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HARNACK INEQUALITIES FOR FUNCTIONAL SDES DRIVEN BY SUBORDINATE FRACTIONAL BROWNIAN MOTION

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Abstract. Being base on coupling by change of measure and an approximation technique, the Harnack inequalities for a class of stochastic functional differential equations driven by subordinate fractional Brownian motion with Hurst parameter $0 < H < 1/2$ are established. By using a transformation formulas for fractional Brownian motion, the Harnack inequalities for stochastic functional differential equations driven by subordinate fractional Brownian motion with Hurst parameter $1/2 < H < 1$ are established.

1. Introduction

The dimension-free Harnack inequality with powers introduced in [35] and the log-Harnack inequality introduced in [29] have attracted more and more attentions because of its extensive applications in stochastic analysis, such as strong Feller property and contractivity properties (see [27, 28, 36]); heat kernel estimates (see [17, 32, 38, 40]); transportation-cost inequalities and properties of invariant measures (see [4, 24, 39]). Up to now, the dimension-free Harnack inequality and log-Harnack inequality have been intensively investigated for various stochastic (partial) differential equations driven by several different kinds of noise. For example, Bao et al, [2] and [3] for functional SDEs and SPDEs driven by Brownian motion, respectively; Shao et al, [33] for SDEs driven by Brownian motion with non-Lipschitz coefficients and Huang and Zhang [18] for functional SDEs by Brownian motion with Dini drifts. In addition, Bass and Levin [5] for the pure jump Markov processes; Wang and Wang [37] for SDEs driven by Lévy noise; Zhang [46] for SDEs driven by α-stable processes and Wang and Zhang [41] for SDEs driven by cylindrical α-stable processes.

The theory of subordinate Brownian motion recently received increasing attentions since they may describe some mathematical models in finance. There also exists several results on the Harnack inequality for subordinate Brownian and the time changed Brownian motion. For example, Rao et al, [26] and Mimica and Kim [23] studied the Harnack inequality for subordinate Brownian motion; Deng [14] established the Harnack inequalities for the inhomogeneous semigroup associated with a class of SDEs

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with Lévy noise containing a subordinate Brownian motion. Very recently, by using the coupling argument, the Girsanov transformations and the an approximation argument, Deng and Huang [12] established Harnack inequalities for the following stochastic differential equation driven by subordinate Brownian motion

$$X(t) = \xi + \int_0^t b(X(s))ds + \int_0^t B(X_s)ds + W_{S(t)}, \quad t \geqslant 0,$$

where $W = \{W_t\}_{t\geqslant 0}$ is a d-dimensional Brownian motion, $S = \{S(t)\}_{t\geqslant 0}$ is a subordinator and independent of W.

It is a natural question whether one can still establish the Harnack inequality when the driving noise is a more general, maybe non-Markovian process. As far as I know that the fractional Brownian motion (in short fBm) becomes the standard Brownian motion when $H = 1/2$, and the fBm B^H neither is a semimartingale nor a Markov process if $H \neq 1/2$. However, the fBm B^H, $H > 1/2$ is a long-memory process and presents an aggregation behavior. The long-memory property make fBm as a potential candidate to model noise in mathematical finance (see [8]); in biology (see [7, 10]); in communication networks (see, for instance [42]); the analysis of global temperature anomaly [30] electricity markets [34] etc. There are several frontier works on the Harnack inequalities for stochastic (partial) differential equations driven by fractional Brownian motion, see [15, 16, 20, 21, 22, 43, 44, 45].

However, there is only a few result on the stochastic differential equations driven by subordinat fBm and we can only find that Deng and Schilling [13] established Harnack inequalities stochastic differential equations driven by subordinate fBm with $H \in (0,1/2)$. The main aim of this work is to establish Harnack inequalities for functional SDEs driven by subordinate fBm with Hurst parameter $H \in (0,1/2) \cup (1/2,1)$. It turns out that our results cover the corresponding ones in the case without delay derived by [13] when $H \in (0,1/2)$ and extend the corresponding ones in the case subordinate Brownian motion derived by [12] to subordinate fBm with Hurst parameter $H \in (0,1/2) \cup (1/2,1)$. To this end, we consider the following stochastic differential equation on $\mathbb{R}^d$

$$X(t) = \xi + \int_0^t b(X(s))ds + \int_0^t B(X_s)ds + W^H_{S(t)}, \quad t \geqslant 0, \tag{1.1}$$

where $b : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d$, $B : [0,\infty) \times \mathscr{L} \to \mathbb{R}^d$ are some appropriate functions; W^H is a d-dimensional fBm with Hurst parameter $H \in (0,1/2) \cup (1/2,1)$, S is a subordinator and independent of W^H. We firstly establish the Harnack inequality for (1.1) with $H \in (0,1/2)$ by using the coupling argument, the Girsanov transformations of fBm with Hurst parameter $H \in (0,1/2)$ and the an approximation argument.

On the other hand, in virtue of irregularity of the operator K_H^{-1}, it is very difficult to obtain the Harnack inequality by using directly Girsanov transformations when the Hurst parameter $1/2 < H < 1$. To overcome this difficulty, motivating mainly by [19], we will transform the fractional Brownian motion with Hurst parameter $1/2 < H < 1$ into some integral with respect to the fractional Brownian motion with Hurst parameter $0 < H < 1/2$ by using of the transformation formula for fractional Brownian motion.

The rest of this paper is organized as follows. In Section 2, we introduce some necessary notations and preliminaries. In Section 3, we devote ourselves to establish the Harnack inequalities for SDEs driven subordinat fBm with $H \in (0,1/2)$. In Section 4, we consider the Harnack inequalities for SDEs driven subordinate fBm with $H \in (1/2,1)$.

2. Preliminaries

In this section, we recall briefly some basic facts on fractional Brownian motion (fBm) which will be used later on. Let $W^H = \{W_t^H, t \geqslant 0\}$ on $\mathbb{R}^d$ be a fractional Brownian motion with Hurst parameter $H \in (0,1)$ defined on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$, i.e., W^H is a centered Gauss process with covariance function

$$R_H(t,s) = \mathbb{E}(W_t^{H,i} W_s^{H,j}) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})\delta_{ij}, \quad t,s \geqslant 0, \ 1 \leqslant i,j \leqslant d,$$

where δ_{ij} denotes Kronecker's delta. In particular, if $H = \frac{1}{2}$, W^H is a Brownian motion. It is well known that if $H \neq \frac{1}{2}$, W^H does not have independent increments and has α-order Hölder continuous path for all $\alpha \in (0,H)$. For more details of fBm and proofs we refer the readers, for instance, to [6, 11].

On the other hand, from [11], we know the covariance kernel $R_H(t,s)$ can be written as

$$R_H(t,s) = \int_0^{t\wedge s} K_H(t,r) K_H(s,r) dr,$$

where K_H is a square integrable kernel given by

$$K_H(t,s) = \Gamma\Big(H + \frac{1}{2}\Big)^{-1} (t-s)^{H-\frac{1}{2}} F\Big(H - \frac{1}{2}, \frac{1}{2} - H, H + \frac{1}{2}, 1 - \frac{t}{s}\Big),$$

in which $F(\cdot,\cdot,\cdot,\cdot)$ is the Gauss hypergeometric function and $\Gamma(\cdot)$ denotes the Euler Gamma function. Moreover, according to [1], fractional Brownian motion has the following integral representation with respect to the usual d-dimensional standard Brownian motion $W = (W_t)_{t\geqslant 0}$

$$W_t^H = \int_0^t K_H(t,s) dW_s.$$

By [11], the operator $K_H : L^2([0,T];\mathbb{R}^d) \to I_{0+}^{H+\frac{1}{2}}(L^2[0,T];\mathbb{R}^d)$ associated with the square integrable kernel $K_H(\cdot,\cdot)$ is defined as follows

$$(K_H f)(t) := \int_0^t K_H(t,s) f(s) ds, \ f \in L^2([0,T];\mathbb{R}^d),$$

where $I_{0+}^{H+\frac{1}{2}}$ is the $H + \frac{1}{2}$-order left fractional Riemann-Liouville operator on $[0,T]$, one can see [31]. It is an isomorphism and for each $f \in L^2([0,T];\mathbb{R}^d)$,

$$(K_H f)(s) = I_{0+}^{2H} s^{\frac{1}{2}-H} I_{0+}^{\frac{1}{2}-H} s^{H-\frac{1}{2}} f, \ H \leqslant \frac{1}{2},$$

$$(K_H f)(s) = I^1_{0+} s^{H-\frac{1}{2}} I^{H-\frac{1}{2}}_{0+} s^{\frac{1}{2}-H} f, \ H \geqslant \frac{1}{2}.$$

As a consequence, for every $h \in I^{H+\frac{1}{2}}_{0+}(L^2[0,T];\mathbb{R}^d)$, the inverse operator K_H^{-1} is of the following form

$$(K_H^{-1} h)(s) = s^{H-\frac{1}{2}} D^{H-\frac{1}{2}}_{0+} s^{\frac{1}{2}-H} h', \ H > \frac{1}{2},$$

$$(K_H^{-1} h)(s) = s^{\frac{1}{2}-H} D^{\frac{1}{2}-H}_{0+} s^{H-\frac{1}{2}} D^{2H}_{0+} h, \ H < \frac{1}{2},$$

where $D^{H-\frac{1}{2}}_{0+}(D^{\frac{1}{2}-H}_{0+})$ is $H-\frac{1}{2}(\frac{1}{2}-H)$-order left-sided Riemann-Liouville derivative, one can see [31]. In particular, if h is absolutely continuous, we have

$$(K_H^{-1} h)(s) = s^{H-\frac{1}{2}} I^{\frac{1}{2}-H}_{0+} s^{\frac{1}{2}-H} h', \ H < \frac{1}{2}. \tag{2.1}$$

Fix a constant $r > 0$. Denote by $\mathscr{L}$ the family of all right continuous functions $f : [-r,0] \to \mathbb{R}^d$ with left limits equipped with the norm $\|\cdot\|_2$

$$\|f\|_2^2 := \int_{-r}^0 |f(s)|^2 ds + |f(0)|^2.$$

For $f : [-r,\infty) \to \mathbb{R}^d$, we will denote $f_t \in \mathscr{L}$, $t \geqslant 0$, the corresponding segment process by

$$f_t(s) := f(t+s), \quad s \in [-r,0].$$

Let $S = \{S(t)\}_{t\geqslant 0}$ be a subordinator (without killing), i.e. a nondecreasing Lévy process in $[0,\infty)$ starting at $S(0) = 0$. Due to the independent and stationary increments property, it is uniquely determined by the Laplace transform

$$\mathbb{E} e^{-uS(t)} = e^{-t\phi(u)}, \quad u > 0, \ t \geqslant 0,$$

where the characteristic (Laplace) exponent $\phi : (0,\infty) \to (0,\infty)$ is a Bernstein function with $\phi(0+) := \lim_{\delta\to 0} = 0$, i.e. a C^∞-function such that $(-1)^{n-1}\phi^{(n)} \geqslant 0$ for all $n \in \mathbb{N}$. Every such ϕ has a unique Lévy-Khintchine representation

$$\phi(u) = \kappa u + \int_{(0,\infty)} (1-e^{-ux})\nu(dx), \quad u > 0, \tag{2.2}$$

where $\kappa \geqslant 0$ is the drift parameter and ν is a Lévy measure on $(0,\infty)$ satisfying

$$\int_{(0,\infty)} (1\wedge x)\nu(dx) < \infty.$$

It is clear that $\tilde{\phi} := \phi(u) - \kappa u$ is the Bernstein function of the subordinator $\tilde{S}(t) = S(t) - \kappa t$ having zero drift and Lévy measure ν.

