Journal of Mathematical Inequalities
Volume 16, Number 4 (2022), 1477–1491 doi:10.7153/jmi-2022-16-96

A REFINED HARDY–LITTLEWOOD–POLYA INEQUALITY AND THE EQUIVALENT FORMS

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(*Communicated by Q.-H. Ma*)

Abstract. In this article, by the Euler-Maclaurin summation formula, we construct proper weight coefficients and use them to establish a refined Hardy-Littlewood-Polya inequality with multi parameters. Based on this inequality, the equivalent statements of the best possible constant factor related to several parameters are discussed. The equivalent forms, some particular inequalities and the operator expressions of the obtained inequalities are considered.

1. Introduction

Suppose that $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $a_m,b_n \geqslant 0$, $0<\sum_{m=1}^{\infty} a_m^p<\infty$ and $0<\sum_{n=1}^{\infty} b_n^q<\infty$. The Hardy-Hilbert inequality with the best possible constant factor $\frac{\pi}{\sin(\pi/p)}$ was provided as follows (cf. [5], Theorem 315):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n}<\frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^q\right)^{\frac{1}{q}}. \tag{1}$$

With the same assumption, we still have the following Hardy-Lottlewood-Polya inequality with the best possible constant factor pq (cf. [5], Theorem 341):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{\max\{m,n\}}<pq\left(\sum_{m=1}^{\infty}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^q\right)^{\frac{1}{q}}. \tag{2}$$

Also, a refined form of (1) was provided as follows (cf. [5], Theorem 323):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n-1}<\frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^q\right)^{\frac{1}{q}}, \tag{3}$$

where, the constant factor $\frac{\pi}{\sin(\pi/p)}$ is still the best possible.

Mathematics subject classification (2020): 26D15, 26D10, 47A05.
Keywords and phrases: Weight coefficient, the Hardy-Littlewood-Polya inequality, Euler-Maclaurin summation formula, parameter.
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By introducing multi parameters $\lambda_i \in (0,2]$ $(i=1,2)$, $\lambda_1+\lambda_2=\lambda \in (0,4]$, and using the Euler-Maclaurin summation formula, an extension of (1) was provided by Krnić et al. [17] in 2006 as follows:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} < B(\lambda_1,\lambda_2)\left[\sum_{m=1}^{\infty} m^{p(1-\lambda_1)-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1}b_n^q\right]^{\frac{1}{q}}, \tag{4}$$

where, the constant factor $B(\lambda_1,\lambda_2)$ is the best possible ($B(u,v)=\int_0^{\infty}\frac{t^{u-1}dt}{(1+t)^{u+v}}$ $(u,v>0)$ is the beta function). For $p=q=2$, $\lambda_1=\lambda_2=\frac{\lambda}{2}$, (3) reduces to Yang's published inequality in [27]. In 2019, applying inequality (2), Adiyasuren et al. [1] gave a new Hardy-Hilbert inequality with the kernel of (4) involving two partial sums.

If $f(x),g(y)\geqslant 0$, $0<\int_0^{\infty}f^p(x)dx<\infty$, and $0<\int_0^{\infty}g^q(y)dy<\infty$, then we still have the following integral analogue of (1) named in the Hardy -Hilbert integral inequality (cf. [5], Theorem 316):

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y}dxdy<\frac{\pi}{\sin(\pi/p)}\left(\int_0^{\infty}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^{\infty}g^q(y)dy\right)^{\frac{1}{q}}. \tag{5}$$

Inequalities (1)–(5) have many extensions (cf. [28], [18], [20], [6], [7], [23], [24], [32], [25], [2], [3] and [21]).

A half-discrete Hilbert-type inequality was given in 1934 as follows (cf. [5], Theorem 351): Assuming that $K(t)$ $(t>0)$ is decreasing, $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $0<\phi(s)=\int_0^{\infty}K(t)t^{s-1}dt<\infty$, we have

$$\int_0^{\infty}x^{p-2}\left(\sum_{n=1}^{\infty}K(nx)a_n\right)^p dx<\phi^p\left(\frac{1}{q}\right)\sum_{n=1}^{\infty}a_n^p. \tag{6}$$

Some new extensions of (6) were provided by [30], [29], [22] and [31]. In 2016, by means of the technique of real analysis, Hong [8]considered some equivalent statements of the extensions of (1) with the best possible constant factor related to several parameters. Some other similar results on the extensions of (1), (5) and (6) were given by [9], [10], [26], [11], [12], [13], [4], [14], [15] and [16].

