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COMBINATORIAL INEQUALITIES ARISING FROM THE INCLUSION–EXCLUSION PRINCIPLE

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Abstract. We study a new inequality arising from the principle of inclusion and exclusion by mixing the idea from Hlawka's inequality and Tverberg's combinatorial sum. We obtain sharp lower bounds for the sum when the number of variables is small.

1. Introduction

A simple version of the triangle inequality can be stated as $|x|+|y|-|x+y| \geqslant 0$ for every $x,y \in \mathbb{R}$, and it can be generalized to a higher dimensional version as $\|x\|+\|y\|-\|x+y\| \geqslant 0$ for $x,y \in V$, where V is an inner product space. If $V = \mathbb{R}^n$ and the norm in V is the ℓ^p-norm, then it is also called Minkowski's inequality. In addition, Hlawka's inequality states that for any x,y,z in an inner product space, we have

$$\|x\|+\|y\|+\|z\|-(\|x+y\|+\|x+z\|+\|y+z\|)+\|x+y+z\| \geqslant 0. \tag{1}$$

In the inclusion-exclusion principle and its dual, one has

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{\emptyset \neq J \subseteq \{1,2,\ldots,n\}} (-1)^{|J|-1} \left|\bigcap_{j\in J} A_j\right|, \quad \text{and}$$
$$\left|\bigcap_{i=1}^{n} A_i\right| = \sum_{\emptyset \neq J \subseteq \{1,2,\ldots,n\}} (-1)^{|J|-1} \left|\bigcup_{j\in J} A_j\right|. \tag{2}$$

If we formally replace the set A_j by vectors, the union $\bigcup$ by the addition $+$, and the cardinality by a norm, and let $n=3$, the right-hand side of (2) is the same as the left-hand side of (1).

For the purpose of this paper, we focus on real numbers, and thus it may be useful to study the version of n real numbers as follows.

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DEFINITION 1. For each $n \geqslant 2$ and $x_1, x_2, \dots, x_n \in \mathbb{R}$, define

$$f_n(x_1, x_2, \dots, x_n) = \sum_{S \subseteq \{1,2,\dots,n\}} (-1)^{|S|-1} \left| \sum_{i \in S} x_i \right|,$$

where the empty sum is defined to be zero. Equivalently,

$$\begin{aligned} f_n(x_1, x_2, \dots, x_n) = \sum_{1 \leqslant i \leqslant n} |x_i| - \sum_{1 \leqslant i_1 < i_2 \leqslant n} |x_{i_1} + x_{i_2}| + \sum_{1 \leqslant i_1 < i_2 < i_3 \leqslant n} |x_{i_1} + x_{i_2} + x_{i_3}| \\ - \cdots + (-1)^{n+1} |x_1 + x_2 + x_3 + \cdots + x_n|. \end{aligned}$$

One of our motivations comes from arithmetic and combinatorial sums defined by Jacobsthal [12], investigated more by Carlitz [4, 5] and Grimson [9], and recently generalized and studied further by Tverberg [27], Onphaeng and Pongsriiam [19], and Thanatipanonda and Wong [26], where the absolute value in Definition 1 is replaced by the floor function–a function that is very useful in combinatorics and number theory. For more information on the floor function, we refer the reader to the book by Graham, Knuth, and Patashnik [8], and the recent articles by Aursukaree et.al. [3], Kawsumarng et.al. [13], and Palatsang et.al. [20].

By the triangle inequality, we know that $f_2(x_1, x_2) \geqslant 0$ for every $x_1, x_2 \in \mathbb{R}$ and this is sharp since it becomes equality when $x_1 x_2 \geqslant 0$. In addition, a sharp lower bound for the case $n = 3$ is given as an exercise in the book by Manfrino, Ortega, and Delgado [14], and its generalization in an inner product space (but only for $n = 3$) is called Hlawka's inequality, which first appeared in a paper of Hornich [11] and extended in various ways by many mathematicians; see for example in [1, 2, 6, 7, 10, 15, 16, 18, 21, 22, 23, 24, 25]. Those extensions are perhaps from an analytic point of view while our sum in Definition 1 is motivated by Tverberg's sum [27], and so it is more combinatorial. Nevertheless, a generalization of Hlawka's inequality in a form similar to ours in the case $n = 4$ has recently been provided by Munteanu [17]. However, as far as we are aware, our results for the case $n \geqslant 5$ are new.

In this article, we give a sharp lower bound for the case $n \in \{4, 5\}$ and a lower bound, which we think, is close to being sharp for the case $n = 6$. In addition, a sum similar to f_n but focusing on a single point y_0 comes up often in our calculation. The definition of that sum is as follows.

DEFINITION 2. For $n \geqslant 2$ and $y_0, x_1, x_2, \dots, x_n \in \mathbb{R}$, define

$$g_n(y_0, x_1, x_2, \dots, x_n) = \sum_{S \subseteq \{1,2,\dots,n\}} (-1)^{|S|} \left| y_0 + \sum_{i \in S} x_i \right|,$$

or equivalently,

$$\begin{aligned} g_n(y_0, x_1, x_2, \dots, x_n) = |y_0| - \sum_{1 \leqslant i \leqslant n} |y_0 + x_i| + \sum_{1 \leqslant i_1 < i_2 \leqslant n} |y_0 + x_{i_1} + x_{i_2}| \\ - \sum_{1 \leqslant i_1 < i_2 < i_3 \leqslant n} |y_0 + x_{i_1} + x_{i_2} + x_{i_3}| + \cdots + (-1)^n |y_0 + x_1 + x_2 + \cdots + x_n|. \end{aligned}$$

We also give sharp lower bounds for g_n when $n=4,5$. Finally, we remark that our results can be written in a notation similar to that in an inner product space as follows. For $\boldsymbol{x}=(x_1,x_2,\dots,x_n)\in\mathbb{R}^n$, let $\|\boldsymbol{x}\|_\infty$ be defined by

$$\|\boldsymbol{x}\|_\infty=\max\{|x_i| : i=1,2,3,\dots,n\}.$$

Then the inequalities in Theorems 1, 5, and 8 may be rewritten as $f_4(\boldsymbol{x})\geqslant-\|\boldsymbol{x}\|_\infty$ for all $\boldsymbol{x}\in\mathbb{R}^4$, $f_5(\boldsymbol{x})\geqslant-2\|\boldsymbol{x}\|_\infty$ for all $\boldsymbol{x}\in\mathbb{R}^5$, and $f_6(\boldsymbol{x})\geqslant-5\|\boldsymbol{x}\|_\infty$ for all $\boldsymbol{x}\in\mathbb{R}^6$. Furthermore, if we let

$$A_+^n=\{\boldsymbol{x}=(x_1,x_2,\dots,x_n)\in\mathbb{R}^n \mid \ x_i>0 \text{ for some } i \text{ and } \|\boldsymbol{x}\|_\infty\leqslant 1\},$$

then Theorems 3 and 6 may be written as $g_4(1,\boldsymbol{x})\geqslant-\|\boldsymbol{x}\|_\infty$ for all $\boldsymbol{x}\in A_+^4$ and $g_5(1,\boldsymbol{x})\geqslant-3\|\boldsymbol{x}\|_\infty$ for all $\boldsymbol{x}\in A_+^5$. In general, it may be interesting to determine appropriate constants $c_n,d_n\in\mathbb{R}$, that depend at most on n and are best possible such that $f_n(\boldsymbol{x})\geqslant c_n\|\boldsymbol{x}\|_\infty$ for all $\boldsymbol{x}\in\mathbb{R}^n$ and $g_n(1,\boldsymbol{x})\geqslant d_n\|\boldsymbol{x}\|_\infty$ for all $\boldsymbol{x}\in A_+^n$ (or for all $\boldsymbol{x}\in\mathbb{R}^n$). The calculation for sharp lower bounds of f_n and g_n when n is large seems much more complicated, so we postpone it for future research.

2. Preliminaries and lemmas

In this section, we give some auxiliary results which will be used in the proof of main theorems. Throughout this article, n is a positive integer larger than 1.