Throughout this paper we assume that the coefficients b and B satisfy the following Hypothesis:

(H) There exist constants $K \in \mathbb{R}$ and $K_1 \geqslant 0$ such that

$$\langle x-y, b(x)-b(y)\rangle \leqslant K|x-y|^2, \quad x,y \in \mathbb{R}^d,$$

and

$$|B(\xi)-B(\eta)| \leqslant K_1\|\xi-\eta\|_2, \quad \xi,\eta \in \mathscr{L}.$$

REMARK 2.1. The Hypothesis (H) ensures the existence, uniqueness and non-explosion of the solution to (1.1). Indeed, letting $L(t) = W^H_{S(t)}$, $\hat{b}(t,x) = b(x+L(t))$ and $\hat{B}(t,\xi) = B(\xi+L_t)$, one has

$$\langle x-y, \hat{b}(t,x)-\hat{b}(t,y)\rangle \leqslant K|x-y|^2, \quad x,y \in \mathbb{R}^d,\ t \geqslant 0,$$

and

$$|\hat{B}(t,\xi)-\hat{B}(t,\eta)| \leqslant K_1\|\xi-\eta\|_2, \quad \xi,\eta \in \mathscr{L},\ t \geqslant 0.$$

Then the following ordinary functional differential equation

$$d\hat{X}(t) = \hat{b}(t,\hat{X}(t))dt + \hat{B}(t,\hat{X}_t)dt$$

has a unique solution which does not explode in finite time; setting $X(t) := \hat{X}(t)+L(t)$, we know that (1.1) has a unique non-explosive solution.

3. Harnack inequality for (1.1) with $H \in (0,1/2)$

For $\xi \in \mathscr{L}$, let X_t^ξ be the solution to (1.1) with $X_0 = \xi$. Let P_t be the semigroup associated to X_t^ξ, i.e.

$$P_t f(\xi) = \mathbb{E} f(X_t^\xi),\ t \geqslant 0,\ f \in \mathscr{B}_b(\mathscr{L}), \tag{3.1}$$

where $\mathscr{B}_b(\mathscr{L})$ denotes the set of all bounded measurable functions on $\mathscr{L}$.

THEOREM 3.1. *Let $H \in (0,1/2)$, $T > r$, the Hypothesis (H) holds and S be a subordinator with Bernstein function of ϕ of the form (2.2). Then,*

(i) for any $\xi,\eta \in \mathscr{L}$ and $f \in \mathscr{B}_b(\mathscr{L})$ with $f \geqslant 1$,

$$\begin{aligned} &P_T \log f(\eta) \\ &\leqslant \log P_T f(\xi) + \Big(C_1(K_1,H,\kappa)\|\xi-\eta\|_2^2 + C_2(K,K_1,H,T,r,\kappa)|\xi(0)-\eta(0)|^2\Big) \\ &\quad \frac{B^2(\frac{3}{2}-H,\frac{1}{2}-H)}{4-4H} T^{2-2H}; \end{aligned}$$

(ii) for any $p>1$, $\xi,\eta \in \mathscr{L}$ and non-negative $f \in \mathscr{B}_b(\mathscr{L})$,

$$\begin{aligned} &\Big(P_T f(\eta)\Big)^p \\ &\leqslant P_T f^p(\xi) \exp\left[\frac{p}{2(p-1)^2}\Big(C_1(K_1,H,\kappa)\|\xi-\eta\|_2^2 + C_2(K,K_1,H,T,r,\kappa)|\xi(0)-\eta(0)|^2\Big)\right. \\ &\quad \left.\frac{B^2(\frac{3}{2}-H,\frac{1}{2}-H)}{4-4H} T^{2-2H}\right], \end{aligned}$$

where

$$C_1(K_1,H,\kappa)=\Big(\frac{K_1}{\Gamma(\frac{1}{2}-H)}\Big)^2\frac{2}{\kappa^2},$$

$$C_2(K,K_1,H,T,r,\kappa)=\Big(\frac{K_1}{\Gamma(\frac{1}{2}-H)}\Big)^2\Big(\frac{e^{2K(T-r)}-1}{K\kappa^2}+2\mathbb{E}\Big(\int_0^{T-r}e^{-2Kt}dS(t)\Big)^{-2}\Big),$$

and $B(\cdot,\cdot)$ *and* $\Gamma(\cdot)$ *are the standard Beta and Gamma functions.*

REMARK 3.1. If $B=0$, then we can choose $r=0$ and $K_1=0$, and thus the assertions in the Theorem 2.1 reduce to the ones derived in [13] for the case without delay.

For a measurable space $(E,\mathscr{F})$, let $\mathscr{P}(E)$ denote the family of all probability measures on $(E,\mathscr{F})$. For $\mu,\nu\in\mathscr{P}(E)$, the entropy $\mathrm{Ent}(\nu|\mu)$ is defined by

$$\mathrm{Ent}(\nu|\mu):=\begin{cases}\int\ln\frac{d\nu}{d\mu}d\nu, & \nu\ll\mu,\\ +\infty, & \text{otherwise.}\end{cases}$$

The total variation distance $\|\mu-\nu\|_{\mathrm{var}}$ is defined by

$$\|\mu-\nu\|_{\mathrm{var}}:=\sup_{A\in\mathscr{F}}|\mu(A)-\nu(A)|.$$

By Pinsker's inequality (see [9]),

$$\|\mu-\nu\|_{\mathrm{var}}^2\leqslant\frac{1}{2}\mathrm{Ent}(\nu|\mu),\quad \mu,\nu\in\mathscr{P}(E).$$

For $\xi\in\mathscr{L}$, let $P_T(\xi,\cdot)$ be the distribution of X_T^ξ. The following corollary is a direct consequence of Theorem 3.1, see [37] for the proof.

COROLLARY 3.1. *Let the assumptions in Theorem 3.1 hold. Then the following assertions hold.*

(i) For any $\xi,\eta\in\mathscr{L}$ *and* $P_T(\xi,\cdot)$ *is equivalent to* $P_T(\eta,\cdot)$ *and*

$$\begin{aligned}&Ent(P_T(\xi,\cdot)|P_T(\eta,\cdot))\\ \leqslant&\Big(C_1(K_1,H,\kappa)\|\xi-\eta\|_2^2+C_2(K,K_1,H,T,r,\kappa)|\xi(0)-\eta(0)|^2\Big)\\ &\frac{B^2(\frac{3}{2}-H,\frac{1}{2}-H)}{4-4H}T^{2-2H},\end{aligned}$$

which together with Pinsker's inequality implies that

$$\begin{aligned}&2\|P_T(\xi,\cdot)-P_T(\eta,\cdot)\|_{var}^2\\ \leqslant&\Big(C_1(K_1,H,\kappa)\|\xi-\eta\|_2^2+C_2(K,K_1,H,T,r,\kappa)|\xi(0)-\eta(0)|^2\Big)\\ &\frac{B^2(\frac{3}{2}-H,\frac{1}{2}-H)}{4-4H}T^{2-2H}.\end{aligned}$$

(ii) For any $p>1$, $\xi,\eta\in\mathscr{L}$

$$\begin{aligned}&P_T\Big\{\Big(\frac{dP_T(\xi,\cdot)}{dP_T(\eta,\cdot)}\Big)^{1/(p-1)}\Big\}(\xi)\\ \leqslant&\mathbb{E}\Big\{\exp\Big[\frac{p}{2(p-1)^2}\Big(C_1(K_1,H,\kappa)\|\xi-\eta\|_2^2+C_2(K,K_1,H,T,r,\kappa)|\xi(0)-\eta(0)|^2\Big)\\ &\frac{B^2(\frac{3}{2}-H,\frac{1}{2}-H)}{4-4H}T^{2-2H}\Big]\Big\}.\end{aligned}$$

Let $\ell:[0,\infty)\to[0,\infty)$ be a sample path of S, which is a non-decreasing and càdlàg function with $\ell(0)=0$. By (H) and the same explanation as in Remark 2.1, for any $\xi\in\mathscr{L}$, the following functional SDE has a unique non-explosive solution with $X_0^\ell=\xi$:

$$dX^\ell(t)=b(X^\ell(t))dt+B(X_t^\ell)dt+dW^H_{\ell(t)}. \tag{3.2}$$

We denote the solution by $X_t^{\ell,\xi}$. Let

$$P_t^\ell f(\xi)=\mathbb{E}f(X_t^{\ell,\xi}),\ \ t\geqslant 0,\ \ f\in\mathscr{B}_b(\mathscr{L}). \tag{3.3}$$

PROPOSITION 3.1. *Let* $H\in(0,1/2)$, $T>r$ *and the Hypothesis (H) hold. Then,*

(i) for any $\xi,\eta\in\mathscr{L}$ *and* $f\in\mathscr{B}_b(\mathscr{L})$ *with* $f\geqslant 1$,

$$\begin{aligned}&P_T^\ell\log f(\eta)\\ \leqslant&\log P_T^\ell f(\xi)+\Big(C_1(K_1,H,\kappa)\|\xi-\eta\|_2^2+C_3(K,K_1,H,T,r,\kappa)|\xi(0)-\eta(0)|^2\Big)\\ &\frac{B^2(\frac{3}{2}-H,\frac{1}{2}-H)}{4-4H}T^{2-2H};\end{aligned}$$

(ii) for any $p>1$, $\xi,\eta\in\mathscr{L}$ *and non-negative* $f\in\mathscr{B}_b(\mathscr{L})$,

$$\begin{aligned}&\Big(P_T^\ell f(\eta)\Big)^p\\ \leqslant&P_T^\ell f^p(\xi)\exp\Big[\frac{p}{2(p-1)^2}\Big(C_1(K_1,H,\kappa)\|\xi-\eta\|_2^2+C_3(K,K_1,H,T,r,\kappa)|\xi(0)-\eta(0)|^2\Big)\\ &\frac{B^2(\frac{3}{2}-H,\frac{1}{2}-H)}{4-4H}T^{2-2H}\Big],\end{aligned}$$

where

$$C_3(K,K_1,H,T,r,\kappa)=\Big(\frac{K_1}{\Gamma(\frac{1}{2}-H)}\Big)^2\Big(\frac{e^{2K(T-r)}-1}{K\kappa^2}+2\Big(\int_0^{T-r}e^{-2Kt}d\ell(t)\Big)^{-2}\Big),$$

and $B(\cdot,\cdot)$ *and* $\Gamma(\cdot)$ *are the standard Beta and Gamma functions.*

For $\varepsilon \in (0,1)$, consider the following regularization of ℓ:

$$\ell^\varepsilon := \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \ell(s)ds + \varepsilon t = \int_0^t \ell(\varepsilon s + t)ds + \varepsilon t, \ t \geqslant 0.$$

It is clear that for each $\varepsilon \in (0,1)$, the function ℓ^ε is a absolutely continuous, strictly increasing and satisfies for any $t \geqslant 0$

$$\ell^\varepsilon(t) \downarrow \ell \quad \text{as } \varepsilon \downarrow 0. \tag{3.4}$$

For $\xi \in \mathscr{L}$, let $X_t^{\ell^\varepsilon,\xi}$ be the solution to the following functional SDE with initial value ξ:

$$dX^{\ell^\varepsilon,\xi}(t) = b(X^{\ell^\varepsilon,\xi}(t))dt + B(X^{\ell^\varepsilon,\xi})dt + dW^H_{\ell^\varepsilon(t)-\ell^\varepsilon(0)}. \tag{3.5}$$

The associated semigroup is denoted by $P_t^{\ell^\varepsilon}$. Note that this SDE is indeed driven by fBm and thus the method of coupling and Girsanov's transformation can be used to establish the dimension-free Harnack inequalities for $P_t^{\ell^\varepsilon}$.