In this paper, following the way of [17] and [8], by the Euler-Maclaurin summation formula and the techniques of real analysis, we will establish a refined Hardy-Littlewood-Polya inequality with multi parameters. Based on this inequality, the equivalent statements of the best possible constant factor related to several parameters are discussed. The equivalent forms and the operator expressions of the obtained inequalities are considered. We also illustrate how the new equivalent inequalities obtained can generate some interested particular cases.

2. Some lemmas

In what follows, we suppose that $p > 1\ (q > 1)$, $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha \in [0, \frac{1}{4}]$. $\lambda \in (\mathbf{0}, \frac{9}{4}]$, $\lambda_i \in (0,1] \cap (0,\lambda)\ (i = 1,2)$, $\widehat{\lambda}_1 := \frac{\lambda - \lambda_2}{p} + \frac{\lambda_1}{q}$, $\widehat{\lambda}_2 := \frac{\lambda - \lambda_1}{q} + \frac{\lambda_2}{p}$. We also assume that $a_m, b_n \geqslant 0\ (m,n \in N = \{1,2,\cdots\})$, such that

$$0 < \sum_{m=1}^{\infty} (m-\alpha)^{p(1-\widehat{\lambda}_1)-1} a_m^p < \infty \text{ and } 0 < \sum_{n=1}^{\infty} (n-\alpha)^{q(1-\widehat{\lambda}_2)-1} b_n^q < \infty.$$

LEMMA 1. *For* $\lambda \in (\mathbf{0}, \frac{9}{4}]$, $\alpha \in [0, \frac{1}{4}]$. $\lambda_2 \in (0,1] \cap (0,\lambda)$, *define the following weight coefficient:*

$$\varpi(\lambda_2, m) := (m-\alpha)^{\lambda - \lambda_2} \sum_{n=1}^{\infty} \frac{(n-\alpha)^{\lambda_2 - 1}}{(\max\{m,n\} - \alpha)^{\lambda}} \quad (m \in \mathbf{N}). \tag{7}$$

We have the following inequalities:

$$\begin{aligned} & k_\lambda(\lambda_2)\left[1 - \frac{(\lambda - \lambda_2)(1-\alpha)^{\lambda_2}}{\lambda (m-\alpha)^{\lambda_2}}\right] \\ & < \varpi(\lambda_2, m) < k_\lambda(\lambda_2) := \frac{\lambda}{\lambda_2(\lambda - \lambda_2)} \quad (m \in \mathbf{N}), \end{aligned} \tag{8}$$

Proof. For fixed $m \in \mathbf{N}$, we set the following real function:

$$g_m(t) := \frac{(t-\alpha)^{\lambda_2 - 1}}{(\max\{m,t\} - \alpha)^{\lambda}} \quad (t > \alpha).$$

Then we find

$$\begin{aligned} g_m(t) &= \begin{cases} \frac{(t-\alpha)^{\lambda_2-1}}{(m-\alpha)^{\lambda}}, & \alpha < t \leqslant m, \\ (t-\alpha)^{\lambda_2 - \lambda - 1}, & t > m \end{cases}, \\ g_m'(t) &= \begin{cases} \frac{(\lambda_2 - 1)(t-\alpha)^{\lambda_2-2}}{(m-\alpha)^{\lambda}}, & \alpha < t < m, \\ (\lambda_2 - \lambda - 1)(t-\alpha)^{\lambda_2 - \lambda - 2}, & t > m \end{cases}, \\ g_m(1) &= \frac{(1-\alpha)^{\lambda_2 - 1}}{(m-\alpha)^{\lambda}}, \\ \int_0^1 g_m(t)dt &= \int_0^1 \frac{(t-\alpha)^{\lambda_2-1}}{(m-\alpha)^{\lambda}} dt = \frac{(1-\alpha)^{\lambda_2}}{\lambda_2 (m-\alpha)^{\lambda}}. \\ \int_\alpha^\infty g_m(t)dt &= \int_\alpha^m \frac{(t-\alpha)^{\lambda_2-1}}{(m-\alpha)^{\lambda}} dt + \int_m^\infty (t-\alpha)^{\lambda_2 - \lambda - 1} dt \\ &= \frac{\lambda}{\lambda_2(\lambda - \lambda_2)(m-\alpha)^{\lambda - \lambda_2}}. \end{aligned}$$