LEMMA 1. *If $x_i=0$ for some $i\in\{1,2,\dots,n\}$, then $f_n(x_1,x_2,\dots,x_n)=0$.*

Proof. Observe that $f_n(x_1,x_2,\dots,x_n)=f_n\left(x_{\sigma(1)},x_{\sigma(2)},\dots,x_{\sigma(n)}\right)$ for any permutation σ of $1,2,\dots,n$. So for convenience, we can suppose $x_n=0$. For $1<k<n$, we write

$$\begin{aligned}&\sum_{1\leqslant i_1<i_2<\cdots<i_k\leqslant n}\left|x_{i_1}+x_{i_2}+\cdots+x_{i_k}\right|\\ &=\sum_{1\leqslant i_1<i_2<\cdots<i_{k-1}<n}\left|x_{i_1}+x_{i_2}+\cdots+x_{i_{k-1}}\right|+\sum_{1\leqslant i_1<i_2<\cdots<i_k<n}\left|x_{i_1}+x_{i_2}+\cdots+x_{i_k}\right|.\end{aligned}$$

Then $f_n(x_1,x_2,\dots,x_n)$ is equal to

$$\begin{aligned}&\left(\sum_{1\leqslant i<n}|x_i|\right)-\left(\sum_{1\leqslant i<n}|x_i|+\sum_{1\leqslant i_1<i_2<n}|x_{i_1}+x_{i_2}|\right)\\ &\qquad+\left(\sum_{1\leqslant i_1<i_2<n}|x_{i_1}+x_{i_2}|+\sum_{1\leqslant i_1<i_2<i_3<n}|x_{i_1}+x_{i_2}+x_{i_3}|\right)-\cdots\\ &\qquad+(-1)^{n+1}|x_1+x_2+\cdots+x_n|,\end{aligned}$$

which is a telescopic sum. We see that $f_n(x_1,x_2,\dots,x_n)=0$, as required. $\square$

LEMMA 2. *If $x_i \geqslant 0$ for every i or $x_i \leqslant 0$ for every i, then $f_n(x_1,x_2,\ldots,x_n)=0$.*

Proof. Suppose $x_i \geqslant 0$ for every i. Then $f_n(x_1,x_2,\ldots,x_n)$ is equal to

$$\begin{aligned}
&\sum_{i=1}^{n} x_i - \binom{n-1}{1}\sum_{i=1}^{n} x_i + \binom{n-1}{2}\sum_{i=1}^{n} x_i - \binom{n-1}{3}\sum_{i=1}^{n} x_i \\
&\quad + \cdots + (-1)^{n-1}\binom{n-1}{n-1}\sum_{i=1}^{n} x_i \\
&= \left(\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\right)\left(\sum_{i=1}^{n} x_i\right) = 0.
\end{aligned}$$

Since $f_n(x_1,x_2,\ldots,x_n) = f_n(-x_1,-x_2,\ldots,-x_n)$, the result also holds if $x_i \leqslant 0$ for all i. □

We need to improve the triangle inequality $f_2(x_1,x_2) \geqslant 0$ a bit further as follows.

LEMMA 3. *For $x,y \in \mathbb{R}$, we have*

$$f_2(x,y) = \begin{cases} 0, & \text{if } xy \geqslant 0; \\ 2\min\{|x|,|y|\}, & \text{if } xy < 0. \end{cases}$$

Proof. If $xy \geqslant 0$, then $|x+y|^2 = x^2+2xy+y^2 = |x|^2+2|x||y|+|y|^2 = (|x|+|y|)^2$, and so $|x+y| = |x|+|y|$, which implies $f_2(x,y)=0$. Suppose $xy<0$. Without loss of generality, we can assume that $x>0$ and $y<0$. If $x+y \geqslant 0$, then $|x| \geqslant |y|$ and

$$f_2(x,y) = x-y-(x+y) = -2y = 2|y| = 2\min\{|x|,|y|\}.$$

If $x+y<0$, then $|x|<|y|$ and

$$f_2(x,y) = x-y+x+y = 2x = 2|x| = 2\min\{|x|,|y|\}.$$

This completes the proof. □

Although Lemma 4 is easy, it is useful in our calculation related to f_2.

LEMMA 4. *Let $a,b,c \in \mathbb{R}$ and $a \geqslant b$. Then*

$$\min\{a,c\} - \min\{b,c\} \geqslant 0 \tag{3}$$

Proof. If $c \leqslant b$, then the left-hand side of (3) is $c-c=0$. If $b<c<a$, then it is $c-b>0$. If $c \geqslant a$, then it is $a-b \geqslant 0$. This completes the proof. □

The inequality $f_3(x_1,x_2,x_3) \geqslant 0$ is already given as an exercise [14] but we give a proof for completeness as in Lemma 5.

LEMMA 5. *For every* $x_1,x_2,x_3 \in \mathbb{R}$, $f_3(x_1,x_2,x_3) \geqslant 0$. *If* $x_i = 0$ *for some* $i = 1,2,3$, *then the inequality becomes equality.*

Proof. If $x_i = 0$ for some $i = 1,2,3$, then the result follows from Lemma 1. Without loss of generality, we can assume that $|x_1| \geqslant |x_2| \geqslant |x_3| > 0$. Let $b = \frac{x_2}{x_1}$ and $c = \frac{x_3}{x_1}$. Then

$$\frac{1}{|x_1|} f(x_1,x_2,x_3) = 1 + |b| + |c| - (|1+b| + |1+c| + |b+c|) + |1+b+c|. \tag{4}$$

Since $|b|,|c| \leqslant 1$, we have $1+b \geqslant 0$ and $1+c \geqslant 0$, so $|1+b| = 1+b$, $|1+c| = 1+c$. Therefore the right-hand side of (4) is equal to $f_2(b,c) + |1+b+c| - (1+b+c) \geqslant 0$, which implies the desired result. $\square$

The calculation for the inequality $f_4(x_1,x_2,x_3,x_4) \geqslant 0$ can be reduced to the consideration of the following sum.

LEMMA 6. *Let* $b,c,d \in \mathbb{R}$ *and* $|b|,|c|,|d| \leqslant 1$. *Let*

$$A = |1+b+c| + |1+b+d| + |1+c+d| - |1+b+c+d| - (2+b+c+d).$$

If $b \geqslant 0$, *or* $c \geqslant 0$, *or* $d \geqslant 0$, *then* $A \geqslant 0$.

Proof. The permutation of b,c,d does not change A. So for convenience, we can suppose that $d \geqslant 0$. Since $|b|,|c| \leqslant 1$, we have $1+b+d \geqslant 0$ and $1+c+d \geqslant 0$. Therefore, A is equal to $f_2(1+b+c,d)$, which is nonnegative as desired. $\square$

Sums similar to A in Lemma 6 arise often in our calculation. So we study the function g as defined earlier in the introduction.

LEMMA 7. *If* $x_i = 0$ *for some* $i = 1,2,\ldots,n$, *then* $g_n(1,x_1,x_2,\ldots,x_n) = 0$.

Proof. Since $g_n(1,x_1,x_2,\ldots,x_n)$ is invariant under the permutation of $x_1,x_2,\ldots,x_n$, we can assume that $x_n = 0$. Then, for $1 < k < n$, we can write

$$\begin{aligned}\sum_{1\leqslant i_1<i_2<\cdots<i_k\leqslant n} |y_0 + x_{i_1} + x_{i_2} + \cdots + x_{i_k}| = &\sum_{1\leqslant i_1<i_2<\cdots<i_{k-1}<n} |y_0 + x_{i_1} + x_{i_2} + \cdots + x_{i_{k-1}}| \\ &+ \sum_{1\leqslant i_1<i_2<\cdots<i_k<n} |y_0 + x_{i_1} + x_{i_2} + \cdots + x_{i_k}|.\end{aligned}$$

Similar to the proof of Lemma 1, $g_n(1,x_1,x_2,\ldots,x_n)$ can be written as a telescopic sum which is equal to zero, as desired. $\square$

LEMMA 8. *Let* $n \geqslant 3$, $y_0,x_1,x_2,\ldots,x_n \in \mathbb{R}$, *and* $y_0 + x_n \geqslant 0$. *Then*

$$g_n(y_0,x_1,x_2,\ldots,x_n) = g_{n-1}(y_0,x_1,x_2,\ldots,x_{n-1}) - g_{n-1}(y_0+x_n,x_1,x_2,\ldots,x_{n-1}).$$