PROPOSITION 3.2. *Let $H \in (0,1/2)$, $T > r$ and the Hypothesis (H) hold. Then*

(i) for any $\xi,\eta \in \mathscr{L}$ and $f \in \mathscr{B}_b(\mathscr{L})$ with $f \geqslant 1$,

$$\begin{aligned} &P_T^{\ell^\varepsilon} \log f(\eta) \\ &\leqslant \log P_T^{\ell^\varepsilon} f(\xi) + \Big(C_1(K_1,H,\kappa)\|\xi-\eta\|_2^2 + C_4(K,K_1,H,T,r,\kappa,\varepsilon)|\xi(0)-\eta(0)|^2\Big) \\ &\quad \frac{B^2(\frac{3}{2}-H,\frac{1}{2}-H)}{4-4H}T^{2-2H}; \end{aligned}$$

(ii) for any $p > 1$, $\xi,\eta \in \mathscr{L}$ and non-negative $f \in \mathscr{B}_b(\mathscr{L})$,

$$\begin{aligned} &\Big(P_T^{\ell^\varepsilon} f(\eta)\Big)^p \\ &\leqslant P_T^{\ell^\varepsilon} f^p(\xi)\exp\Big[\frac{p}{2(p-1)^2}\Big(C_1(K_1,H,\kappa)\|\xi-\eta\|_2^2 + C_4(K,K_1,H,T,r,\kappa,\varepsilon)|\xi(0)-\eta(0)|^2\Big) \\ &\quad \frac{B^2(\frac{3}{2}-H,\frac{1}{2}-H)}{4-4H}T^{2-2H}\Big], \end{aligned}$$

where

$$C_4(K,K_1,H,T,r,\kappa,\varepsilon) = \Big(\frac{K_1}{\Gamma(\frac{1}{2}-H)}\Big)^2\Big(\frac{e^{2K(T-r)}-1}{K\kappa^2} + 2\Big(\int_0^{T-r} e^{-2Kt}d\ell^\varepsilon(t)\Big)^{-2}\Big),$$

and $B(\cdot,\cdot)$ and $\Gamma(\cdot)$ are the standard Beta and Gamma functions.

Proof. First of all, we will construct coupling as follows. Let Y_t solve the equation

$$\begin{aligned} dY(t) &= b(Y(t))dt + B(X_t^{\ell^\varepsilon,\xi})dt + \lambda(t)\mathscr{I}_{[0,\tau)}(t)\frac{X^{\ell^\varepsilon,\xi}(t)-Y(t)}{|X^{\ell^\varepsilon,\xi}(t)-Y(t)|}|\xi(0)-\eta(0)|d\ell^\varepsilon(t) \\ &\quad + dW^H_{\ell^\varepsilon(t)-\ell^\varepsilon(0)} \end{aligned} \tag{3.6}$$

with $Y_0=\eta$, where

$$\lambda(t):=\frac{e^{-Kt}}{\int_0^{T-r}e^{-2Ks}d\ell^\varepsilon(s)},\ \ t\geqslant 0,$$

and

$$\tau:=T\wedge\inf\{t\geqslant 0;X^{\ell^\varepsilon,\xi}(t)=Y(t)\}$$

is the coupling time. It is clear that $(X^{\ell^\varepsilon,\xi}(t),Y(t))$ is well defined for $t<\tau$. By (H), we have

$$d|X^{\ell^\varepsilon,\xi}(t)-Y(t)|\leqslant K|X^{\ell^\varepsilon,\xi}(t)-Y(t)|dt-\lambda(t)|\xi(0)-\eta(0)|d\ell^\varepsilon(t),\ \ t\in[0,\tau).$$

Thus, for $t\in[0,\tau)$,

$$\begin{aligned}|X^{\ell^\varepsilon,\xi}(t)-Y(t)|&\leqslant e^{Kt}|\xi(0)-\eta(0)|\Big(1-\int_0^t e^{-Ks}\lambda(s)d\ell^\varepsilon(s)\Big)\\&\leqslant\frac{e^{Kt}\int_t^{T-r}e^{-2Ks}d\ell^\varepsilon(s)}{\int_0^{T-r}e^{-2Ks}d\ell^\varepsilon(s)}|\xi(0)-\eta(0)|\\&=:\gamma(t)|\xi(0)-\eta(0)|.\end{aligned}\tag{3.7}$$

If $\tau(\omega)>T-r$ for some $\omega\in\Omega$, we can take $t=T-r$ in the above inequality to get

$$0<|X^{\ell^\varepsilon,\xi}(t)(\omega)-Y(t)(\omega)|\leqslant 0,$$

which is absurd. Therefore, $\tau\leqslant T-r$. Letting $Y(t)=X^{\ell^\varepsilon,\xi}(t)$ for $t\in[\tau,T]$, $Y(t)$ solves (3.6) for $t\in[\tau,T]$. In particular, $X_T^{\ell^\varepsilon,\xi}=Y_T$. Moreover, by (3.7) and $\tau\leqslant T-r$, we have

$$|X^{\ell^\varepsilon,\xi}(t)-Y(t)|^2\leqslant|\xi(0)-\eta(0)|^2\gamma(t)^2\mathscr{I}_{[0,T-r]}(t),\ \ t\in[0,T].\tag{3.8}$$

Denote by $\zeta^\varepsilon:[\ell^\varepsilon(0),\infty)\to[0,\infty)$ the inverse function of ℓ^ε. Then $\ell^\varepsilon(\zeta^\varepsilon(t))=t$ for $t\geqslant\ell^\varepsilon(0)$, $\zeta^\varepsilon(\ell^\varepsilon(t))=t$ for $t\geqslant 0$, and $t\to\zeta^\varepsilon(t)$ is absolutely continuous and strictly increasing. Let

$$\Psi(u):=\Phi\circ\zeta^\varepsilon(u+\ell^\varepsilon(0)),$$

where

$$\Phi(u):=[B(X_u^{\ell^\varepsilon,\xi})-B(Y_u)]\frac{1}{(\ell^\varepsilon)'(u)}+\lambda(u)\mathscr{I}_{[0,\tau)}(u)\frac{X^{\ell^\varepsilon,\xi}(u)-Y(u)}{|X^{\ell^\varepsilon,\xi}(u)-Y(u)|}|\xi(0)-\eta(0)|.$$

A simple calculation shows $\int_0^{\ell^\varepsilon(T)-\ell^\varepsilon(0)}|\Psi(u)|^2du<\infty$. Then, by using $H\in(0,1/2)$ we have $\int_0^\cdot\Phi(u)du\in I_{0+}^{H+1/2}(L^2([0,\ell^\varepsilon(T)-\ell^\varepsilon(0)];\mathbb{R}^d))$. Therefore, the following stochastic integral defines a martingale

$$M_t:=-\int_0^t\langle\Upsilon(s),dW_s\rangle,\ \ \ t\geqslant 0,$$

where $\Upsilon(s) := K_H^{-1}\Big(\int_0^{\cdot}\Psi(r)dr\Big)(s)$, $s \geqslant 0$ and $W = \{W_t\}_{t\geqslant 0}$ is a d-dimensional standard Brownian motion. Because of (2.1), we have

$$K_H^{-1}\Big(\int_0^{\cdot}\Psi(r)dr\Big)(s) = s^{H-\frac{1}{2}}I_{0+}^{\frac{1}{2}-H}s^{\frac{1}{2}-H}\Psi(s),$$

and this yields for any $s \in [0, \ell^{\varepsilon}(T)-\ell^{\varepsilon}(0)]$,

$$|\Upsilon(s)| = \Big|\frac{1}{\Gamma(\frac{1}{2}-H)}s^{H-\frac{1}{2}}\int_0^s u^{\frac{1}{2}-H}(s-u)^{-H-\frac{1}{2}}\Phi(u)du\Big|.$$

By the definition of Φ and the Hypothesis (H) we have

$$|\Upsilon(s)| \leqslant \frac{K_1}{\Gamma(\frac{1}{2}-H)}s^{H-\frac{1}{2}}\Big|\int_0^s u^{\frac{1}{2}-H}(s-u)^{-H-\frac{1}{2}}\Big(\frac{\|X_u^{\ell^{\varepsilon},\xi}-Y_u\|_2}{(\ell^{\varepsilon})'(u)}+\lambda(u)|\xi(0)-\eta(0)|\Big)du\Big|.$$

Recalling that ℓ is an sample path of the subordinator S with drift parameter $\kappa \geqslant 0$, one have

$$(\ell^{\varepsilon})'(t) = \frac{\ell(t+\varepsilon)-\ell(t)}{\varepsilon}+\varepsilon > \kappa,$$

and therefore

$$|\Upsilon(s)| \leqslant \frac{K_1}{\Gamma(\frac{1}{2}-H)}s^{H-\frac{1}{2}}\Big|\int_0^s u^{\frac{1}{2}-H}(s-u)^{-H-\frac{1}{2}}\Big(\frac{1}{\kappa}\|X_u^{\ell^{\varepsilon},\xi}-Y_u\|_2+\lambda(u)|\xi(0)-\eta(0)|\Big)du\Big|.$$

On the other hand, by view of the definition of $\|\cdot\|_2$ we have for all $t \geqslant 0$

$$\begin{aligned}
\|X_t^{\ell^{\varepsilon},\xi}-Y_t\|_2^2 &= \int_{-r}^0 |X^{\ell^{\varepsilon},\xi}(t+s)-Y(t+s)|^2ds+|\xi(0)-\eta(0)|^2\\
&= \int_{t-r}^t |X^{\ell^{\varepsilon},\xi}(s)-Y(s)|^2ds+|\xi(0)-\eta(0)|^2\\
&\leqslant \int_{-r}^0 |\xi(s)-\eta(s)|^2ds+\int_0^t |X^{\ell^{\varepsilon},\xi}(s)-Y(s)|^2ds+|\xi(0)-\eta(0)|^2\\
&= \|\xi-\eta\|_2^2+\int_0^t |X^{\ell^{\varepsilon},\xi}(s)-Y(s)|^2ds.
\end{aligned}$$