For $\lambda_2 \in (0,1] \cap (0,\lambda)$, $0 < \varepsilon, \varepsilon_1 < 1$, by the Euler-Maclaurin summation formula (cf. [17], [28]), we find

$$\begin{aligned}
\sum_{n=1}^{m} g_m(n) &= \int_1^m g_m(t)dt + \frac{1}{2} g_m(t)|_1^m + \int_1^m P_1(t) g_m'(t)dt \\
&= \int_1^m g_m(t)dt + \frac{1}{2} g_m(t)|_1^m + \frac{\lambda_2 - 1}{(m-\alpha)^{\lambda}} \int_1^m P_1(t)(t-\alpha)^{\lambda_2 - 2} dt \\
&= \int_1^m g_m(t)dt + \frac{1}{2} g_m(t)|_1^m + \frac{\lambda_2 - 1}{(m-\alpha)^{\lambda}} \frac{\varepsilon}{12} (t-\alpha)^{\lambda_2 - 2}|_1^m \\
\sum_{n=m+1}^{\infty} g_m(n) &= \int_m^{\infty} g_m(t)dt + \frac{1}{2} g_m(t)|_m^{\infty} + \int_m^{\infty} P_1(t) g_m'(t)dt \\
&= \int_m^{\infty} g_m(t)dt + \frac{1}{2} g_m(t)|_m^{\infty} + \frac{\varepsilon_1}{12} (\lambda_2 - \lambda - 1)(t-\alpha)^{\lambda_2 - \lambda - 2}|_m^{\infty},
\end{aligned}$$

where, $P_1(t) = t - [t] - \frac{1}{2}$ $(t \in \mathbf{R} = (-\infty, \infty))$ is the Bernoulli function of 1-order. For the assumption, $(-1)^{i+1} \frac{d^i}{dt^i} g_m'(t) > 0$ $(i = 0,1,2,3;\ t \in [1,m)$ or $t \in (m,\infty))$, it follows that

$$\begin{aligned}
\int_1^m P_1(t) g_m'(t)dt &= \frac{\varepsilon}{12} g_m'(t)|_1^m, \\
\int_m^{\infty} P_1(t) g_m'(t)dt &= \frac{-\varepsilon_1}{12} g_m'(m) \quad (0 < \varepsilon, \varepsilon_1 < 1).
\end{aligned}$$

Then, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} g_m(n) &= \int_1^{\infty} g_m(t)dt + \frac{1}{2} g_m(1) \\
&\quad + \left[\frac{1-\lambda_2}{12(1-\alpha)^{1-\lambda_2}(m-\alpha)^{\lambda}} - \frac{1-\lambda_2}{12(m-\alpha)^{1+\lambda-\lambda_2}} \right] \varepsilon \\
&\quad + \frac{\lambda - \lambda_2 + 1}{12(m-\alpha)^{2+\lambda-\lambda_2}} \varepsilon_1 .
\end{aligned} \tag{9}$$

By (9), we have

$$\begin{aligned}
\sum_{n=1}^{\infty} g_m(n) &< \int_1^{\infty} g_m(t)dt + \frac{1}{2} g_m(1) + \frac{1-\lambda_2}{12(1-\alpha)^{1-\lambda_2}(m-\alpha)^{\lambda}} \\
&\quad - \frac{1-\lambda_2}{12(m-\alpha)^{1+\lambda-\lambda_2}} + \frac{\lambda - \lambda_2 + 1}{12(m-\alpha)^{2+\lambda-\lambda_2}} \\
&\leqslant \int_{\alpha}^{\infty} g_m(t)dt - \left[\int_{\alpha}^{1} g_m(t)dt \right] - \frac{1}{2} g_m(1) \\
&\quad - \frac{(1-\lambda_2)(1-\alpha)^{\lambda_2 - 1}}{12(m-\alpha)^{\lambda}} - \frac{\lambda - \lambda_2 + 1}{12(m-\alpha)^{2+\lambda-\lambda_2}} \Bigg] \\
&= \frac{1}{(m-\alpha)^{\lambda - \lambda_2}} \left[\frac{\lambda}{(\lambda - \lambda_2)\lambda_2} - h_m(\lambda_2) \right],
\end{aligned}$$

$$h_m(\lambda_2) := \left(\frac{1-\alpha}{\lambda_2} - \frac{1}{2} - \frac{1-\lambda_2}{12}\right)\frac{(1-\alpha)^{\lambda_2-1}}{(m-\alpha)^{\lambda_2}} - \frac{\lambda+1-\lambda_2}{12(m-\alpha)^2}.$$