Proof. For $1 < k < n$, we write

$$\begin{aligned}
\sum_{1\leqslant i_1<i_2<\cdots<i_k\leqslant n} & |y_0+x_{i_1}+x_{i_2}+\cdots+x_{i_k}| \\
&= \sum_{1\leqslant i_1<i_2<\cdots<i_k<n} |y_0+x_{i_1}+x_{i_2}+\cdots+x_{i_k}| \\
&\quad + \sum_{1\leqslant i_1<i_2<\cdots<i_{k-1}<n} |y_0+x_{i_1}+x_{i_2}+\cdots+x_{i_{k-1}}+x_n| \\
&= \sum_{1\leqslant i_1<i_2<\cdots<i_k<n} |y_0+x_{i_1}+x_{i_2}+\cdots+x_{i_k}| \\
&\quad + \sum_{1\leqslant i_1<i_2<\cdots<i_{k-1}<n} |(y_0+x_n)+x_{i_1}+x_{i_2}+\cdots+x_{i_{k-1}}|.
\end{aligned}$$

When $k=1$, we write

$$\sum_{1\leqslant i\leqslant n} |y_0+x_i| = \left(\sum_{1\leqslant i<n} |y_0+x_i|\right) + |y_0+x_n|.$$

From this, it is not difficult to see that the result holds. $\square$

3. Main results

We begin with a sharp lower bound for $f_4(x_1,x_2,x_3,x_4)$.

THEOREM 1. *For every* $x_1,x_2,x_3,x_4\in\mathbb{R}$,

$$f_4(x_1,x_2,x_3,x_4) \geqslant -\max\{|x_1|,|x_2|,|x_3|,|x_4|\}. \tag{5}$$

Furthermore, the lower bound is sharp in the sense that there are infinitely many x_1, x_2, x_3, x_4 *such that the inequality becomes equality. In particular, if* $x_1\in\mathbb{R}$ *is arbitrary and* $x_2=x_3=x_4=-\frac{x_1}{2}$, *then*

$$f_4(x_1,x_2,x_3,x_4) = -\max\{|x_1|,|x_2|,|x_3|,|x_4|\}.$$

Proof. Let $x_1,x_2,x_3,x_4\in\mathbb{R}$. Without loss of generality, we can assume that $|x_1|\geqslant|x_2|\geqslant|x_3|\geqslant|x_4|$. If $x_4=0$, then (5) follows immediately from Lemma 1. So assume that $|x_4|>0$. Dividing both sides of (5) by $|x_1|$ and replacing $\frac{x_2}{x_1}$ by b, $\frac{x_3}{x_1}$ by c and $\frac{x_4}{x_1}$ by d, we see that it is enough to prove $f_4(1,b,c,d)\geqslant-1$, where $1\geqslant|b|\geqslant|c|\geqslant|d|>0$.

Since $-1\leqslant b,c,d\leqslant 1$, we have $1+b,1+c,1+d\geqslant 0$, and so $|1+b|=1+b$, $|1+c|=1+c$, and $|1+d|=1+d$. Therefore, $f_4(1,b,c,d)$ is equal to

$$\begin{aligned}
&1+|b|+|c|+|d|-(|1+b|+|1+c|+|1+d|+|b+c|+|b+d|+|c+d|) \\
&\quad +|1+b+c|+|1+b+d|+|1+c+d|+|b+c+d|-|1+b+c+d| \\
&= f_3(b,c,d)+A,
\end{aligned} \tag{6}$$

where A is the sum in Lemma 6. If $b \geqslant 0$, or $c \geqslant 0$, or $d \geqslant 0$, then (6), Lemma 5, and Lemma 6 imply

$$f_4(1,b,c,d) = f_3(b,c,d) + A \geqslant 0.$$

So assume that $b < 0$, $c < 0$, and $d < 0$. Then by Lemma 2, $f_3(b,c,d) = 0$, and therefore $f_4(1,b,c,d) = A$. If $1+b+c+d \geqslant 0$, then $1+b+c$, $1+b+d$, $1+c+d \geqslant 0$ and $f_4(1,b,c,d) = A = 0$. So assume that $1+b+c+d < 0$. Then

$$f_4(1,b,c,d) = |1+b+c| + |1+b+d| + |1+c+d| - 1 \geqslant -1,$$

as required. It is straightforward to verify that (5) becomes equality if $x_1 \in \mathbb{R}$ is arbitrary and $x_2 = x_3 = x_4 = -x_1/2$. Hence the proof is complete. $\square$

It is interesting to observe that the condition $x_2 = x_3 = x_4 = -\frac{x_1}{2}$, where $x_1 \in \mathbb{R}$ is arbitrary, is also a necessary condition, under a natural restriction on permutations, for (5) to becomes equality. More precisely, we have the following theorem.

THEOREM 2. *Let $x_1,x_2,x_3,x_4 \in \mathbb{R}$. Then*

$$f_4(x_1,x_2,x_3,x_4) = -\max\{|x_1|,|x_2|,|x_3|,|x_4|\} \tag{7}$$

if and only if (x_1,x_2,x_3,x_4) is a permutation of 4-tuples of the form $\left(y,-\frac{y}{2},-\frac{y}{2},-\frac{y}{2}\right)$ where $y \in \mathbb{R}$ is arbitrary.

Proof. We follow closely the proof of Theorem 1. Assume that (7) holds. If $x_i = 0$ for some i, then Lemma 1 and (7) imply that $x_i = 0$ for all i, so we can choose $y = 0$. Suppose that $x_i \neq 0$ for any i. Without loss of generality, we can assume that $|x_1| \geqslant |x_2| \geqslant |x_3| \geqslant |x_4| > 0$. Dividing both sides of (7) by $|x_1|$, letting $b = \frac{x_2}{x_1}$, $c = \frac{x_3}{x_1}$, $d = \frac{x_4}{x_1}$, and following the calculation in the proof of Theorem 1, we obtain

$$f_3(b,c,d) + A = -1, \tag{8}$$

where A is the sum in Lemma 6. Again by the proof of Theorem 1, we see that $b < 0$, $c < 0$, and $d < 0$; otherwise the left-hand side of (8) is nonnegative, which is not the case. By Lemma 2, we see that (8) reduces to $A = -1$. Again by the proof of Theorem 1, if $1+b+c+d \geqslant 0$, then $A = 0$, which is not the case. So $1+b+c+d < 0$. Therefore

$$A = |1+b+c| + |1+b+d| + |1+c+d| - 1.$$

Thus

$$|1+b+c| + |1+b+d| + |1+c+d| = 0$$

Therefore $1+b+c = 1+b+d = 1+c+d = 0$. Solving for b,c,d, we obtain $b = c = d = -1/2$. So $x_2 = -\frac{x_1}{2}$, $x_3 = -\frac{x_1}{2}$, $x_4 = -\frac{x_1}{2}$. Therefore (x_1,x_2,x_3,x_4) can be written as $\left(y,-\frac{y}{2},-\frac{y}{2},-\frac{y}{2}\right)$ where $y = x_1$. The converse is easy to check. So the proof is complete. $\square$

REMARK 1. By considering the proof of Lemma 5 again, we notice that

$$f_3(x_1,x_2,x_3)=0 \text{ if and only if}$$
$$(x_1,x_2,x_3) \text{ is a permutation of 3-tuples of the form } (0,y,z) \text{ or } (y,by,cy)$$

where $y,z\in\mathbb{R}$ are arbitrary and $b,c\in\mathbb{R}$ satisfy $|b|,|c|\leqslant 1$, $bc\geqslant 0$, and $b+c\geqslant -1$. The idea is to follow the proof of Lemma 5. Then at the end (or almost the end), we have $f_2(b,c)+|1+b+c|-(1+b+c)=0$, which implies $f_2(b,c)=0$ and $|1+b+c|=1+b+c$. This implies $bc\geqslant 0$ and $1+b+c\geqslant 0$, and the proof is complete.

In order to obtain a sharp lower bound for f_5, we first give a sharp lower bound for g_4 as follows.