Then, by (3.8) we have for all $t \geqslant 0$

$$\begin{aligned}
\|X_t^{\ell^{\varepsilon},\xi}-Y_t\|_2^2 &\leqslant \|\xi-\eta\|_2^2+|\xi(0)-\eta(0)|^2\int_0^{T-r}\gamma(s)^2ds\\
&\leqslant \|\xi-\eta\|_2^2+\frac{e^{2K(T-r)}-1}{2K}|\xi(0)-\eta(0)|^2,
\end{aligned} \tag{3.9}$$

where in the last inequality we have used $\gamma(s) \leqslant e^{Ks}$ for $s \in [0, T-r]$. By the definition of $\lambda(t)$, it is easy to see that for all $t \geqslant 0$

$$|\lambda(t)| \leqslant \Big(\int_0^{T-r} e^{-2Kt}d\ell^{\varepsilon}(t)\Big)^{-1}. \tag{3.10}$$

Thus, by (3.9) and (3.10) the compensator of the martingale M_t satisfies for all $t \geqslant 0$,

$$\begin{aligned}\langle M\rangle_t &= \int_0^t |\Upsilon(s)|^2 ds \leqslant \Big(C_1(K_1,H,\kappa)\|\xi-\eta\|_2^2 + C_4(K,K_1,H,T,r,\kappa,\varepsilon)|\xi(0)-\eta(0)|^2\Big)\\ &\quad \cdot \int_0^t s^{2H-1}\Big|\int_0^s u^{\frac{1}{2}-H}(s-u)^{-H-\frac{1}{2}}du\Big|^2 ds\\ &= \Big(C_1(K_1,H,\kappa)\|\xi-\eta\|_2^2 + C_4(K,K_1,H,T,r,\kappa,\varepsilon)|\xi(0)-\eta(0)|^2\Big)\\ &\quad \cdot B^2\Big(\frac{3}{2}-H,\frac{1}{2}-H\Big)\int_0^t s^{1-2H}ds\\ &\leqslant \Big(C_1(K_1,H,\kappa)\|\xi-\eta\|_2^2 + C_4(K,K_1,H,T,r,\kappa,\varepsilon)|\xi(0)-\eta(0)|^2\Big)\\ &\quad \cdot \frac{B^2(\frac{3}{2}-H,\frac{1}{2}-H)}{2-2H}T^{2-2H}.\end{aligned} \tag{3.11}$$

Let

$$R := \exp\Big[M(\ell^\varepsilon(T)) - \frac{1}{2}\langle M\rangle_{\ell^\varepsilon(T)-\ell^\varepsilon(0)}\Big].$$

By Novikov's criterion, we have $\mathbb{E}R = 1$. According to Girsanov's theorem, $\widetilde{W}_t := \int_0^t \Psi(u)du + W_t$ is a d-dimensional Brownian motion and $\widetilde{W}_t^H := \int_0^t \Psi(u)du + W_t^H$ is a d-dimensional fractional Brownian with $H \in (0,1/2)$ under the new probability measure $R\mathbb{P}$. Rewrite (3.6) as

$$dY(t) = b(Y(t))dt + B(Y_t)dt + d\widetilde{W}^H_{\ell^\varepsilon(t)-\ell^\varepsilon(0)}.$$

Thus, the distribution of $\{Y_t\}_{0\leqslant t\leqslant T}$ under $R\mathbb{P}$ coincides with that of $\{X_t^{\ell^\varepsilon,\eta}\}$ under $\mathbb{P}$; in particular, it holds that for any $f \in \mathscr{B}_b(\mathscr{L})$,

$$\mathbb{E}f(X_T^{\ell^\varepsilon,\eta}) = \mathbb{E}_{R\mathbb{P}}f(Y_T) = \mathbb{E}[Rf(Y_T)] = \mathbb{E}[Rf(X_t^{\ell^\varepsilon,\xi})]. \tag{3.12}$$

By (3.12) and the Young inequality, and the observation that

$$\begin{aligned}\log R &= -\int_0^{\ell^\varepsilon(T)-\ell^\varepsilon(0)} \langle \Upsilon(s), dW_s\rangle - \frac{1}{2}\int_0^{\ell^\varepsilon(T)-\ell^\varepsilon(0)} |\Upsilon(s)|^2 ds\\ &= -\int_0^{\ell^\varepsilon(T)-\ell^\varepsilon(0)} \langle \Upsilon(s), d\widetilde{W}_s\rangle + \frac{1}{2}\langle M\rangle_{\ell^\varepsilon(T)-\ell^\varepsilon(0)},\end{aligned}$$

we get that, for any $f \in \mathscr{B}_b(\mathscr{L})$ with $f \geqslant 1$,

$$\begin{aligned}P_T^{\ell^\varepsilon}\log f(\eta) &= \mathbb{E}\log f(X_T^{\ell^\varepsilon,\eta})\\ &= \mathbb{E}[R\log f(X_T^{\ell^\varepsilon,\xi})]\\ &\leqslant \log \mathbb{E}f(X_T^{\ell^\varepsilon,\xi}) + \mathbb{E}[R\log R]\\ &= \log P_T^{\ell^\varepsilon}f(\xi) + \mathbb{E}_{R\mathbb{P}}\log R\\ &= \log P_T^{\ell^\varepsilon}f(\xi) + \frac{1}{2}\mathbb{E}_{R\mathbb{P}}\langle M\rangle_{\ell^\varepsilon(T)-\ell^\varepsilon(0)}.\end{aligned}$$

Combining this with (3.11), we obtain the desired log-Harnack inequality.

Next, we prove the second assertion of the proposition. For any non-negative $f \in \mathscr{B}_b(\mathscr{L})$ we can obtain from (3.12) and the Hölder's inequality

$$\begin{aligned}(P_T^{\ell^\varepsilon} f)^p(\eta) &= (\mathbb{E} f(X_T^{\ell^\varepsilon,\eta}))^p \\ &= (\mathbb{E}[Rf(X_T^{\ell^\varepsilon,\xi})])^p \\ &\leqslant P_T^{\ell^\varepsilon} f^p(\xi)\cdot(\mathbb{E}[R^{p/(p-1)}])^{p-1}.\end{aligned} \tag{3.13}$$

Furthermore, by (3.11) we get

$$\begin{aligned}&R^{p/(p-1)}\\ =&\exp\left[\frac{p}{p-1}M_{\ell^\varepsilon(T)-\ell^\varepsilon(0)}-\frac{p}{2(p-1)}\langle M\rangle_{\ell^\varepsilon(T)-\ell^\varepsilon(0)}\right]\\ =&\exp\left[\frac{p}{2(p-1)^2}\langle M\rangle_{\ell^\varepsilon(T)-\ell^\varepsilon(0)}\right]\cdot\exp\left[\frac{p}{p-1}M_{\ell^\varepsilon(T)-\ell^\varepsilon(0)}-\frac{p^2}{2(p-1)^2}\langle M\rangle_{\ell^\varepsilon(T)-\ell^\varepsilon(0)}\right]\\ \leqslant&\exp\left[\frac{p}{2(p-1)^2}\Big(C_1(K_1,H,\kappa)\|\xi-\eta\|_2^2+C_4(K,K_1,H,T,r,\kappa,\varepsilon)|\xi(0)-\eta(0)|^2\Big)\right.\\ &\left.\frac{B^2(\frac{3}{2}-H,\frac{1}{2}-H)}{2-2H}T^{2-2H}\right]\cdot\exp\left[\frac{p}{p-1}M_{\ell^\varepsilon(T)-\ell^\varepsilon(0)}-\frac{p^2}{2(p-1)^2}\langle M\rangle_{\ell^\varepsilon(T)-\ell^\varepsilon(0)}\right],\end{aligned}$$

and noting the fact that $\exp\left[\frac{p}{p-1}M_{\ell^\varepsilon(T)-\ell^\varepsilon(0)}-\frac{p^2}{2(p-1)^2}\langle M\rangle_{\ell^\varepsilon(T)-\ell^\varepsilon(0)}\right]$, $0\leqslant t\leqslant T$ is a martingale with mean 1 due to Novikov's criterion. Then, we have

$$\begin{aligned}&\mathbb{E}\left[R^{p/(p-1)}\right]\\ \leqslant&\exp\left[\frac{p}{2(p-1)^2}\Big(C_1(K_1,H,\kappa)\|\xi-\eta\|_2^2+C_4(K,K_1,H,T,r,\kappa,\varepsilon)|\xi(0)-\eta(0)|^2\Big)\right.\\ &\left.\frac{B^2(\frac{3}{2}-H,\frac{1}{2}-H)}{2-2H}T^{2-2H}\right].\end{aligned}$$

Inserting this estimate into (3.13), we get the power-Harnack inequality. □

Proof of Proposition 3.1. Fix $T>r$. By a standard approximation argument, we may assume that $f\in C_b(\mathscr{L})$.

Step 1: First, we assume that b is globally Lipschitzian: there exists a constant $C>0$ such that

$$|b(x)-b(y)|\leqslant C|x-y|,\ \ x,y\in\mathbb{R}^d.$$

By the Lipschitz continuity of b and B, and noting that $|X^{\ell^\varepsilon,\xi}(u)-X^{\ell,\xi}(u)|\leqslant\|X^{\ell^\varepsilon,\xi}(u)-X^{\ell,\xi}(u)\|_2$, we have for $t\geqslant 0$

$$\begin{aligned}|X^{\ell^\varepsilon,,\xi}(t)-X^{\ell,\xi}(t)|\leqslant&C\int_0^t|X^{\ell^\varepsilon,\xi}(u)-X^{\ell,\xi}(u)|du+K_1\int_0^t\|X_u^{\ell^\varepsilon,\xi}-X_u^{\ell,\xi}\|_2du\\ &+|W^H_{\ell^\varepsilon(t)-\ell^\varepsilon(0)}-W^H_{\ell(t)}|\\ \leqslant&(C+K_1)\int_0^t\|X^{\ell^\varepsilon,\xi}(u)-X^{\ell,\xi}(u)\|_2du+|W^H_{\ell^\varepsilon(t)-\ell^\varepsilon(0)}-W^H_{\ell(t)}|.\end{aligned}$$

By the Hölder's inequality we have for $t \in [0,T]$

$$
\begin{aligned}
|X^{\ell^\varepsilon,\xi}(t) - X^{\ell,\xi}(t)|^2 &\leqslant 2(C+K_1)t\int_0^t \|X^{\ell^\varepsilon,\xi}(u) - X^{\ell,\xi}(u)\|_2^2 du + 2|W^H_{\ell^\varepsilon(t)-\ell^\varepsilon(0)} - W^H_{\ell(t)}|^2 \\
&\leqslant 2(C+K_1)T\int_0^t \|X^{\ell^\varepsilon,\xi}(u) - X^{\ell,\xi}(u)\|_2^2 du + 2|W^H_{\ell^\varepsilon(t)-\ell^\varepsilon(0)} - W^H_{\ell(t)}|^2.
\end{aligned}
$$

Applying the Lemma 3.3 of [12] with $g^{(\varepsilon)}(t) = |X^{\ell^\varepsilon,\xi}(t) - X^{\ell,\xi}(t)|$ and $h^{(\varepsilon)}(t) = 2|W^H_{\ell^\varepsilon(t)-\ell^\varepsilon(0)} - W^H_{\ell(t)}|^2$, we conclude that $X_T^{\ell^\varepsilon,\xi} \to X_T^{\ell,\xi}$ in $\mathscr{L}$ as $\varepsilon \downarrow 0$, and so

$$\lim_{\varepsilon\downarrow 0} P_T^{\ell^\varepsilon} f = P_T^\ell f, \quad f \in C_b(\mathscr{L}).$$

Since ℓ is of bounded variation, it is easy to get from (3.4) that

$$\lim_{\varepsilon\downarrow 0} \int_0^{T-r} e^{-2Kt} d\ell^\varepsilon(t) = \int_0^{T-r} e^{-2Kt} d\ell(t).$$

Letting $\varepsilon \downarrow 0$ in the Proposition 3.2, we obtain the desired results.