We obtain that for $\lambda_2 \in (0,1] \cap (0,\lambda)$, $\alpha \in [0,\frac{1}{4}]$, $\lambda \in (0,\frac{9}{4}]$,

$$\begin{aligned}
h_m(\lambda_2) &\geqslant \frac{12(1-\alpha)-7+1}{12\lambda_2}\frac{(m-\alpha)^{\lambda_2-1}}{(m-\alpha)^{\lambda_2}} - \frac{\frac{13}{4}-\lambda_2}{12(m-\alpha)^2} \\
&= \frac{6-12\alpha}{12\lambda_2(m-\alpha)} - \frac{\frac{13}{4}-\lambda_2}{12(m-\alpha)^2} = \frac{(6-12\alpha)(m-\alpha)-\frac{13}{4}\lambda_2+\lambda_2^2}{12\lambda_2(m-\alpha)^2} \\
&\geqslant \frac{(6-12\alpha)(1-\alpha)-\frac{13}{4}\lambda_2+\lambda_2^2}{12\lambda_2(m-\alpha)^2} \geqslant \frac{(6-12\alpha)(1-\alpha)-\frac{13}{4}+1}{12\lambda_2(m-\alpha)^2} \\
&\geqslant \frac{(6-12\times\frac{1}{4})(1-\frac{1}{4})-\frac{9}{4}}{12\lambda_2(m-\alpha)^2} = 0.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\varpi(\lambda_2,m) &= (m-\alpha)^{\lambda-\lambda_2}\sum_{n=1}^{\infty} g_m(n) \\
&< (m-\alpha)^{\lambda-\lambda_2}\int_{\alpha}^{\infty} g_m(t)dt = k_\lambda(\lambda_2) = \frac{\lambda}{(\lambda-\lambda_2)\lambda_2}.
\end{aligned}$$

On the other hand, by (19), we find

$$\begin{aligned}
\sum_{n=1}^{\infty} g_m(n) &> \int_1^{\infty} g_m(t)dt \\
&= \int_{\alpha}^{\infty} g_m(t)dt - \int_{\alpha}^{1} g_m(t)dt \\
&= \frac{\lambda}{(\lambda-\lambda_2)\lambda_2(m-\alpha)^{\lambda-\lambda_2}} - \frac{(1-\alpha)^{\lambda_2}}{\lambda_2(m-\alpha)^{\lambda}} \\
&= \frac{\lambda}{(\lambda-\lambda_2)\lambda_2(m-\alpha)^{\lambda-\lambda_2}}\left[1-\frac{(\lambda-\lambda_2)(1-\alpha)^{\lambda_2}}{\lambda(m-\alpha)^{\lambda_2}}\right].
\end{aligned}$$

Therefore, we obtain (8).

The lemma is proved. □

LEMMA 2. *We have the following refined Hardy-Littlewood-Polya inequality:*

$$\begin{aligned}
I := \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(\max\{m,n\}-\alpha)^{\lambda}} &< k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1) \\
&\times\left[\sum_{m=1}^{\infty}(m-\alpha)^{p(1-\widehat{\lambda}_1)-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}(n-\alpha)^{q(1-\widehat{\lambda}_2)-1}b_n^q\right]^{\frac{1}{q}}. \quad (10)
\end{aligned}$$

Proof. In the same way of obtaining (8), for $n \in \mathbf{N}$, $\alpha \in [0, \frac{1}{4}]$, $\lambda \in (0, \frac{9}{4}]$, $\lambda_1 \in (0,1) \cap (0,\lambda)$, we have the following inequalities for the another weight coefficient:

$$\left[1 - \frac{(\lambda-\lambda_1)(1-\alpha)^{\lambda_1}}{\lambda(n-\alpha)^{\lambda_1}}\right] < \omega(\lambda_1, n) := (n-\alpha)^{\lambda-\lambda_1} \sum_{m=1}^{\infty} \frac{(m-\alpha)^{\lambda_1-1}}{(\max\{m,n\}-\alpha)^{\lambda}} < k_\lambda(\lambda_1). \tag{11}$$

By the Hölder inequality (cf. [19]), we obtain

$$\begin{aligned}
I &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{(\max\{m,n\}-\alpha)^{\lambda}} \left[\frac{(n-\alpha)^{(\lambda_2-1)/p}}{(m-\alpha)^{(\lambda_1-1)/q}} a_m\right] \left[\frac{(m-\alpha)^{(\lambda_1-1)/q}}{(n-\alpha)^{(\lambda_2-1)/p}} b_n\right] \\
&\leqslant \left[\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{(\max\{m,n\}-\alpha)^{\lambda}} \frac{(n-\alpha)^{\lambda_2-1}}{(m-\alpha)^{(\lambda_1-1)(p-1)}} a_m^p\right]^{\frac{1}{p}} \\
&\quad \times \left[\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{(\max\{m,n\}-\alpha)^{\lambda}} \frac{(m-\alpha)^{\lambda_1-1}}{(n-\alpha)^{(\lambda_2-1)(q-1)}} b_n^q\right]^{\frac{1}{q}} \\
&= \left[\sum_{m=1}^{\infty} \varpi(\lambda_2, m)(m-\alpha)^{p(1-\widehat{\lambda}_1)-1} a_m^p\right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} \omega(\lambda_1, n)(n-\alpha)^{q(1-\widehat{\lambda}_2)-1} b_n^q\right]^{\frac{1}{q}}.
\end{aligned}$$

Then by (8) and (11), we have (10).

