*

$$|E_r(X) - [E(X)]^r| \leqslant \frac{r\left[\chi_1(\varepsilon, \epsilon)|\varepsilon|^{r-1} + \chi_2(\varepsilon, \epsilon)|\epsilon|^{r-1}\right]}{\epsilon - \varepsilon} - r\chi_3(\varepsilon, \epsilon), \tag{3.2}$$

where

$$\chi_1(\varepsilon, E(X), \epsilon) = \frac{1}{2} \int_{\varepsilon}^{E(X)} (E(X) - v)(\epsilon - v)\eta(v)\, dv$$
$$- \frac{1}{2} \int_{E(X)}^{\epsilon} (E(X) - v)(\epsilon - v)\eta(v)\, dv,$$

$$\chi_2(\varepsilon, E(X), \epsilon) = \frac{1}{2} \int_{E(X)}^{\epsilon} (E(X) - v)(v - \varepsilon)\eta(v)\, dv$$
$$- \frac{1}{2} \int_{\varepsilon}^{E(X)} (E(X) - v)(v - \varepsilon)\eta(v)\, dv$$

and

$$\chi_3(\varepsilon, E(X), \epsilon) = \frac{1}{6} \int_{\varepsilon}^{E(X)} (v - E(X)) \left[2\left(v^2 + vE(X) + [E(X)]^2\right)\right.$$
$$\left. - 3(\varepsilon + \epsilon)(v - E(X)) + 6\varepsilon\,\epsilon\right]\eta(v)\, dv$$
$$- \frac{1}{6} \int_{E(X)}^{\epsilon} (v - E(X)) \left[2\left(v^2 + vE(X) + [E(X)]^2\right)\right.$$
$$\left. - 3(\varepsilon + \epsilon)(v - E(X)) + 6\varepsilon\,\epsilon\right]\eta(v)\, dv.$$

Proof. Let $h(v) = v^r$ on $[\varepsilon, \epsilon]$ for $r \geqslant 2$, we have $|h'(v)| = rv^{r-1}$ is convex. Therefore, from the inequality (2.3), we obtain the inequality (3.2). $\square$

COROLLARY 5. *If the assumptions of Theorem 7 and if* $\eta : [\varepsilon, \epsilon] \to [0, \infty)$ *is symmetric with respect to* $\frac{\varepsilon + \epsilon}{2}$, *then the following inequality holds*

$$\left| E_r(X) - \left(\frac{\varepsilon + \epsilon}{2} \right)^r \right| \leqslant \frac{r \left[\chi_1(\varepsilon, \epsilon) |\varepsilon|^{r-1} + \chi_2(\varepsilon, \epsilon) |\epsilon|^{r-1} \right]}{\epsilon - \varepsilon} - r\chi_3(\varepsilon, \epsilon), \qquad (3.3)$$

where

$$\chi_1(\varepsilon, \epsilon) = \frac{1}{2} \int_\varepsilon^{\frac{\varepsilon + \epsilon}{2}} \left(\frac{\varepsilon + \epsilon}{2} - v \right) (\epsilon - v) \eta(v) dv$$
$$- \frac{1}{2} \int_{\frac{\varepsilon + \epsilon}{2}}^\epsilon \left(\frac{\varepsilon + \epsilon}{2} - v \right) (\epsilon - v) \eta(v) dv,$$

$$\chi_2(\varepsilon, \epsilon) = \frac{1}{2} \int_{\frac{\varepsilon + \epsilon}{2}}^\epsilon \left(\frac{\varepsilon + \epsilon}{2} - v \right) (v - \varepsilon) \eta(v) dv$$
$$- \frac{1}{2} \int_\varepsilon^{\frac{\varepsilon + \epsilon}{2}} \left(\frac{\varepsilon + \epsilon}{2} - v \right) (v - \varepsilon) \eta(v) dv$$

and

$$\chi_3(\varepsilon, \epsilon) = \frac{1}{6} \int_\varepsilon^{\frac{\varepsilon + \epsilon}{2}} \left(v - \frac{\varepsilon + \epsilon}{2} \right) \left[2 \left(v^2 + v \left(\frac{\varepsilon + \epsilon}{2} \right) + \left(\frac{\varepsilon + \epsilon}{2} \right)^2 \right) \right.$$
$$\left. -3(\varepsilon + \epsilon) \left(v - \frac{\varepsilon + \epsilon}{2} \right) + 6\varepsilon \epsilon \right] \eta(v) dv$$
$$- \frac{1}{6} \int_{\frac{\varepsilon + \epsilon}{2}}^\epsilon \left(v - \frac{\varepsilon + \epsilon}{2} \right) \left[2 \left(v^2 + v \left(\frac{\varepsilon + \epsilon}{2} \right) + \left(\frac{\varepsilon + \epsilon}{2} \right)^2 \right) \right.$$
$$\left. -3(\varepsilon + \epsilon) \left(v - \frac{\varepsilon + \epsilon}{2} \right) + 6\varepsilon \epsilon \right] \eta(v) dv.$$

Proof. Since $\eta : [\varepsilon, \epsilon] \to [0, \infty)$ is symmetric with respect to $\frac{\varepsilon + \epsilon}{2}$, hence $E(X) = \frac{\varepsilon + \epsilon}{2}$. $\square$

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