Step 2: For the general case, we shall make use of the approximation argument proposed in [37]. Let

$$\tilde{b}(x) := b(x) - Kx, \quad x \in \mathbb{R}^d.$$

Then $\tilde{b}$ satisfies the dissipative condition:

$$\langle \tilde{b}(x) - \tilde{b}(y), x-y\rangle \leqslant 0, \quad x,y \in \mathbb{R}^d,$$

and it is easy to see that the mapping $id - \varepsilon b : \mathbb{R}^d \to \mathbb{R}^d$ is injective for any $\varepsilon > 0$. For $\varepsilon > 0$, let $\tilde{b}^{(\varepsilon)}$ be the Yoshida approximation of $\tilde{b}$, i.e.

$$\tilde{b}^{(\varepsilon)}(x) := \frac{1}{\varepsilon}\left[\Big(id - \varepsilon b\Big)^{-1}(x) - x\right], \quad x \in \mathbb{R}^d.$$

Then $\tilde{b}^{(\varepsilon)}$ is dissipative and globally Lipschitzian, $|\tilde{b}^{(\varepsilon)}| \leqslant |\tilde{b}|$ and $\lim_{\varepsilon\downarrow 0} \tilde{b}^{(\varepsilon)} = \tilde{b}$. Let $b^{(\varepsilon)}(x) := \tilde{b}^{(\varepsilon)}(x) + Kx$. Then $b^{(\varepsilon)}$ is also Lipschitzian and

$$\langle x-y, b^{(\varepsilon)}(x) - b^{(\varepsilon)}(y)\rangle \leqslant K|x-y|^2, \quad x,y \in \mathbb{R}^d.$$

Let $X_t^{\ell,(\varepsilon),\xi}$ solve the SDE (1.1) with b replaced by $b^{(\varepsilon)}$ and $X_0^{\ell,(\varepsilon),\xi} = \xi \in \mathscr{L}$. Denote by $P_t^{\ell,(\varepsilon)}$ the associated semigroup. Due to the first step of the proof, the statements of the Proposition 3.1 hold with P_t^ℓ replaced by $P_t^{\ell,(\varepsilon)}$. If

$$\lim_{\varepsilon\downarrow 0} P_T^{\ell,(\varepsilon)} f = P_T^\ell f, \quad f \in C_b(\mathscr{L}), \tag{3.14}$$

we complete the proof by applying the Proposition 3.1 with P_t^ℓ replaced by $P_t^{\ell,(\varepsilon)}$ and letting $\varepsilon \downarrow 0$. Indeed, noticing that

$$\begin{aligned}
& d|X^{\ell^\varepsilon,\xi}(t) - X^{\ell,\xi}(t)|^2 \\
=& 2\langle X^{\ell^\varepsilon,\xi}(t) - X^{\ell,\xi}(t), b^{(\varepsilon)}(X^{\ell^\varepsilon,\xi}(t)) - b^{(\varepsilon)}(X^{\ell,\xi}(t))\rangle dt \\
& + 2\langle X^{\ell^\varepsilon,\xi}(t) - X^{\ell,\xi}(t), b^{(\varepsilon)}(X^{\ell,\xi}(t)) - b(X^{\ell,\xi}(t))\rangle dt \\
& + 2\langle X^{\ell^\varepsilon,\xi}(t) - X^{\ell,\xi}(t), B^{(\varepsilon)}(X_t^{\ell^\varepsilon,\xi}) - B^{(\varepsilon)}(X_t^{\ell,\xi})\rangle dt \\
\leqslant& (2K+1)|X^{\ell^\varepsilon,\xi}(t) - X^{\ell,\xi}(t)|^2 dt + |b^{(\varepsilon)}(X^{\ell,\xi}(t)) - b(X^{\ell,\xi}(t))|^2 dt \\
& + 2K_1 \|X_t^{\ell^\varepsilon,\xi} - X_t^{\ell,\xi}\|_2^2 dt \\
\leqslant& (2|K| + 2K_1 + 1)\|X_t^{\ell^\varepsilon,\xi} - X_t^{\ell,\xi}\|_2^2 dt + |b^{(\varepsilon)}(X^{\ell,\xi}(t)) - b(X^{\ell,\xi}(t))|^2 dt,
\end{aligned}$$

one has for $t \in [0,T]$

$$\begin{aligned}
& |X^{\ell^\varepsilon,\xi}(t) - X^{\ell,\xi}(t)|^2 \\
\leqslant& (2|K| + 2K_1 + 1)\int_0^t \|X_s^{\ell^\varepsilon,\xi} - X_s^{\ell,\xi}\|_2^2 ds + \int_0^t |b^{(\varepsilon)}(X^{\ell,\xi}(s)) - b(X^{\ell,\xi}(s))|^2 ds \\
=& (2|K| + 2K_1 + 1)\int_0^t \|X_s^{\ell^\varepsilon,\xi} - X_s^{\ell,\xi}\|_2^2 ds + \int_0^t |\tilde{b}^{(\varepsilon)}(X^{\ell,\xi}(s)) - \tilde{b}(X^{\ell,\xi}(s))|^2 ds.
\end{aligned}$$

Applying the Lemma 3.3 of [12] with $g^{(\varepsilon)}(t) = |X^{\ell^\varepsilon,\xi}(t) - X^{\ell,\xi}(t)|$ and $h^{(\varepsilon)}(t) = \int_0^t |\tilde{b}^{(\varepsilon)}(X^{\ell,\xi}(s)) - \tilde{b}(X^{\ell,\xi}(s))|^2 ds$, we know that $X_T^{\ell^\varepsilon,\xi} \to X_T^{\ell,\xi}$ in $\mathscr{L}$ as $\varepsilon \downarrow 0$, and thus (3.14) follows. $\square$

Proof of Theorem 3.1. Since the processes S and W^H are independent, we have

$$P_T f(\cdot) = \mathbb{E}\Big[P_T^\ell f(\cdot)|\ell = S\Big], f \in \mathscr{B}_b(\mathscr{L}). \tag{3.15}$$

By the first assertion of the Proposition 3.1, for all $f \in \mathscr{B}_b(\mathscr{L})$ with $f \geqslant 1$,

$$\begin{aligned}
& P_T \log f(\eta) \\
=& \mathbb{E}\Big[P_T^\ell \log f(\eta)|\ell = S\Big] \\
\leqslant& \mathbb{E}\Big[P_T^\ell \log f(\xi)|\ell = S\Big] + \Big(C_1(K_1,H,\kappa)\|\xi - \eta\|_2^2 + C_2(K,K_1,H,T,r,\kappa)|\xi(0) - \eta(0)|^2\Big) \\
& \frac{B^2(\frac{3}{2} - H, \frac{1}{2} - H)}{4 - 4H} T^{2-2H},
\end{aligned}$$

which, together with the Jensen's inequality and (3.15), implies the log-Harnack inequality. Analogously, by the second assertion of Proposition 3.1, for all non-negative $f \in \mathscr{B}_b(\mathscr{L})$

$$
\begin{aligned}
P_T f(\eta) =& \mathbb{E}\Big[P_T^{\ell} f(\eta)|\ell = S\Big] \\
\leqslant& \mathbb{E}\Big[(P_T^{\ell} f^p(\xi))^{1/p} \exp\Big[\frac{1}{2(p-1)}\Big(C_1(K_1,H,\kappa)\|\xi-\eta\|_2^2 \\
&+ |\xi(0)-\eta(0)|^2 C_3(K,K_1,H,T,r,\kappa)\Big|_{\ell=S}\Big)\frac{B^2(\frac{3}{2}-H,\frac{1}{2}-H)}{4-4H}T^{2-2H}\Big]\Big].
\end{aligned}
$$

It remains to use the Hölder inequality and (3.15) to derive the power-Harnack inequality. □

4. Harnack inequality for (1.1) with $H \in (1/2,1)$

In this section, we consider the Harnack inequality for (1.1) with $H \in (1/2,1)$.

THEOREM 4.1. *Let $H \in (1/2,1)$, $T > r$, the Hypothesis (H) holds and S be a subordinator with Bernstein function of ϕ of the form (2.2). Then for any $\sigma > 1$ such that* $\max\{0, 2H-\frac{3}{2}\} < \frac{1}{\sigma} < H - \frac{1}{2}$ *we have*

(i) for any $\xi,\eta \in \mathscr{L}$ and $f \in \mathscr{B}_b(\mathscr{L})$ with $f \geqslant 1$,

$$
P_T \log f(\eta) \leqslant \log P_T f(\xi) + \frac{1}{2}\mathbb{E}[\rho(K,K_1,H,\kappa,r,T,\sigma,\xi,\eta)];
$$

(ii) for any $p > 1$, $\xi,\eta \in \mathscr{L}$ and non-negative $f \in \mathscr{B}_b(\mathscr{L})$,

$$
\Big(P_T f(\eta)\Big)^p \leqslant P_T f^p(\xi)\Big(\mathbb{E}\exp\Big[\frac{p}{2(p-1)^2}\rho(K,K_1,H,\kappa,r,T,\sigma,\xi,\eta)\Big]\Big),
$$

where

$$
\begin{aligned}
&\rho(K,K_1,H,\kappa,r,T,\sigma,\xi,\eta) \\
&= C(K,K_1,H,\kappa,r,T,\sigma)\Big[\frac{S(T)^{2-2H}}{H-\frac{1}{\sigma}} + 2S(T)^{2-2H}B\Big(2H-\frac{2}{\sigma}, 3-4H+\frac{2}{\sigma}\Big)\Big], \\
&C(K,K_1,H,\kappa,r,T,\sigma) \\
&= \frac{C(K,K_1,H,\kappa,r,T)}{\Gamma^2(H-\frac{1}{2})|\sigma(1-2H)+1|^{\frac{2}{\sigma}}}\tilde{c}_H^{-2}B^{\frac{2(\sigma-1)}{\sigma}}\Big(\frac{\sigma(H-\frac{1}{2})}{\sigma-1}+1, \frac{\sigma(-\frac{3}{2}+H)}{\sigma-1}+1\Big), \\
&\tilde{c}_H = \Big(\frac{2H}{\Gamma(2H)\Gamma(3-2H)}\Big)^{\frac{1}{2}}, \\
&C(K,K_1,H,\kappa,r,T) = \Big(\frac{K_1}{\Gamma(\frac{1}{2}-H)}\Big)^{-2}C_1(K_1,H,\kappa)\|\xi-\eta\|^2 \\
&\qquad + \Big(\frac{K_1}{\Gamma(\frac{1}{2}-H)}\Big)^{-2}C_2'(K,K_1,H,T,r,\kappa)|\xi(0)-\eta(0)|^2,
\end{aligned}
$$

$$C_2'(K,K_1,H,T,r,\kappa)=\Big(\frac{K_1}{\Gamma(\frac{1}{2}-H)}\Big)^2\Big(\frac{e^{2K(T-r)}-1}{K\kappa^2}+2\Big(\int_0^{T-r}e^{-2Kt}dS(t)\Big)^{-2}\Big)$$

and $B(\cdot,\cdot)$ and $\Gamma(\cdot)$ are the standard Beta and Gamma functions.