The lemma is proved. □

REMARK 1. By (10), for $\lambda_1 + \lambda_2 = \lambda \in (0,2]$ $(\subset (0, \frac{9}{4}])$, $\lambda_i \in (0,1] \cap (0,\lambda)$ we find

$$0 < \sum_{m=1}^{\infty} (m-\alpha)^{p(1-\lambda_1)-1} a_m^p < \infty, 0 < \sum_{n=1}^{\infty} (n-\alpha)^{q(1-\lambda_2)-1} b_n^q < \infty.$$

and the following refined Hardy-Littlewood-Polya inequality:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{(\max\{m,n\}-\alpha)^{\lambda}} < \frac{\lambda}{\lambda_1\lambda_2} \left[\sum_{m=1}^{\infty} (m-\alpha)^{p(1-\lambda_1)-1} a_m^p\right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} (n-\alpha)^{q(1-\lambda_2)-1} b_n^q\right]^{\frac{1}{q}}. \tag{12}$$

LEMMA 3. *The constant factor $\frac{\lambda}{\lambda_1\lambda_2}$ in (12) is the best possible.*

Proof. For any $0 < \varepsilon < p\lambda_1$, we set $\widetilde{a}_m := m^{\lambda_1 - \frac{\varepsilon}{p} - 1}$, $\widetilde{b}_n := n^{\lambda_2 - \frac{\varepsilon}{q} - 1}$ $(m, n \in \mathbf{N})$. If there exists a constant $M \leqslant \frac{\lambda}{\lambda_1\lambda_2}$, such that (12) is valid when we replace $\frac{\lambda}{\lambda_1\lambda_2}$ by M, then in particular, for $\alpha = 0$, substitution of $a_m = \widetilde{a}_m$ and $b_n = \widetilde{b}_n$ in (12), we have

$$\widetilde{I} := \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\widetilde{a}_m \widetilde{b}_n}{(\max\{m,n\})^{\lambda}} < M\left[\sum_{m=1}^{\infty} m^{p[1-\lambda_1)-1}\widetilde{a}_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty} n^{q[1-\lambda_2)-1}\widetilde{b}_n^q\right]^{\frac{1}{q}}. \tag{13}$$

By (13) and the decreasingness property of series, we obtain

$$\begin{aligned}\widetilde{I} &< M\left[\sum_{m=1}^{\infty} m^{p[1-\lambda_1)-1} m^{p(\lambda_1-\frac{\varepsilon}{p}-1)}\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty} n^{q[1-\lambda_2)-1} n^{q(\lambda_2-\frac{\varepsilon}{q}-1)}\right]^{\frac{1}{q}} \\ &= M\left(\sum_{m=1}^{\infty} m^{-\varepsilon-1}\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} n^{-\varepsilon-1}\right)^{\frac{1}{p}} = M\left(1+\sum_{m=2}^{\infty} m^{-\varepsilon-1}\right) \\ &< M\left(1+\int_1^{\infty} x^{-\varepsilon-1}dx\right) = \frac{M}{\varepsilon}(\varepsilon+1).\end{aligned}$$

By (11) (for $\alpha = 0$, setting $\widetilde{\lambda}_1 := \lambda_1 - \frac{\varepsilon}{p} \in (0,1)\cap(0,\lambda)$ $(0 < \widetilde{\lambda}_2 = \lambda_2 + \frac{\varepsilon}{p} = \lambda - \widetilde{\lambda}_1 < \lambda))$, we find