THEOREM 3. *If $|x_1|,|x_2|,|x_3|,|x_4|\in[0,1]$ and $x_i>0$ for some $i=1,2,3,4$, then*

$$g_4(1,x_1,x_2,x_3,x_4)\geqslant -1. \tag{9}$$

In addition, if $\frac{1}{2}\leqslant x_4\leqslant 1$ is arbitrary and $x_1,x_2,x_3=-\frac{1}{2}$, then (9) becomes equality.

Proof. For convenience, we sometimes write g_4 instead of $g_4(1,x_1,x_2,x_3,x_4)$. Since g_4 is invariant under the permutation of x_1, x_2, x_3, x_4, we can suppose that $x_4>0$. Since $|x_i|\leqslant 1$, we obtain $1+x_i\geqslant 0$ for every i. In addition, $1+x_i+x_4\geqslant 0$ for every $i=1,2,3$. Therefore g_4 is equal to

$$\begin{aligned}2x_4+\sum_{1\leqslant i<j\leqslant 3}|1+x_i+x_j|-\sum_{1\leqslant i<j<k\leqslant 4}|1+x_i+x_j+x_k|+|1+x_1+x_2+x_3+x_4|\\ =f_2(1+x_1+x_2,x_4)+f_2(1+x_1+x_3,x_4)\\ +f_2(1+x_2+x_3,x_4)-f_2(1+x_1+x_2+x_3,x_4).\end{aligned} \tag{10}$$

By Lemma 3, if $1+x_1+x_2+x_3\geqslant 0$, then $f_2(1+x_1+x_2+x_3,x_4)=0$, and so $g_4\geqslant 0$. Therefore we assume throughout that $1+x_1+x_2+x_3<0$. For convenience, let

$$A=\{(i,j)\mid 1\leqslant i<j\leqslant 3 \text{ and } 1+x_i+x_j<0\}$$
$$B=\{(i,j)\mid 1\leqslant i<j\leqslant 3 \text{ and } 1+x_i+x_j\geqslant 0\}.$$

Case 1 A is empty, that is, $1+x_i+x_j\geqslant 0$ for every $1\leqslant i<j\leqslant 3$. Adding $1+x_1+x_2$, $1+x_1+x_3$, and $1+x_2+x_3$, we obtain $3+2(x_1+x_2+x_3)\geqslant 0$, and so $1+x_1+x_2+x_3\geqslant -\frac{1}{2}$. Therefore $|1+x_1+x_2+x_3|\leqslant\frac{1}{2}$. By Lemma 3,

$$g_4=-f_2(1+x_1+x_2+x_3,x_4)=-2\min\{|1+x_1+x_2+x_3|,|x_4|\}\geqslant -1,$$

as desired. From this point on, we apply Lemma 3 without reference.

Case 2 A is not empty, that is, there exists at least one pair (i,j) such that $1\leqslant i<j\leqslant 3$ and $1+x_i+x_j<0$. Then

$$\begin{aligned}g_4(1,x_1,x_2,x_3,x_4)&=\sum_{(i,j)\in A}f_2(1+x_i+x_j,x_4)-f_2(1+x_1+x_2+x_3,x_4)\\ &=2\sum_{(i,j)\in A}\min\{|1+x_i+x_j|,|x_4|\}-2\min\{|1+x_1+x_2+x_3|,|x_4|\}.\end{aligned}$$

If $|x_4| \leqslant |1+x_i+x_j|$ for some $(i,j) \in A$, then

$$g_4(1,x_1,x_2,x_3,x_4) \geqslant 2|x_4| - 2\min\{|1+x_1+x_2+x_3|,|x_4|\} \geqslant 0.$$

So suppose that $|x_4| > |1+x_i+x_j|$ for all $(i,j) \in A$. Then

$$g_4(1,x_1,x_2,x_3,x_4) = 2\sum_{(i,j)\in A} |1+x_i+x_j| - 2\min\{|1+x_1+x_2+x_3|,|x_4|\}.$$

Observe that

$$\sum_{(i,j)\in A}(1+x_i+x_j) + \sum_{(i,j)\in B}(1+x_i+x_j) = \sum_{1\leqslant i<j\leqslant 3}(1+x_i+x_j) = 1+2(1+x_1+x_2+x_3).$$

This observation is used in Case 2.1 and Case 2.2 as follows.

Case 2.1 $|1+x_1+x_2+x_3| \leqslant |x_4|$. Then

$$\begin{aligned} g_4 &= 2\sum_{(i,j)\in A} |1+x_i+x_j| - 2|1+x_1+x_2+x_3| \\ &= -2\sum_{(i,j)\in A}(1+x_i+x_j) + 2(1+x_1+x_2+x_3) \\ &= \sum_{(i,j)\in B}(1+x_i+x_j) - \sum_{(i,j)\in A}(1+x_i+x_j) - 1 \geqslant -1. \end{aligned}$$

Case 2.2 $|1+x_1+x_2+x_3| > |x_4|$. Then $1+x_1+x_2+x_3+x_4 < 0$ and so g_4 is

$$\begin{aligned} &2\sum_{(i,j)\in A} |1+x_i+x_j| - 2|x_4| = -2\sum_{(i,j)\in A}(1+x_i+x_j) - 2x_4 \\ &= -2\sum_{(i,j)\in A}(1+x_i+x_j) + 2(1+x_1+x_2+x_3) - 2(1+x_1+x_2+x_3+x_4) \\ &= \sum_{(i,j)\in B}(1+x_i+x_j) - \sum_{(i,j)\in A}(1+x_i+x_j) - 1 - 2(1+x_1+x_2+x_3+x_4), \end{aligned}$$

which is larger than -1.

In any case, $g_4 \geqslant -1$. The second part is straightforward. So the proof is complete. $\square$

THEOREM 4. *If* $|x_1|,|x_2|,|x_3|,|x_4| \in [0,1], x_i \leqslant 0$ *for every* $i = 1,2,3,4$ *and* $1 + x_1+x_2+x_3+x_4 < 0$, *then*

$$g_4(1,x_1,x_2,x_3,x_4) \geqslant -2.$$

In addition, if $x_1 = x_2 = x_3 = x_4 = -\frac{1}{2}$, *then* $g_4(1,x_1,x_2,x_3,x_4) = -2$

Proof. Since $1+x_1+x_2+x_3+x_4<0$ and $1+x_i \geqslant 0$ for every i, $g_4(1,x_1,x_2,x_3,x_4)$ is equal to

$$\begin{aligned}
&-4+2\sum_{1\leqslant i\leqslant 4}|x_i|+\sum_{1\leqslant i<j\leqslant 4}|1+x_i+x_j|-\sum_{1\leqslant i<j<k\leqslant 4}|1+x_i+x_j+x_k|\\
&\quad = f_2(1+x_1+x_2,x_3)+f_2(1+x_2+x_3,x_4)+f_2(1+x_3+x_4,x_1)\\
&\qquad + f_2(1+x_1+x_4,x_2)+A_2,
\end{aligned}$$

where

$$A_2 = |x_1|+|x_2|+|x_3|+|x_4|+|1+x_1+x_3|+|1+x_2+x_4|-4.$$

By the triangle inequality, $g_4(1,x_1,x_2,x_3,x_4) \geqslant A_2$ and

$$A_2 \geqslant |x_1+x_2+x_3+x_4|+|1+x_1+x_3+1+x_2+x_4|-4 = |A|+|A+2|-4, \tag{11}$$

where $A = x_1+x_2+x_3+x_4$. Recall that the sum of the form

$$|b-y_1|+|b-y_2|+\cdots+|b-y_n|$$

is minimal if b is the median of $y_1,y_2,\ldots,y_n$. (If n is even, we can take b to be any number lying between the two middle points.) In our case, $|A|+|A+2|$ is minimal when $A\in[-2,0]$, and the minimum value is 2 which occurs, for example, when $x_1 = x_2 = x_3 = x_4 = -\frac{1}{2}$. Hence, we obtain from (11) that $A_2 \geqslant -2$, and so

$$g_4(1,x_1,x_2,x_3,x_4) \geqslant -2$$

as desired. $\square$

We now give a sharp lower bound for f_5 as follows.