COROLLARY 4.1. *Let the assumptions in Theorem 4.1 hold. Then the following assertions hold.*

(i) For any $\xi,\eta\in\mathscr{L}$ and $P_T(\xi,\cdot)$ is equivalent to $P_T(\eta,\cdot)$ and

$$Ent(P_T(\xi,\cdot)|P_T(\eta,\cdot))\leqslant\frac{1}{2}\mathbb{E}[\rho(K,K_1,H,\kappa,r,T,\sigma,\xi,\eta)],$$

which together with Pinsker's inequality implies that

$$2\|P_T(\xi,\cdot)-P_T(\eta,\cdot)\|_{var}^2\leqslant\frac{1}{2}\mathbb{E}[\rho(K,K_1,H,\kappa,r,T,\sigma,\xi,\eta)].$$

(ii) For any $p>1$, $\xi,\eta\in\mathscr{L}$

$$P_T\Big\{\Big(\frac{dP_T(\xi,\cdot)}{dP_T(\eta,\cdot)}\Big)^{1/(p-1)}\Big\}(\xi)\leqslant\mathbb{E}\Big\{\exp\Big[\frac{p}{2(p-1)^2}[\rho(K,K_1,H,\kappa,r,T,\sigma,\xi,\eta)]\Big]\Big\}.$$

PROPOSITION 4.1. *Let $H\in(1/2,1)$, $T>r$ and the Hypothesis (H) holds. Then for any $\sigma>1$ such that $\max\{0,2H-\frac{3}{2}\}<\frac{1}{\sigma}<H-\frac{1}{2}$ we have*

(i) for any $\xi,\eta\in\mathscr{L}$ and $f\in\mathscr{B}_b(\mathscr{L})$ with $f\geqslant 1$,

$$P_T^\ell\log f(\eta)\leqslant\log P_T^\ell f(\xi)+\frac{1}{2}\rho(K,K_1,H,\kappa,r,T,\ell,\sigma,\xi,\eta);$$

(ii) for any $p>1$, $\xi,\eta\in\mathscr{L}$ and non-negative $f\in\mathscr{B}_b(\mathscr{L})$,

$$\Big(P_T^\ell f(\eta)\Big)^p\leqslant P_T^\ell f^p(\xi)\exp\Big[\frac{p}{2(p-1)^2}\rho(K,K_1,H,\kappa,r,T,\ell,\sigma,\xi,\eta)\Big],$$

where

$$\rho(K,K_1,H,\kappa,r,T,\ell,\sigma,\xi,\eta)=C(K,K_1,H,\kappa,r,T,\ell,\sigma)\Big[\frac{\ell(T)^{2-2H}}{H-\frac{1}{\sigma}}+2\ell(T)^{2-2H}B\Big(2H-\frac{2}{\sigma},3-4H+\frac{2}{\sigma}\Big)\Big],$$

$$C(K,K_1,H,\kappa,r,T,\ell,\sigma)=\frac{C(K,K_1,H,\kappa,r,T,\ell)}{\Gamma^2(H-\frac{1}{2})|\sigma(1-2H)+1|^{\frac{2}{\sigma}}}\tilde{c}_H^{-2}B^{\frac{2(\sigma-1)}{\sigma}}\Big(\frac{\sigma(H-\frac{1}{2})}{\sigma-1}+1,\frac{\sigma(-\frac{3}{2}+H)}{\sigma-1}+1\Big),$$

$$\tilde{c}_H = \Big(\frac{2H}{\Gamma(2H)\Gamma(3-2H)}\Big)^{\frac{1}{2}},$$

$$C(K,K_1,H,\kappa,r,T,\ell) = \Big(\frac{K_1}{\Gamma(\frac{1}{2}-H)}\Big)^{-2} C_1(K_1,H,\kappa)\|\xi-\eta\|^2 + \Big(\frac{K_1}{\Gamma(\frac{1}{2}-H)}\Big)^{-2} C_3(K,K_1,H,T,r,\kappa)|\xi(0)-\eta(0)|^2,$$

and $B(\cdot,\cdot)$ *and* $\Gamma(\cdot)$ *are the standard Beta and Gamma functions.*

PROPOSITION 4.2. *Let* $H \in (1/2,1)$*,* $T > r$ *and the Hypothesis (H) holds. Then for any* $\sigma > 1$ *such that* $\max\{0, 2H-\frac{3}{2}\} < \frac{1}{\sigma} < H-\frac{1}{2}$ *we have*

(i) for any $\xi,\eta \in \mathscr{L}$ *and* $f \in \mathscr{B}_b(\mathscr{L})$ *with* $f \geqslant 1$*,*

$$P_T^{\ell^\varepsilon}\log f(\eta) \leqslant \log P_T^{\ell^\varepsilon} f(\xi) + \frac{1}{2}\rho(K,K_1,H,\kappa,r,T,\varepsilon,\sigma,\xi,\eta);$$

(ii) for any $p>1$*,* $\xi,\eta \in \mathscr{L}$ *and non-negative* $f \in \mathscr{B}_b(\mathscr{L})$*,*

$$\Big(P_T^{\ell^\varepsilon} f(\eta)\Big)^p \leqslant P_T^{\ell^\varepsilon} f^p(\xi)\exp\Big[\frac{p}{2(p-1)^2}\rho(K,K_1,H,\kappa,r,T,\varepsilon,\sigma,\xi,\eta)\Big],$$

where

$$\rho(K,K_1,H,\kappa,r,T,\varepsilon,\sigma,\xi,\eta) = C(K,K_1,H,\kappa,r,T,\varepsilon,\sigma)\Big[\frac{\ell^\varepsilon(T)^{2-2H}}{H-\frac{1}{\sigma}} + 2\ell^\varepsilon(T)^{2-2H} B\Big(2H-\frac{2}{\sigma}, 3-4H+\frac{2}{\sigma}\Big)\Big],$$

$$C(K,K_1,H,\kappa,r,T,\varepsilon,\sigma) = \frac{C(K,K_1,H,\kappa,r,T,\varepsilon)}{\Gamma^2(H-\frac{1}{2})|\sigma(1-2H)+1|^{\frac{2}{\sigma}}}\tilde{c}_H^{-2} B^{\frac{2(\sigma-1)}{\sigma}}\Big(\frac{\sigma(H-\frac{1}{2})}{\sigma-1}+1, \frac{\sigma(-\frac{3}{2}+H)}{\sigma-1}+1\Big),$$

$$\tilde{c}_H = \Big(\frac{2H}{\Gamma(2H)\Gamma(3-2H)}\Big)^{\frac{1}{2}},$$

$$C(K,K_1,H,\kappa,r,T,\varepsilon) = \Big(\frac{K_1}{\Gamma(\frac{1}{2}-H)}\Big)^{-2} C_1(K_1,H,\kappa)\|\xi-\eta\|^2 + \Big(\frac{K_1}{\Gamma(\frac{1}{2}-H)}\Big)^{-2} C_4(K,K_1,H,T,r,\kappa,\varepsilon)|\xi(0)-\eta(0)|^2,$$

and $B(\cdot,\cdot)$ *and* $\Gamma(\cdot)$ *are the standard Beta and Gamma functions.*

Proof. First of all, we will construct coupling as follows. Let Y_t solve the equation

$$dY(t) = b(Y(t))dt + B(X_t^{\ell^\varepsilon,\xi})dt + \lambda(t)\mathscr{I}_{[0,\tau)}(t)\frac{X^{\ell^\varepsilon,\xi}(t)-Y(t)}{|X^{\ell^\varepsilon,\xi}(t)-Y(t)|}|\xi(0)-\eta(0)|d\ell^\varepsilon(t) + dW^H_{\ell^\varepsilon(t)-\ell^\varepsilon(0)} \tag{4.1}$$

with $Y_0 = \eta$, where

$$\lambda(t) := \frac{e^{-Kt}}{\int_0^{T-r} e^{-2Ks} d\ell^\varepsilon(s)}, \ \ t \geqslant 0,$$

and

$$\tau := T \wedge \inf\{t \geqslant 0; X^{\ell^\varepsilon,\xi}(t) = Y(t)\}$$

is the coupling time. It is clear that $(X^{\ell^\varepsilon,\xi}(t), Y(t))$ is well defined for $t < \tau$. By (H), we have

$$d|X^{\ell^\varepsilon,\xi}(t) - Y(t)| \leqslant K|X^{\ell^\varepsilon,\xi}(t) - Y(t)|dt - \lambda(t)|\xi(0) - \eta(0)|d\ell^\varepsilon(t), \ \ t \in [0,\tau).$$

Thus, for $t \in [0,\tau)$,

$$\begin{aligned} |X^{\ell^\varepsilon,\xi}(t) - Y(t)| &\leqslant e^{Kt}|\xi(0) - \eta(0)|\Big(1 - \int_0^t e^{-Ks}\lambda(s)d\ell^\varepsilon(s)\Big) \\ &\leqslant \frac{e^{Kt}\int_t^{T-r} e^{-2Ks}d\ell^\varepsilon(s)}{\int_0^{T-r} e^{-2Ks}d\ell^\varepsilon(s)}|\xi(0) - \eta(0)| \\ &=: \gamma(t)|\xi(0) - \eta(0)|. \end{aligned} \tag{4.2}$$

If $\tau(\omega) > T - r$ for some $\omega \in \Omega$, we can take $t = T - r$ in the above inequality to get

$$0 < |X^{\ell^\varepsilon,\xi}(t)(\omega) - Y(t)(\omega)| \leqslant 0,$$

which is absurd. Therefore, $\tau \leqslant T - r$. Letting $Y(t) = X^{\ell^\varepsilon,\xi}(t)$ for $t \in [\tau, T]$, $Y(t)$ solves (3.6) for $t \in [\tau, T]$. In particular, $X_T^{\ell^\varepsilon,\xi} = Y_T$. Moreover, by (3.7) and $\tau \leqslant T - r$, we have

$$|X^{\ell^\varepsilon,\xi}(t) - Y(t)|^2 \leqslant |\xi(0) - \eta(0)|^2\gamma(t)^2\mathscr{I}_{[0,T-r]}(t), \ \ t \in [0,T]. \tag{4.3}$$