$$\begin{aligned}\widetilde{I} &= \sum_{n=1}^{\infty}\left[n^{\lambda_2+\frac{\varepsilon}{p}}\sum_{m=1}^{\infty}\frac{m^{(\lambda_1-\frac{\varepsilon}{p})-1}}{(\max\{m,n\})^{\lambda}}\right]n^{-\varepsilon-1} = \sum_{n=1}^{\infty}\omega(\widetilde{\lambda}_1,n)n^{-\varepsilon-1} \\ &> k_{\lambda}(\widetilde{\lambda}_1)\sum_{n=1}^{\infty}\left(1-\frac{\widetilde{\lambda}_2}{\lambda n^{\widetilde{\lambda}_1}}\right)n^{-\varepsilon-1} \\ &= k_{\lambda}(\widetilde{\lambda}_1)\left(\sum_{n=1}^{\infty} n^{-\varepsilon-1} - \frac{\widetilde{\lambda}_2}{\lambda}\sum_{n=1}^{\infty}\frac{1}{n^{\lambda_1+\frac{\varepsilon}{q}+1}}\right) \\ &> k_{\lambda}(\widetilde{\lambda}_1)\left(\int_1^{\infty} y^{-\varepsilon-1}dy - O(1)\right) = \frac{k_{\lambda}(\widetilde{\lambda}_1)}{\varepsilon}(1-\varepsilon O(1)).\end{aligned}$$

Then we have the following inequality:

$$k_{\lambda}(\lambda_1 - \frac{\varepsilon}{p})(1-\varepsilon O(1)) \leqslant \varepsilon\widetilde{I} < M(\varepsilon+1).$$

For $\varepsilon \to 0^+$, we have $k_{\lambda}(\lambda_1) = \frac{\lambda}{\lambda_1\lambda_2} \leqslant M$. Hence, $M = \frac{\lambda}{\lambda_1\lambda_2}$ is the best possible constant factor of (12).

The lemma is proved. $\square$

REMARK 2. For $\widehat{\lambda}_1 = \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q}$, $\widehat{\lambda}_2 = \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p}$, we find

$$\widehat{\lambda}_1 + \widehat{\lambda}_2 = \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q} + \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p} = \lambda,$$
$$0 < \widehat{\lambda}_i < \lambda \ (i = 1,2).$$

If $\lambda-\lambda_i \leqslant 1\ (i=1,2)$, then we have $\widehat{\lambda}_i \leqslant 1\ (i=1,2)$, and we can rewrite (12) as follows:

$$
\begin{aligned}
I &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(\max\{m,n\}-\alpha)^{\lambda}} \\
&< k_{\lambda}(\widehat{\lambda}_1)\left[\sum_{m=1}^{\infty}(m-\alpha)^{p(1-\widehat{\lambda}_1)-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}(n-\alpha)^{q(1-\widehat{\lambda}_2)-1}b_n^q\right]^{\frac{1}{q}}. \qquad (14)
\end{aligned}
$$

LEMMA 4. *If the constant factor* $k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1)$ *in* (11) *is the best possible, then for* $\lambda-\lambda_i \leqslant 1\ (i=1,2)$, *we have* $\lambda_1+\lambda_2=\lambda$.

Proof. If the constant factor $k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1)$ in (10) is the best possible, then in view of the assumption and (14), we have the following inequality:

$$
k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1) \leqslant k_{\lambda}(\widehat{\lambda}_1) = \frac{\lambda}{\widehat{\lambda}_1\widehat{\lambda}_2}(\in \mathbf{R}_+ = (0.\infty)) \qquad (15)
$$

By the Hölder inequality (cf. [19]), we find

$$
\begin{aligned}
0 &< k_{\lambda}(\widehat{\lambda}_1) = k_{\lambda}\left(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}\right) \\
&= \int_0^{\infty}\frac{1}{(\max\{1,u\})^{\lambda}}u^{\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}-1}du \\
&= \int_0^{\infty}\frac{1}{(\max\{1,u\})^{\lambda}}(u^{\frac{\lambda-\lambda_2-1}{p}})(u^{\frac{\lambda_1-1}{q}})du \\
&\leqslant \left[\int_0^{\infty}\frac{u^{\lambda-\lambda_2-1}}{(\max\{1,u\})^{\lambda}}du\right]^{\frac{1}{p}}\left[\int_0^{\infty}\frac{u^{\lambda_1-1}}{(\max\{1,u\})^{\lambda}}du\right]^{\frac{1}{q}} \\
&= \left[\int_0^{\infty}\frac{v^{\lambda_2-1}}{(\max\{1,v\})^{\lambda}}dv\right]^{\frac{1}{p}}\left[\int_0^{\infty}\frac{u^{\lambda_1-1}}{(\max\{1,u\})^{\lambda}}du\right]^{\frac{1}{q}} \\
&= k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1). \qquad (16)
\end{aligned}
$$

In view of (15), we have

$$
k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1) = k_{\lambda}(\widehat{\lambda}_1),
$$

namely, (16) keeps the form of equality. We observe that (16) keeps the form of equality if and only if there exist constants A and B, such that they are not both zero and (cf. [19])

$$
Au^{\lambda-\lambda_2-1} = Bu^{\lambda_1-1} \text{ a.e. in } \mathbf{R}_+.
$$

Assuming that $A \neq 0$, we have

$$u^{\lambda-\lambda_2-\lambda_1} = \frac{B}{A} \text{ a.e. in } \mathbf{R}_+,$$

and $\lambda - \lambda_2 - \lambda_1 = 0$, namely, $\lambda_1 + \lambda_2 = \lambda$.