THEOREM 5. *For every $x_1,x_2,x_3,x_4,x_5 \in \mathbb{R}$,*

$$f_5(x_1,x_2,x_3,x_4,x_5) \geqslant -2\max\{|x_i| : i=1,2,\ldots,5\}.$$

In particular, if $x_1\in\mathbb{R}$ is arbitrary and $x_2=x_3=x_4=x_5=-\frac{x_1}{2}$, then

$$f_5(x_1,x_2,x_3,x_4,x_5) = -2\max\{|x_i| : i=1,2,\ldots,5\}.$$

Proof. It is straightforward to verify that if $x_2=x_3=x_4=x_5=-\frac{x_1}{2}$, then the above equality holds. So it remains to prove the inequality. Similar to the proof of Theorem 1, we can assume that $|x_1|\geqslant|x_2|\geqslant\cdots\geqslant|x_5|>0$, and after dividing by $|x_1|$ and changing variables, it is enough to show that

$$f_5(1,x_1,x_2,x_3,x_4) \geqslant -2,$$

where $1\geqslant|x_1|\geqslant|x_2|\geqslant|x_3|\geqslant|x_4|>0$. In addition, we can write $f_5(1,x_1,x_2,x_3,x_4)$ as

$$f_5(1,x_1,x_2,x_3,x_4) = f_4(x_1,x_2,x_3,x_4)+g_4(1,x_1,x_2,x_3,x_4). \tag{12}$$

Case 1 $x_i > 0$ for some $i = 1,2,3,4$. Then by Theorem 3, $g_4(1,x_1,x_2,x_3,x_4) \geqslant -1$. In addition, $f_4(x_1,x_2,x_3,x_4) \geqslant -1$ by Theorem 1. Therefore, (12) implies

$$f_5(x_1,x_2,x_3,x_4,x_5) \geqslant -2. \tag{13}$$

Case 2 $x_i < 0$ for every $i = 1,2,3,4$. By Lemma 2, $f_4(x_1,x_2,x_3,x_4) = 0$, and so

$$f_5(x_1,x_2,x_3,x_4,x_5) = g_4(1,x_1,x_2,x_3,x_4). \tag{14}$$

If $1+x_1+x_2+x_3+x_4 \geqslant 0$, then every term in the absolute value in the sum defining $g_4(1,x_1,x_2,x_3,x_4)$ is nonnegative, which leads to $g_4(1,x_1,x_2,x_3,x_4) = 0$, and we are done. So suppose that $1+x_1+x_2+x_3+x_4 < 0$, then we apply Theorem 4 to obtain

$$f_5(x_1,x_2,x_3,x_4,x_5) = g_4(1,x_1,x_2,x_3,x_4) \geqslant -2.$$

This completes the proof. $\square$

We now have sharp lower bounds for f_4 and f_5. Next, we give a lower bound for g_5 and use it to obtain a lower bound for f_6

THEOREM 6. *If* $|x_1|,|x_2|,|x_3|,|x_4|,|x_5| \in [0,1]$ *and* $x_i > 0$ *for some* $i = 1,2,3,4,5$, *then*

$$g_5(1,x_1,x_2,x_3,x_4,x_5) \geqslant -3.$$

In addition, if $x_1 = x_2 = x_3 = x_4 = -\frac{1}{2}$ *and* $x_5 = \frac{1}{2}$, *then the above inequality becomes equality.*

Proof. For convenience, we write g_5 instead of $g_5(1,x_1,x_2,x_3,x_4,x_5)$. Similar to the proof of Theorem 3, we can suppose that $x_5 > 0$, obtain $1+x_i \geqslant 0$ for every $i \leqslant 5$ and $1+x_i+x_5 \geqslant 0$ for every $i \leqslant 4$. Therefore

$$\begin{aligned} g_5 &= 3x_5 + \sum_{1\leqslant i_1<i_2\leqslant 4} |1+x_{i_1}+x_{i_2}| - \sum_{1\leqslant i_1<i_2<i_3\leqslant 5} |1+x_{i_1}+x_{i_2}+x_{i_3}| \\ &\quad + \sum_{1\leqslant i_1<i_2<i_3<i_4\leqslant 5} |1+x_{i_1}+x_{i_2}+x_{i_3}+x_{i_4}| - |1+x_1+x_2+x_3+x_4+x_5| \\ &= x - y + z, \text{ where} \end{aligned}$$

$$x = \sum_{1\leqslant i<j\leqslant 4} f_2(1+x_i+x_j,x_5), \quad y = \sum_{1\leqslant i<j<k\leqslant 4} f_2(1+x_i+x_j+x_k,x_5),$$

and

$$z = f_2(1+x_1+x_2+x_3+x_4,x_5).$$

For convenience, let

$$A = \left\{(i,j,k) \mid 1 \leqslant i < j < k \leqslant 4 \text{ and } 1+x_i+x_j+x_k < 0\right\}.$$

Similar to Theorem 3, we use Lemma 3 throughout the proof without reference. If $A = \varnothing$, then $y = 0$ and so $g_5 = x + z \geqslant 0$. If $|A| = 1$, say $(i, j, k) \in A$, then

$$g_5 \geqslant -y = -2\min\{|1 + x_i + x_j + x_k|, |x_5|\} \geqslant -2|x_5| \geqslant -2.$$

So it remains to consider the case $2 \leqslant |A| \leqslant 4$.

Case 1 $|A| = 2$. Let (i, j, k) and (i_2, j_2, k_2) be the two elements of A. Then

$$y \leqslant f_2(1 + x_i + x_j + x_k, x_5) + 2|x_5| \leqslant f_2(1 + x_i + x_j + x_k, x_5) + 2.$$

In addition, we observe that $g_4(1, x_i, x_j, x_k, x_5)$ is equal to

$$f_2(1 + x_i + x_j, x_5) + f_2(1 + x_i + x_k, x_5) + f_2(1 + x_j + x_k, x_5) - f_2(1 + x_i + x_j + x_k, x_5). \tag{15}$$

Therefore

$$g_5 = x - y + z \geqslant x - y \geqslant g_4(1, x_i, x_j, x_k, x_5) - 2 \geqslant -3.$$

Case 2 $|A| = 3$. Without loss of generality, we can assume that $(1, 2, 3) \notin A$. Then $1 + x_1 + x_2 + x_3 \geqslant 0$, $f_2(1 + x_1 + x_2 + x_3, x_5) = 0$, and $1 + x_i + x_j + x_k < 0$ for $(i, j, k) \neq (1, 2, 3)$.

Case 2.1 $x_1 \geqslant 0$. Then $1 + x_2 + x_4 \leqslant 1 + x_1 + x_2 + x_4 < 0$, and so $|1 + x_2 + x_4| \geqslant |1 + x_1 + x_2 + x_4|$. By Lemma 4, we obtain

$$f_2(1 + x_2 + x_4, x_5) - f_2(1 + x_1 + x_2 + x_4, x_5) \geqslant 0.$$

Similarly, since $1 + x_3 + x_4 \leqslant 1 + x_1 + x_3 + x_4 < 0$, we obtain

$$f_2(1 + x_3 + x_4, x_5) - f_2(1 + x_1 + x_3 + x_4, x_5) \geqslant 0.$$

Therefore

$$g_5 \geqslant -f_2(1 + x_2 + x_3 + x_4, x_5) \geqslant -2|x_5| \geqslant -2.$$

Case 2.2 $x_1 < 0$. Then $1 + x_1 + x_2 + x_3 + x_4 < 1 + x_2 + x_3 + x_4 < 0$ and so

$$|1 + x_1 + x_2 + x_3 + x_4| > |1 + x_2 + x_3 + x_4|.$$

Lemma 4 implies that

$$f_2(1 + x_1 + x_2 + x_3 + x_4, x_5) - f_2(1 + x_2 + x_3 + x_4, x_5) \geqslant 0. \tag{16}$$

In addition,

$$-f_2(1 + x_1 + x_3 + x_4, x_5) \geqslant -2 \tag{17}$$

Furthermore, by a similar calculation as in (15), we have

$$\left(\sum_{\substack{i,j\in\{1,2,4\}\\ i<j}} f_2(1+x_i+x_j,x_5)\right) - f_2(1+x_1+x_2+x_4,x_5) = g_4(1,x_1,x_2,x_4,x_5) \geqslant -1 \tag{18}$$

Adding (16), (17), and (18), we see that $x-y+z \geqslant -3$, and thus $g_5 \geqslant -3$.