Denote by $\zeta^\varepsilon : [\ell^\varepsilon(0),\infty) \to [0,\infty)$ the inverse function of ℓ^ε. Then $\ell^\varepsilon(\zeta^\varepsilon(t)) = t$ for $t \geqslant \ell^\varepsilon(0)$, $\zeta^\varepsilon(\ell^\varepsilon(t)) = t$ for $t \geqslant 0$, and $t \to \zeta^\varepsilon(t)$ is absolutely continuous and strictly increasing. Let

$$\Psi(u) := \Phi \circ \zeta^\varepsilon(u + \ell^\varepsilon(0)),$$

where

$$\Phi(u) := [B(X_u^{\ell^\varepsilon,\xi}) - B(Y_u)]\frac{1}{(\ell^\varepsilon)'(u)} + \lambda(u)\mathscr{I}_{[0,\tau)}(u)\frac{X^{\ell^\varepsilon,\xi}(u) - Y(u)}{|X^{\ell^\varepsilon,\xi}(u) - Y(u)|}|\xi(0) - \eta(0)|.$$

Then, we have

$$\begin{aligned} &\int_0^{\ell^\varepsilon(T)} (\ell^\varepsilon(T) - v)^{1-2H}\Phi(v)dv \\ &= \frac{1}{\Gamma(\frac{3}{2} - H)}\int_0^{\ell^\varepsilon(T)} (\ell^\varepsilon(T) - v)^{\frac{1}{2}-H}\Gamma\Big(\frac{3}{2} - H\Big)(\ell^\varepsilon(T) - v)^{\frac{1}{2}-H}\Phi(v)dv \\ &= (I_{0+}^{\frac{3}{2}-H}\overline{\Phi})(\ell^\varepsilon(T)) \end{aligned}$$

where

$$\overline{\Phi}(v) = \Gamma\Big(\frac{3}{2} - H\Big)(\ell^{\varepsilon}(T) - v)^{\frac{1}{2}-H}\Phi(v).$$

Recalling that ℓ is an sample path of the subordinator S with drift parameter $\kappa \geqslant 0$, one have for all $t \in [0, \ell^{\varepsilon}(T)]$

$$(\ell^{\varepsilon})'(t) = \frac{\ell(t+\varepsilon) - \ell(t)}{\varepsilon} + \varepsilon > \kappa.$$

by view of the definition of $\|\cdot\|_2$ we have for all $t \geqslant 0$

$$\begin{aligned}
\|X_t^{\ell^{\varepsilon},\xi} - Y_t\|_2^2 &= \int_{-r}^{0} |X^{\ell^{\varepsilon},\xi}(t+s) - Y(t+s)|^2 ds + |\xi(0) - \eta(0)|^2 \\
&= \int_{t-r}^{t} |X^{\ell^{\varepsilon},\xi}(s) - Y(s)|^2 ds + |\xi(0) - \eta(0)|^2 \\
&\leqslant \int_{-r}^{0} |\xi(s) - \eta(s)|^2 ds + \int_{0}^{t} |X^{\ell^{\varepsilon},\xi}(s) - Y(s)|^2 ds + |\xi(0) - \eta(0)|^2 \\
&= \|\xi - \eta\|_2^2 + \int_{0}^{t} |X^{\ell^{\varepsilon},\xi}(s) - Y(s)|^2 ds.
\end{aligned}$$

Then, by (4.2) we have for all $t \geqslant 0$

$$\begin{aligned}
\|X_t^{\ell^{\varepsilon},\xi} - Y_t\|_2^2 &\leqslant \|\xi - \eta\|_2^2 + |\xi(0) - \eta(0)|^2 \int_0^{T-r} \gamma(s)^2 ds \\
&\leqslant \|\xi - \eta\|_2^2 + \frac{e^{2K(T-r)} - 1}{2K} |\xi(0) - \eta(0)|^2,
\end{aligned} \tag{4.4}$$

where in the last inequality we have used $\gamma(s) \leqslant e^{Ks}$ for $s \in [0, T-r]$. By the definition of $\lambda(t)$, it is easy to see that for all $t \geqslant 0$

$$|\lambda(t)| \leqslant \Big(\int_0^{T-r} e^{-2Kt} d\ell^{\varepsilon}(t)\Big)^{-1}. \tag{4.5}$$

Thus, combining (4.4) and (4.5) we have for all $t \in [0, \ell^{\varepsilon}(T)]$

$$\begin{aligned}
|\Phi(t)|^2 &\leqslant \Big(\frac{K_1}{\Gamma(\frac{1}{2} - H)}\Big)^{-2} C_1(K_1, H, \kappa)\|\xi - \eta\|_2^2 \\
&\quad + \Big(\frac{K_1}{\Gamma(\frac{1}{2} - H)}\Big)^{-2} C_4(K, K_1, H, T, r, \kappa, \varepsilon)|\xi(0) - \eta(0)|^2 \\
&=: C(K, K_1, H, \kappa, r, T, \varepsilon).
\end{aligned} \tag{4.6}$$

Since $1 - 2H > -1$ and $\Phi(v)$ is bound on $[0, \ell^{\varepsilon}(T)]$, we know $\overline{\Phi}(v) \in L^2([0, \ell^{\varepsilon}(T)]; \mathbb{R}^d)$. Thus we have that $\int_0^{\cdot} \tilde{c}_H^{-1}(\ell^{\varepsilon}(T) - v)^{1-2H}\Phi(v)dv \in I_{0+}^{\frac{3}{2}-H}(L^2([0, \ell^{\varepsilon}(T)]; \mathbb{R}))$. Note that by means of the integral representation of fractional Brownian motion, the definition of the operator K_H and transformation formulas for fractional Brownian motion (see [19]), we get for any $0 \leqslant \ell^{\varepsilon}(t) \leqslant \ell^{\varepsilon}(T)$,

$$\begin{aligned}
\widetilde{W}^H_{\ell^{\varepsilon}(t)} &= \int_0^{\ell^{\varepsilon}(t)} \Phi(v)dv + W^H_{\ell^{\varepsilon}(t)} \\
&= \int_0^{\ell^{\varepsilon}(t)} \Phi(v)dv + \tilde{c}_H \int_0^{\ell^{\varepsilon}(t)} (\ell^{\varepsilon}(t)-v)^{2H-1} dW^{1-H}_v \\
&= \int_0^{\ell^{\varepsilon}(t)} \tilde{c}_H (\ell^{\varepsilon}(t)-v)^{2H-1} [\tilde{c}_H^{-1}(\ell^{\varepsilon}(t)-v)^{1-2H}\Phi(v)dv + dW^{1-H}_v].
\end{aligned} \tag{4.7}$$

For any $0 \leqslant v \leqslant \ell^{\varepsilon}(t)$, let

$$\begin{aligned}
\widetilde{W}^{1-H}_v &= \int_0^v \tilde{c}_H^{-1}(\ell^{\varepsilon}(t)-s)^{1-2H}\Phi(s)ds + W^{1-H}_v \\
&= \int_0^v \tilde{c}_H^{-1}(\ell^{\varepsilon}(t)-s)^{1-2H}\Phi(s)dv + \int_0^v K_{1-H}(v,s)dW_s \\
&= \int_0^v K_{1-H}(v,s)\Big[\Big(K^{-1}_{1-H}\int_0^{\cdot} \tilde{c}_H^{-1}(\ell^{\varepsilon}(t)-z)^{1-2H}\Phi(z)dz\Big)(s)ds + dW_s\Big]
\end{aligned}$$

where $\tilde{c}_H = \left(\frac{2H}{\Gamma(2H)\Gamma(3-2H)}\right)^{\frac{1}{2}}$.

Now, let

$$\begin{aligned}
R(\ell^{\varepsilon}(T)) = \exp\Big[&-\int_0^{\ell^{\varepsilon}(T)} \Big(K^{-1}_{1-H}\int_0^{\cdot} \tilde{c}_H^{-1}(\ell^{\varepsilon}(T)-z)^{1-2H}\Phi(z)dz\Big)(v)dW(v) \\
&-\frac{1}{2}\int_0^{\ell(T)} \Big(K^{-1}_{1-H}\int_0^{\cdot} \tilde{c}_H^{-1}(\ell^{\varepsilon}(T)-z)^{1-2H}\Phi(z)dz\Big)^2(v)dv\Big].
\end{aligned}$$

Using Corollary 5.2 of [19], we immediately know that $(\widetilde{W}^H_{\ell^{\varepsilon}(t)})_{0\leqslant t\leqslant \ell^{\varepsilon}(T)}$ is an $\mathscr{F}^{B^H}_{\ell\varepsilon(t)}$-fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2},1)$ under the new probability $\mathbb{Q}(d\omega) = R(\ell^{\varepsilon}(T))\mathbb{P}(d\omega)$ if $(\widetilde{W}^{1-H}_{\ell^{\varepsilon}(t)})_{0\leqslant t\leqslant \ell^{\varepsilon}(T)}$ is an $\mathscr{F}^{W^{1-H}}_{\ell^{\varepsilon}(t)}$-fractional Brownian motion with Hurst parameter $1-H$ under the new probability $\mathbb{Q}(d\omega) = R(\ell^{\varepsilon}(T))\mathbb{P}(d\omega)$.