The lemma is proved. □

3. Main results

THEOREM 1. *We have the following inequality equivalent to* (10)*:*

$$J := \left\{ \sum_{n=1}^{\infty} (n-\alpha)^{p\widehat{\lambda}_2 - 1} \left[\sum_{m=1}^{\infty} \frac{a_m}{(\max\{m,n\}-\alpha)^{\lambda}} \right]^p \right\}^{\frac{1}{p}} < k_\lambda^{\frac{1}{p}}(\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1) \left[\sum_{m=1}^{\infty} (m-\alpha)^{p(1-\widehat{\lambda}_1)-1} a_m^p \right]^{\frac{1}{p}}. \tag{17}$$

If the constant factor in (10) *is the best possible, then so is the constant factor in* (17).

Proof. Suppose that (10) is valid. By thr Hölder inequality (cf. [19]), we have

$$I = \sum_{n=1}^{\infty} \left[(n-\alpha)^{\widehat{\lambda}_2 - \frac{1}{p}} \sum_{m=1}^{\infty} \frac{a_m}{(\max\{m,n\}-\alpha)^{\lambda}} \right] \left[(n-\alpha)^{\frac{1}{p}-\widehat{\lambda}_2} b_n \right] \leqslant J \left[\sum_{n=1}^{\infty} (n-\alpha)^{q(1-\widehat{\lambda}_2)-1} b_n^q \right]^{\frac{1}{q}}. \tag{18}$$

Then by (17), we obtain (10).

On the other hand, assuming that (10) is valid, we set

$$b_n := (n-\alpha)^{p\widehat{\lambda}_2 - 1} \left[\sum_{m=1}^{\infty} \frac{a_m}{(\max\{m,n\}-\alpha)^{\lambda}} \right]^{p-1}, \quad n \in \mathbf{N}.$$

If $J = 0$, then (10) is naturally valid; if $J = \infty$, then it is impossible that makes (10) valid, namely, $J < \infty$. Suppose that $0 < J < \infty$. By (10), it follows that

$$\sum_{n=1}^{\infty} (n-\alpha)^{q(1-\widehat{\lambda}_2)-1} b_n^q = J^p = I < k_\lambda^{\frac{1}{p}}(\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1) \left[\sum_{m=1}^{\infty} (m-\alpha)^{p(1-\widehat{\lambda}_1)-1} a_m^p \right]^{\frac{1}{p}} \times \left[\sum_{n=1}^{\infty} (n-\alpha)^{q(1-\widehat{\lambda}_2)-1} b_n^q \right]^{\frac{1}{q}},$$

$$J=\left[\sum_{n=1}^{\infty}(n-\alpha)^{q(1-\widehat{\lambda}_2)-1}b_n^q\right]^{\frac{1}{p}}$$
$$<k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1)\left[\sum_{m=1}^{\infty}(m-\alpha)^{p(1-\widehat{\lambda}_1)-1}a_m^p\right]^{\frac{1}{p}},$$

namely, (17) follows, which is equivalent to (10).

If the constant factor in (10) is the best possible, then so is constant factor in (17). Otherwise, by (18), we would reach a contradiction that the constant factor in (10) is not the best possible.

The theorem is proved. □

THEOREM 2. *The following statements (i), (ii), (iii) and (iv) are equivalent:*
(i) Both $k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1)$ *and* $k_{\lambda}(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})$ *are independent of* p,q*;*
(ii)
$$k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1)=k_{\lambda}\left(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}\right); \tag{19}$$
(iii) $k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1)$ *in* (10) *is the best possible constant factor;*
(iv) if $\lambda-\lambda_i\leqslant 1\ (i=1,2)$*, then we have* $\lambda_1+\lambda_2=\lambda$*.*

If the statement (iv) follows, namely, $\lambda_1+\lambda_2=\lambda$*, then we have* (12) *and the following equivalent inequality with the best possible constant factor* $\frac{\lambda}{\lambda_1\lambda_2}$*:*

$$\left\{\sum_{n=1}^{\infty}(n-\alpha)^{p\lambda_2-1}\left[\sum_{m=1}^{\infty}\frac{a_m}{(\max\{m,n\}-\alpha)^{\lambda}}\right]^p\right\}^{\frac{1}{p}}$$
$$<\frac{\lambda}{\lambda_1\lambda_2}\left[\sum_{m=1}^{\infty}(m-\alpha)^{p(1-\lambda_1)-1}a_m^p\right]^{\frac{1}{p}}. \tag{20}$$