Case 3 $|A|=4$. We first suppose that $x_i \geqslant 0$ for some $i \leqslant 4$. Then the calculation is similar to Case 2.1. If $x_1 \geqslant 0$, then $1+x_i+x_j \leqslant 1+x_1+x_i+x_j < 0$ for each $2 \leqslant i < j \leqslant 4$, which implies $f_2(1+x_i+x_j,x_5) - f_2(1+x_1+x_i+x_j,x_5) \geqslant 0$ for such i,j, and therefore

$$g_5 \geqslant -f_2(1+x_2+x_3+x_4,x_5) \geqslant -2|x_5| \geqslant -2.$$

Similarly, if $x_j \geqslant 0$ for some $j \in \{2,3,4\}$, then

$$g_5 \geqslant -f_2(1+x_1+x_2+x_3+x_4-x_j,x_5) \geqslant -2|x_5| \geqslant -2.$$

So we assume throughout that $x_i < 0$ for all $i \leqslant 4$. Recall that $g_5 \geqslant x-y+z$ and since g_5,x,y,z are invariant under the permutation of x_1,x_2,x_3,x_4, we can assume that $x_1 \leqslant x_2 \leqslant x_3 \leqslant x_4 < 0$. Then $1+x_1+x_2+x_3+x_4 \leqslant 1+x_1+x_2+x_3 \leqslant 1+x_1+x_2+x_4 \leqslant 1+x_1+x_3+x_4 \leqslant 1+x_2+x_3+x_4 < 0$, where the last inequality is obtained from the fact that $(2,3,4) \in A$. Therefore

$$\begin{aligned}|1+x_1+x_2+x_3+x_4| \geqslant |1+x_1+x_2+x_3| \geqslant |1+x_1+x_2+x_4|\\ \geqslant |1+x_1+x_3+x_4| \geqslant |1+x_2+x_3+x_4|.\end{aligned} \tag{19}$$

Then

$$f_2(1+x_1+x_2+x_3+x_4,x_5) - f_2(1+x_1+x_2+x_3,x_5) \geqslant 0. \tag{20}$$

If $x_1 \geqslant -\frac{1}{2}$, then $|1+x_i+x_j+x_k| \leqslant \frac{1}{2}$ for every $1 \leqslant i < j < k \leqslant 4$, and thus

$$g_5 \geqslant -2 \sum_{\substack{1\leqslant i<j<k\leqslant 4\\ (i,j,k)\neq(1,2,3)}} \min\{|1+x_i+x_j+x_k|,|x_5|\} \geqslant -3.$$

Therefore we assume that $x_1 < -\frac{1}{2}$. Let $C = \{(i,j) \mid 1 \leqslant i < j \leqslant 4 \text{ and } 1+x_i+x_j < 0\}$.

Case 3.1 $C = \varnothing$. Since $x_1 < -\frac{1}{2}$ and $1+x_1+x_2 \geqslant 0$, we obtain $x_2 > -\frac{1}{2}$. Therefore $x_3,x_4 > -\frac{1}{2}$ and so $|1+x_2+x_3+x_4| < \frac{1}{2}$. Therefore

$$-f_2(1+x_2+x_3+x_4,x_5) \geqslant -1 \tag{21}$$

By a similar calculation as in (15), we have

$$\left(\sum_{\substack{i,j\in\{1,2,4\}\\ i<j}} f_2(1+x_i+x_j,x_5)\right) - f_2(1+x_1+x_2+x_4,x_5) \geqslant -1 \tag{22}$$

$$\left(\sum_{\substack{i,j\in\{1,3,4\}\\ i<j}} f_2(1+x_i+x_j,x_5)\right) - f_2(1+x_1+x_3+x_4,x_5) \geqslant -1 \tag{23}$$

Since $C=\varnothing$, we see that $1+x_2+x_3$ and $1+x_1+x_4$ are nonnegative, and so

$$f_2(1+x_2+x_3,x_5) = f_2(1+x_1+x_4,x_5) = 0.$$

So we can replace $f_2(1+x_1+x_4,x_5)$ by $f_2(1+x_2+x_3,x_5)$ in the sum on the left-hand side of (23). After that adding (20), (21), (22), (23) leads to $x-y+z \geqslant -3$. Therefore, $g_5 \geqslant -3$. Before proceeding to the next case, we first observe that

$$1+x_1+x_2 < 1+x_1+x_3 < \min\{1+x_1+x_4, 1+x_2+x_3\} < 1+x_2+x_4 < 1+x_3+x_4 \tag{24}$$

Case 3.2 $|C|\geqslant 2$. By (24), we know that $1+x_1+x_2 < 1+x_1+x_3 < 0$, and so

$$|1+x_1+x_3| < |1+x_1+x_2|.$$

Case 3.2.1 $x_5 \leqslant |1+x_1+x_3|$. Then $f_2(1+x_1+x_3,x_5) = 2x_5 \geqslant f_2(1+x_1+x_3+x_4,x_5)$. Therefore

$$f_2(1+x_1+x_3,x_5) - f_2(1+x_1+x_3+x_4,x_5) \geqslant 0. \tag{25}$$

Similarly,

$$f_2(1+x_1+x_2,x_5) - f_2(1+x_1+x_2+x_4,x_5) \geqslant 0. \tag{26}$$

Adding (20), (25), (26), we see that

$$g_5 \geqslant -f_2(1+x_2+x_3+x_4,x_5) \geqslant -2x_5 \geqslant -2.$$

Case 3.2.2 $|1+x_1+x_3| < x_5 < |1+x_1+x_2|$. Then (26) still holds. In addition, by a similar calculation as in (15), we have

$$\left(\sum_{\substack{i,j\in\{1,3,4\}\\ i<j}} f_2(1+x_i+x_j,x_5)\right) - f_2(1+x_1+x_3+x_4,x_5) \geqslant -1. \tag{27}$$

Furthermore

$$-f_2(1+x_2+x_3+x_4,x_5) \geqslant -2. \tag{28}$$

Adding (20), (26), (27), (28), we obtain $g_5 \geqslant -3$.

Case 3.2.3 $|x_5| \geqslant |1+x_1+x_3|$. Then $f_2(1+x_1+x_2,x_5) = 2|1+x_1+x_2|$ and

$$f_2(1+x_1+x_3,x_5) = 2|1+x_1+x_3|.$$

Suppose first that $x_4 \geqslant -\frac{1}{2}$. Then the left-hand side of (25) is larger than or equal to

$$\begin{aligned} 2|1+x_1+x_3| - 2|1+x_1+x_3+x_4| &= -2(1+x_1+x_3) + 2(1+x_1+x_3+x_4) \\ &= 2x_4 \geqslant -1 \end{aligned} \tag{29}$$

Similarly, the left-hand side of (26) is larger than or equal to

$$2|1+x_1+x_2| - 2|1+x_1+x_2+x_4| = 2x_4 \geqslant -1 \tag{30}$$

In addition, by a calculation similar to (15), we have

$$\left(\sum_{\substack{i,j\in\{2,3,4\} \\ i<j}} f_2(1+x_i+x_j,x_5) \right) - f_2(1+x_2+x_3+x_4,x_5) \geqslant -1. \tag{31}$$

Adding (20), (29), (30), and (31), we obtain $g_5 \geqslant -3$.