Next we want to show $(\widetilde{W}^{1-H}_t)_{0\leqslant t\leqslant \ell^{\varepsilon}(T)}$ is an $\mathscr{F}^{B^{1-H}}_{\ell^{\varepsilon}(t)}$-fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2},1)$ under the new probability $\mathbb{Q}(d\omega) = R(\ell^{\varepsilon}(T))\mathbb{P}(d\omega)$. Due to [25], it suffices to show that $\mathbb{E}^{\mathbb{P}}R(\ell^{\varepsilon}(T)) = 1$. Since $\int_0^{\cdot} \tilde{c}_H^{-1}(\ell^{\varepsilon}(T)-z)^{1-2H}\Phi(z)dz$ is absolutely continuous, we have by (2.1) that

$$\Big(K^{-1}_{1-H}\int_0^{\cdot} \tilde{c}_H^{-1}(\ell^{\varepsilon}(T)-z)^{1-2H}\Phi(z)dz\Big)(v) = v^{\frac{1}{2}-H} I^{H-\frac{1}{2}}_{0+} v^{H-\frac{1}{2}} \tilde{c}_H^{-1}(\ell^{\varepsilon}(T)-v)^{1-2H}\Phi(v),$$
$$v \in [0,\ell^{\varepsilon}(T)].$$

Hence, by the Hölder's inequality and (4.6) we have further that for $v \in [0, \ell^{\varepsilon}(T)]$,

$$
\begin{aligned}
&\Big|\Big(K_{1-H}^{-1}\int_0^{\cdot}\tilde{c}_H^{-1}(\ell^{\varepsilon}(T)-z)^{1-2H}\Phi(z)dz\Big)(v)\Big| \\
=&\Big|\frac{1}{\Gamma(H-\frac{1}{2})}\tilde{c}_H^{-1}v^{\frac{1}{2}-H}\int_0^v z^{H-\frac{1}{2}}(\ell^{\varepsilon}(T)-z)^{1-2H}\Phi(z)(v-z)^{-\frac{3}{2}+H}dz\Big| \\
\leqslant&\frac{C^{1/2}(K,K_1,H,\kappa,r,T,\varepsilon)}{\Gamma(H-\frac{1}{2})}\tilde{c}_H^{-1}v^{\frac{1}{2}-H}\Big\{\int_0^v\Big(z^{H-\frac{1}{2}}(v-z)^{-\frac{3}{2}+H}\Big)^{\frac{\sigma}{\sigma-1}}dz\Big\}^{\frac{\sigma-1}{\sigma}} \\
&\cdot\Big\{\int_0^v(\ell^{\varepsilon}(T)-z)^{\sigma(1-2H)}dz\Big\}^{\frac{1}{\sigma}} \\
=&\frac{C^{1/2}(K,K_1,H,\kappa,r,T,\varepsilon)}{\Gamma(H-\frac{1}{2})}\tilde{c}_H^{-1}B^{\frac{\sigma-1}{\sigma}}\Big(\frac{\sigma(H-\frac{1}{2})}{\sigma-1}+1,\frac{\sigma(-\frac{3}{2}+H)}{\sigma-1}+1\Big) \\
&\cdot\frac{\ell^{\varepsilon}(T)^{1-2H+\frac{1}{\sigma}}+(\ell^{\varepsilon}(T)-v)^{1-2H+\frac{1}{\sigma}}}{|\sigma(1-2H)+1|^{\frac{1}{\sigma}}}v^{H-\frac{1}{2}-\frac{1}{\sigma}}.
\end{aligned}
\tag{4.8}
$$

Then, we can further obtain that for the fixed $T>0$,

$$
\begin{aligned}
&\int_0^{\ell^{\varepsilon}(T)}\Big|\Big(K_{1-H}^{-1}\int_0^{\cdot}\tilde{c}_H^{-1}(\ell^{\varepsilon}(T)-z)^{1-2H}\Phi(z)dz\Big)(v)\Big|^2dv \\
\leqslant&C(K,K_1,H,\kappa,r,T,\varepsilon,\sigma)\int_0^{\ell^{\varepsilon}(T)}v^{2H-1-\frac{2}{\sigma}}\Big[\ell^{\varepsilon}(T)^{1-2H+\frac{1}{\sigma}}+(\ell^{\varepsilon}(T)-v)^{1-2H+\frac{1}{\sigma}}\Big]^2dv \\
\leqslant&C(K,K_1,H,\kappa,r,T,\varepsilon,\sigma)\Big[\frac{\ell^{\varepsilon}(T)^{2-2H}}{H-\frac{1}{\sigma}}+2\ell^{\varepsilon}(T)^{2-2H}B\Big(2H-\frac{2}{\sigma},3-4H+\frac{2}{\sigma}\Big)\Big] \\
=:&\rho(K,K_1,H,\kappa,r,T,\varepsilon,\sigma,\xi,\eta).
\end{aligned}
\tag{4.9}
$$

As a consequence, we get

$$
\begin{aligned}
&\mathbb{E}\exp\Big[\frac{1}{2}\int_0^{\ell^{\varepsilon}(T)}\Big(\tilde{c}_H^{-1}K_{1-H}^{-1}\int_0^{\cdot}(\ell^{\varepsilon}(T)-z)^{1-2H}\Phi(z)dz\Big)^2(v)dv\Big] \\
\leqslant&\exp\big[\frac{1}{2}\rho(K,K_1,H,\kappa,r,T,\varepsilon,\sigma,\xi,\eta)\big].
\end{aligned}
\tag{4.10}
$$

Using the well-known Novikov criterion, one can have $\mathbb{E}^{\mathbb{P}}R(\ell^{\varepsilon}(T))=1$. Then we can rewrite (4.1) as

$$dY(t)=b(Y(t))dt+B(Y_t)dt+d\widetilde{W}^H_{\ell^{\varepsilon}(t)-\ell^{\varepsilon}(0)}.$$

Thus, the distribution of $\{X_t^{\ell^{\varepsilon},\eta}\}_{0\leqslant t\leqslant T}$ under $\mathbb{P}$ coincides with the law of $\{Y_t\}_{0\leqslant t\leqslant T}$ under $R(\ell^{\varepsilon}(T))\mathbb{P}$. Therefore, we conclude from the definition of $P_t^{\ell^{\varepsilon}}$ that for all bounded Borel functions $f:\mathbb{R}^d\to\mathscr{L}$

$$P_T^{\ell^{\varepsilon}}f(\eta)=\mathbb{E}^{\mathbb{Q}}f(Y_T(\eta))=\mathbb{E}^{\mathbb{Q}}f(X_T^{\ell^{\varepsilon},\xi}(\xi))=\mathbb{E}^{\mathbb{P}}R(\ell^{\varepsilon}(T))f(X_T^{\ell^{\varepsilon},\xi}(\xi)). \tag{4.11}$$

By the Jensen's inequality, we obtain for any random variable $F \geqslant 1$,

$$\begin{aligned}\mathbb{E}\Big[R(\ell^\varepsilon(T))\log\frac{F}{R(\ell^\varepsilon(T))}\Big] &= \mathbb{E}_{R(\ell^\varepsilon(T))\mathbb{P}}\Big[\log\frac{F}{R(\ell^\varepsilon(T))}\Big]\\ &\leqslant \log\mathbb{E}_{R(\ell^\varepsilon(T))\mathbb{P}}\Big[\frac{F}{R(\ell^\varepsilon(T))}\Big] = \log\mathbb{E}[F].\end{aligned}$$

Then we have

$$\mathbb{E}\Big[R(\ell^\varepsilon(T))\log F\Big] \leqslant \log\mathbb{E}[F] + \mathbb{E}[R(\ell^\varepsilon(T))\log R(\ell^\varepsilon(T))].$$

Let $\eta(s) = \Big(K_{1-H}^{-1}\int_0^\cdot \tilde{c}_H^{-1}(\ell^\varepsilon(T)-z)^{1-2H}\Phi(z)dz\Big)(s)$. By the definition of $R(\ell^\varepsilon(T))$, combining with (4.10), we have

$$\begin{aligned}\log R(\ell^\varepsilon(T)) &= -\int_0^{\ell^\varepsilon(T)}\langle\eta(s),dW_s\rangle - \frac{1}{2}\int_0^{\ell^\varepsilon(T)}|\eta(s)|^2ds\\ &= -\int_0^{\ell^\varepsilon(T)}\langle\eta(s),d\widetilde{W}_s\rangle + \frac{1}{2}\int_0^{\ell^\varepsilon(T)}|\eta(s)|^2ds\\ &\leqslant -\int_0^{\ell^\varepsilon(T)}\langle\eta(s),d\widetilde{W}_s\rangle + \frac{1}{2}\rho(K,K_1,H,\kappa,r,T,\varepsilon,\sigma,\xi,\eta).\end{aligned}$$

Thus, for all bounded Borel functions $f \geqslant 1$ on $\mathscr{L}$, we have

$$\begin{aligned}P_T^{\ell^\varepsilon}\log f(\eta) =& \mathbb{E}[\log f(X_T^{\ell^\varepsilon}(\eta))] = \mathbb{E}[R(\ell^\varepsilon(T))\log f(X_T^{\ell^\varepsilon,\xi}(\xi))]\\ \leqslant& \log\mathbb{E}[f(X_T^{\ell^\varepsilon,\xi}(\xi))] + \mathbb{E}[R(\ell^\varepsilon(T))\log R(\ell^\varepsilon(T))]\\ =& \log P_T^{\ell^\varepsilon}f(\xi) + \mathbb{E}_{R(\ell^\varepsilon(T))\mathbb{P}}[\log R(\ell^\varepsilon(T))]\\ \leqslant& \log P_T^{\ell^\varepsilon}f(\xi) + \frac{1}{2}\rho(K,K_1,H,\kappa,r,T,\varepsilon,\sigma,\xi,\eta).\end{aligned}$$

The proof of the log-Harnack inequality is complete.

For all bounded Borel functions $f \geqslant 1$ on $\mathscr{L}$, by (4.11) and the Hölder's inequality, we can obtain

$$(P_T^{\ell^\varepsilon}f(\eta))^p = (\mathbb{E}f(X_T^{\ell^\varepsilon,\xi}(\eta)))^p = (\mathbb{E}Rf(X_T^{\ell^\varepsilon,\xi}(\xi)))^p \leqslant (P_T^{\ell^\varepsilon}f^p(\xi))(\mathbb{E}R^{\frac{p}{p-1}}(\ell^\varepsilon(T)))^{p-1}. \tag{4.12}$$

Let $M_t := -\int_0^t\langle\eta(s),dW_s\rangle$, $t \geqslant 0$. By using (4.10) we get

$$\begin{aligned}&R^{\frac{p}{p-1}}(\ell^\varepsilon(T))\\ =&\exp\Big[\frac{p}{p-1}M_{\ell^\varepsilon(T)} - \frac{p}{2(p-1)}\langle M\rangle_{\ell^\varepsilon(T)}\Big]\\ =&\exp\Big[\frac{p}{2(p-1)^2}\langle M\rangle_{\ell^\varepsilon(T)}\Big]\times\exp\Big[\frac{p}{p-1}M_{\ell^\varepsilon(T)} - \frac{p^2}{2(p-1)^2}\langle M\rangle_{\ell^\varepsilon(T)}\Big]\\ \leqslant&\exp\Big[\frac{p}{2(p-1)^2}\rho(K,K_1,H,\kappa,r,T,\varepsilon,\sigma,\xi,\eta)\Big]\\ &\times\exp\Big[\frac{p}{p-1}M_{\ell^\varepsilon(T)} - \frac{p^2}{2(p-1)^2}\langle M\rangle_{\ell^\varepsilon(T)}\Big].\end{aligned}$$

Notice that $\exp\left[\frac{p}{p-1}M_{\ell^\varepsilon(T)}-\frac{p^2}{2(p-1)^2}\langle M\rangle_{\ell^\varepsilon(T)}\right]$ is a martingale with mean 1. Then, using Novikov's criterion we get

$$\mathbb{E}[R^{\frac{p}{p-1}}(\ell^\varepsilon(T))]\leqslant\exp\left[\frac{p}{2(p-1)^2}\rho(K,K_1,H,\kappa,r,T,\varepsilon,\sigma,\xi,\eta)\right].$$

Thus, we get the power-Harnack inequality by plugging the above expression into (4.12). The proof is complete. □

According to the Proposition 4.2, we can easily prove the Proposition 4.1 and the Theorem 4.1 by using the same way as $H\in(0,1/2)$. Thus, we omit them.

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