Hence it only remains to consider the case that $x_4 < -\frac{1}{2}$. Since $x_1 < x_2 < x_3 < x_4 < -\frac{1}{2}$, we see that $1+x_i+x_j < 0$ for each $1 \leqslant i < j \leqslant 4$. By (24) and the assumption $|x_5| \geqslant |1+x_1+x_2|$, we see that

$$x = 2 \sum_{1\leqslant i<j\leqslant 4} |1+x_i+x_j| = -12 - 6 \sum_{1\leqslant i\leqslant 4} x_i \tag{32}$$

Case 3.2.3.1 $x_5 \geqslant |1+x_i+x_j+x_k|$ for each $1 \leqslant i < j < k \leqslant 4$. Then

$$y = 2 \sum_{1\leqslant i<j<k\leqslant 4} |1+x_i+x_j+x_k| = -8 - 6 \sum_{1\leqslant i\leqslant 4} x_i \tag{33}$$

By (32) and (33), we obtain $x-y+z = z-4$. Since $x_i < -\frac{1}{2}$ for every i, we have

$$x_5 \geqslant |1+x_1+x_2+x_3| \geqslant \frac{1}{2} \text{ and } |1+x_1+x_2+x_3+x_4| \geqslant 1.$$

Therefore $z = 2\min\{|1+x_1+x_2+x_3+x_4|, x_5\} \geqslant 1$. Hence $g_5 = x-y+z = z-4 \geqslant -3$.

Case 3.2.3.2 $x_5 < |1+x_i+x_j+x_k|$ for some $1 \leqslant i < j < k \leqslant 4$. Let

$$D = \{(i,j,k) \mid 1 \leqslant i < j < k \leqslant 4 \text{ and } x_5 < |1+x_i+x_j+x_k|\}.$$

Then $1+x_i+x_j+x_k+x_5<0$ for all $(i,j,k)\in D$. We have

$$
\begin{aligned}
y &= 2\sum_{1\leqslant i<j<k\leqslant 4}|1+x_i+x_j+x_k| - 2\sum_{(i,j,k)\in D}|1+x_i+x_j+x_k| + 2\sum_{(i,j,k)\in D} x_5 \\
&= -2\left(4+3\sum_{1\leqslant i\leqslant 4} x_i\right) + 2\sum_{(i,j,k)\in D}(1+x_i+x_j+x_k+x_5) \qquad (34)\\
&\leqslant -8-6\sum_{1\leqslant i\leqslant 4} x_i \qquad (35)
\end{aligned}
$$

Therefore, if $x_5 \geqslant \frac{1}{2}$, then we obtain from (35) that $x-y+z \geqslant z-4 = 2x_5-4 \geqslant -3$. So assume that $x_5 < \frac{1}{2}$. This implies that D contains all (i,j,k) such that $1\leqslant i<j<k\leqslant 4$. Then we apply (34) to obtain

$$
\begin{aligned}
x-y+z &= -4-2\sum_{\substack{(i,j,k)\in D\\(i,j,k)\neq(1,2,3)}}(1+x_i+x_j+x_k+x_5) - 2(1+x_1+x_2+x_3)\\
&\geqslant -4-2(1+x_1+x_2+x_3) \geqslant -3.
\end{aligned}
$$

Case 3.3 $|C|=1$. That is, $1+x_1+x_2<0$ and $1+x_i+x_j\geqslant 0$ for $1\leqslant i<j\leqslant 4$ with $(i,j)\neq(1,2)$. If $x_5<|1+x_1+x_2|$, then by a similar reason as in Case 3.2, we see that (26), (27) and (28) still hold. Adding (20), (26), (27) and (28), we obtain $g_5\geqslant -3$. So assume that $x_5\geqslant |1+x_1+x_2|$. If $x_4<-\frac{1}{2}$, then $x_i<-\frac{1}{2}$ for all i, and so $1+x_i+x_j<0$ for each $1\leqslant i<j\leqslant 4$, which contradicts the assumption that $|C|=1$. Hence $x_4\geqslant -\frac{1}{2}$. Then $1+x_1+x_3+x_4\geqslant x_4\geqslant -\frac{1}{2}$ and so the left-hand side of (25) is

$$-f_2(1+x_1+x_3+x_4,x_5)\geqslant -2|1+x_1+x_3+x_4|\geqslant -1. \qquad (36)$$

As in (30), the left-hand side of (26) is

$$\geqslant 2|1+x_1+x_2|-2|1+x_1+x_2+x_4| = 2x_4\geqslant -1. \qquad (37)$$

Similarly,

$$-f_2(1+x_2+x_3+x_4,x_5)\geqslant -2|1+x_2+x_3+x_4|\geqslant -1. \qquad (38)$$

Adding (20), (36), (37), and (38), we obtain $g_5\geqslant -3$. This completes the proof. □

THEOREM 7. *If* $0\leqslant y\leqslant 2$, $|x_1|,|x_2|,|x_3|,|x_4|\in(0,1]$ *and* $x_i>0$ *for some* $i=1,2,3,4$, *then*

$$g_4(y,x_1,x_2,x_3,x_4)\leqslant 4.$$

Proof. For convenience, we write g_4 instead of $g_4(y,x_1,x_2,x_3,x_4)$. Since g_4 is invariant under the permutation of x_1,x_2,x_3,x_4, we can suppose that $x_4>0$. Then $x_4+y\geqslant 0$. After a straightforward manipulation, we obtain that

$$
\begin{aligned}
g_4 &= f_2(y+x_1+x_2,x_4)+f_2(y+x_1+x_3,x_4)+f_2(y+x_2+x_3,x_4)\\
&\quad - f_2(y+x_1+x_2+x_3,x_4) - f_2(y+x_1,x_4) - f_2(y+x_2,x_4) - f_2(y+x_3,x_4)\\
&\leqslant f_2(y+x_1+x_2,x_4)+f_2(y+x_1+x_3,x_4)+f_2(y+x_2+x_3,x_4)\\
&\quad - f_2(y+x_1+x_2+x_3,x_4).
\end{aligned}
$$

For convenience, let

$$A = \{(i,j) \mid 1 \leqslant i < j \leqslant 3 \text{ and } y + x_i + x_j < 0\}$$
$$B = \{(i,j) \mid 1 \leqslant i < j \leqslant 3 \text{ and } y + x_i + x_j \geqslant 0\}.$$

Observe that if $y+x_1+x_2<0$, $y+x_1+x_3<0$, and $y+x_2+x_3<0$, then combining all of them, we have $y+x_1+x_2+x_3<-y/2\leqslant 0$. Suppose first that $y+x_1+x_2+x_3\geqslant 0$. Then one of $y+x_1+x_2$, $y+x_1+x_3$, and $y+x_2+x_3$ is nonnegative. Then $|A|\leqslant 2$. By Lemma 3, we obtain $f_2(y+x_1+x_2+x_3,x_4)=0, f_2(y+x_i+x_j,x_4)=0$ for $(i,j)\in B$, and therefore

$$\begin{aligned} g_4 &\leqslant \sum_{(i,j)\in A} f_2\left(y+x_i+x_j,x_4\right) \\ &= 2\sum_{(i,j)\in A} \min\left\{\left|y+x_i+x_j\right|,|x_4|\right\} \leqslant 4|x_4| \leqslant 4. \end{aligned}$$

From this point on, we apply Lemma 3 without reference. Next, we assume throughout that $y+x_1+x_2+x_3<0$. If B is not empty, then $|A|\leqslant 2$ and so

$$\begin{aligned} g_4 &\leqslant \sum_{(i,j)\in A} f_2\left(y+x_i+x_j,x_4\right) \\ &= 2\sum_{(i,j)\in A} \min\left\{\left|y+x_i+x_j\right|,|x_4|\right\} \leqslant 4. \end{aligned}$$

So suppose that B is empty. Then $y+x_i+x_j<0$ for every $1\leqslant i<j\leqslant 3$.

If $|y+x_1+x_2+x_3|\geqslant |x_4|$, then $g_4\leqslant 6|x_4|-2|x_4|=4|x_4|\leqslant 4$. So assume that $|y+x_1+x_2+x_3|<|x_4|$. Adding $y+x_1+x_2$, $y+x_1+x_3$, and $y+x_2+x_3$, we obtain $y+x_1+x_2+x_3<-y/2\leqslant 0$. Since $y\geqslant 0$ and $y+x_i+x_j<0$ for every $1\leqslant i\leqslant j\leqslant 3$, there exists $i\in\{1,2,3\}$ such that $x_i<0$. Without loss of generality, we can assume that $x_3<0$. Then $y+x_1+x_2+x_3\leqslant y+x_1+x_2<0$. This leads to $|y+x_1+x_2|\leqslant |y+x_1+x_2+x_3|<|x_4|$. Therefore

$$g_4 \leqslant 4|x_4| + 2|y+x_1+x_2| - 2|y+x_1+x_2+x_3| \leqslant 4|x_4| \leqslant 4.$$

This completes the proof. $\square$

Finally, we calculate a lower bound, which we think is close to being sharp for f_6 as follows.

THEOREM 8. *For every $x_1,x_2,x_3,x_4,x_5,x_6\in\mathbb{R}$,*

$$f_6(x_1,x_2,x_3,x_4,x_5,x_6) \geqslant -5\max\{|x_i| : i=1,2,\ldots,6\}.$$

Proof. Similarly to the proof of Theorems 1 and 5, it is enough to show that

$$f_6(1,x_1,x_2,x_3,x_4,x_5) \geqslant -5,$$

where $1 \geqslant |x_1| \geqslant |x_2| \geqslant |x_3| \geqslant |x_4| \geqslant |x_5| > 0$. Then we write $f_6 = f_6(1, x_1, x_2, x_3, x_4, x_5)$ as

$$f_6(1, x_1, x_2, x_3, x_4, x_5) = f_5(x_1, x_2, x_3, x_4, x_5) + g_5(1, x_1, x_2, x_3, x_4, x_5).$$

If $x_i > 0$ for some $i = 1, 2, 3, 4, 5$, we apply Theorems 5 and 6, to obtain

$$f_6(1, x_1, x_2, x_3, x_4, x_5) \geqslant -5.$$

So assume that $x_i < 0$ for every $i = 1, 2, 3, 4, 5$. By Lemma 2, $f_5(x_1, x_2, x_3, x_4, x_5) = 0$. So $f_6 = g_5(1, x_1, x_2, x_3, x_4, x_5)$. If $1 + x_1 + x_2 + x_3 + x_4 + x_5 \geqslant 0$, then every term in the absolute value in the sum defining $g_5(1, x_1, x_2, x_3, x_4, x_5)$ is nonnegative, which leads to $g_5(1, x_1, x_2, x_3, x_4, x_5) = 0$. So we suppose that $1 + x_1 + x_2 + x_3 + x_4 + x_5 < 0$. Then $g_5(1, x_1, x_2, x_3, x_4, x_5)$ is equal to

$$\begin{aligned}
&-3 + \sum_{1 \leqslant i < j \leqslant 5} |1 + x_i + x_j| - \sum_{1 \leqslant i < j < k \leqslant 5} |1 + x_i + x_j + x_k| \\
&\quad + \sum_{1 \leqslant i < j < k < \ell \leqslant 5} |1 + x_i + x_j + x_k + x_\ell| \\
&= f_2(1 + x_1 + x_2, x_4) + f_2(1 + x_1 + x_3, x_5) + f_2(1 + x_1 + x_4, x_3) + f_2(1 + x_1 + x_5, x_2) \\
&\quad + f_2(1 + x_2 + x_3, x_1) + f_2(1 + x_2 + x_4, x_5) + f_2(1 + x_2 + x_5, x_3) + f_2(1 + x_3 + x_4, x_2) \\
&\quad + f_2(1 + x_3 + x_5, x_4) + f_2(1 + x_4 + x_5, x_1) + A_1, \qquad (39)
\end{aligned}$$

where

$$A_1 = -3 - 2 \sum_{1 \leqslant i \leqslant 5} |x_i| + \sum_{1 \leqslant i < j < k < \ell \leqslant 5} |1 + x_i + x_j + x_k + x_\ell|.$$

Let $A = x_1 + x_2 + x_3 + x_4 + x_5$. By the triangle inequality, we have

$$A_1 = -3 + 2 \sum_{1 \leqslant i \leqslant 5} x_i + \sum_{1 \leqslant i < j < k < \ell \leqslant 5} |1 + x_i + x_j + x_k + x_\ell| \geqslant -3 + 2A + |5 + 4A|. \quad (40)$$

Let $h : \mathbb{R} \to \mathbb{R}$ be defined by

$$h(x) = -3 + 2x + |5 + 4x| = \begin{cases} 6x + 2, & \text{if } x \geqslant -\frac{5}{4}; \\ -2x - 8, & \text{if } x < -\frac{5}{4}. \end{cases}$$

Next we divide the consideration into two cases. From this point on, we write g_5 instead of $g_5(1, x_1, x_2, x_3, x_4, x_5)$.

Case 1 $1 + x_i + x_j \leqslant 0$ for all $1 \leqslant i < j \leqslant 5$. Summing all of them, we obtain

$$10 + 4x_1 + 4x_2 + 4x_3 + 4x_4 + 4x_5 \leqslant 0,$$

which implies $A \leqslant -\frac{5}{2}$. Since h is decreasing on $(-\infty, -\frac{5}{4}]$ and $A \leqslant -\frac{5}{2} < -\frac{5}{4}$, we obtain by the triangle inequality, (39), and (40) that

$$g_5 \geqslant A_1 \geqslant h(A) \geqslant h\left(-\frac{5}{2}\right) = -2\left(-\frac{5}{2}\right) - 8 = -3.$$

Case 2 there exist integers $1 \leqslant i < j \leqslant 5$ such that $1+x_i+x_j > 0$. Since $1 \geqslant |x_1| \geqslant |x_2| \geqslant |x_3| \geqslant |x_4| \geqslant |x_5| > 0$ and $x_i <$ for every $i = 1,2,3,4,5$, we have $1+x_4+x_5 > 0$. Then $f_2(1+x_4+x_5, x_1) = 2\min\{1+x_4+x_5, |x_1|\}$. We divide the consideration into three cases depending on $1+x_4+x_5$ and $|x_1|$.

Case 2.1 $1+x_4+x_5 \geqslant \frac{1}{4}$ and $|x_1| \geqslant \frac{1}{4}$. Then $f_2(1+x_4+x_5, x_1) \geqslant \frac{1}{2}$. We see that h has the minimum value at $-\frac{5}{4}$. So we obtain by the triangle inequality, (39), and (40) that

$$\begin{aligned} g_5 &\geqslant f_2(1+x_4+x_5, x_1) + A_1 \\ &\geqslant f_2(1+x_4+x_5, x_1) + h(A) \geqslant \frac{1}{2} + h\left(-\frac{5}{4}\right) = -5. \end{aligned}$$

Case 2.2 $|x_1| < \frac{1}{4}$. Then $x_i > -\frac{1}{4}$ for every $i = 1,2,3,4,5$. Therefore $A \geqslant 5x_1 > -\frac{5}{4}$, $1+x_4+x_5 > \frac{1}{2}$, and $f_2(1+x_4+x_5, x_1) = 2|x_1| = -2x_1$. Since h is increasing on $[-\frac{5}{4}, \infty)$, we obtain by the triangle inequality, (39), and (40) that

$$\begin{aligned} g_5 &\geqslant f_2(1+x_4+x_5, x_1) + A_1 \\ &\geqslant -2x_1 + h(A) \geqslant -2x_1 + 6(5x_1) + 2 = 28x_1 + 2 > -5. \end{aligned}$$

Case 2.3 $1+x_4+x_5 < \frac{1}{4}$. Then $2x_4 \leqslant x_4+x_5 < -\frac{3}{4}$. This implies $x_1 \leqslant x_2 \leqslant x_3 \leqslant x_4 < -\frac{3}{8}$ and $|x_1| > \frac{3}{8} > \frac{1}{4}$. So $A < 4x_4 + x_5 < -\frac{3}{2} < -\frac{5}{4}$ and $f_2(1+x_4+x_5, x_1) = 2+2x_4+2x_5$. Since h is decreasing on $(-\infty, -\frac{5}{4}]$, we obtain by the triangle inequality, (39), and (40) that

$$\begin{aligned} g_5 &\geqslant 2+2x_4+2x_5 - 2(4x_4+x_5) - 8 \\ &= -6x_4 - 6 > -\frac{15}{4} > -5. \end{aligned}$$

In any case, we have $g_5 \geqslant -5$. Therefore $f_6(1, x_1, x_2, x_3, x_4, x_5) \geqslant -5$, as desired. □

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