Proof. $(i)\Rightarrow(ii)$. By (i), we have

$$k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1)=\lim_{p\to\infty}\lim_{q\to1^+}k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1)=k_{\lambda}(\lambda_1),$$
$$k_{\lambda}\left(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}\right)=\lim_{p\to\infty}\lim_{q\to1^+}k_{\lambda}\left(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}\right)=k_{\lambda}(\lambda_1),$$

namely, (19) follows.

$(ii)\Rightarrow(iv)$. If (19) follows, then (16) keeps the form of equality. In view of the proof of Lemma 4, it follows that $\lambda_1+\lambda_2=\lambda$.

$(iv)\Rightarrow(i)$. If $\lambda_1+\lambda_2=\lambda$, then

$$k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1)=k_{\lambda}\left(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}\right)=k_{\lambda}(\lambda_1),$$

which is independent of p,q.

Hence, we have $(i) \Leftrightarrow (ii) \Leftrightarrow (iv)$.

$(iii) \Rightarrow (iv)$. By the assumption and Lemma 4, we have $\lambda_1 + \lambda_2 = \lambda$.

$(iv) \Rightarrow (iii)$. By Lemma 3, for $\lambda_1 + \lambda_2 = \lambda$,

$$k_\lambda^{\frac{1}{p}}(\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1) \Big(= \frac{\lambda}{\lambda_1 \lambda_2} \Big)$$

is the best possible constant factor of (12).

Therefore, we have $(iii) \Leftrightarrow (iv)$.

Hence, the statements (i), (ii), (iii) and (iv) are equivalent.

The theorem is proved. □

REMARK 3. (i) For $\lambda = 1$, $\lambda_1 = \frac{1}{q}$, $\lambda_2 = \frac{1}{p}$ in (12) and (20), we have the following equivalent inequalities with the best possible constant factor pq:

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m,n\} - \alpha} < pq \left(\sum_{m=1}^{\infty} a_m^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} b_n^q \right)^{\frac{1}{q}}, \tag{21}$$

$$\left[\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \frac{a_m}{\max\{m,n\} - \alpha} \right)^p \right]^{\frac{1}{p}} < pq \left(\sum_{m=1}^{\infty} a_m^p \right)^{\frac{1}{p}}. \tag{22}$$

For $\alpha = 0$, (21) reduces to (2). Hence, (21) is a refinement of (2).

(ii) For $\lambda = 1$, $\lambda_1 = \frac{1}{p}$, $\lambda_2 = \frac{1}{q}$ in (12) and (20), we have the following equivalent dual inequalities with the best possible constant factor pq:

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m,n\} - \alpha} < pq \left[\sum_{m=1}^{\infty} (m-\alpha)^{p-2} a_m^p \right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} (n-\alpha)^{q-2} b_n^q \right]^{\frac{1}{q}}, \tag{23}$$

$$\left[\sum_{n=1}^{\infty} (n-\alpha)^{p-2} \left(\sum_{m=1}^{\infty} \frac{a_m}{\max\{m,n\} - \alpha} \right)^p \right]^{\frac{1}{p}} < pq \left[\sum_{m=1}^{\infty} (m-\alpha)^{p-2} a_m^p \right]^{\frac{1}{p}}. \tag{24}$$

(iii) For $p = q = 2$, both (21) and (23) reduce to

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m,n\} - \alpha} < 4 \left(\sum_{m=1}^{\infty} a_m^2 \sum_{n=1}^{\infty} b_n^2 \right)^{\frac{1}{2}}, \tag{25}$$

and both (22) and (24) reduce to the equivalent form of (25) as follows:

$$\left[\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{a_m}{\max\{m,n\}-\alpha}\right)^2\right]^{\frac{1}{2}} < 4\left(\sum_{m=1}^{\infty}a_m^2\right)^{\frac{1}{2}}. \tag{26}$$

(iv) For $\alpha=0$ in (12) and (20), we have the following equivalent inequalities with the best possible constant factor $\frac{\lambda}{\lambda_1\lambda_2}$ $(\lambda_1+\lambda_2=\lambda\in(0,2],\ 0<\lambda_i\leqslant 1\ \ (i=1,2))$:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(\max\{m,n\})^{\lambda}} < \frac{\lambda}{\lambda_1\lambda_2}\left[\sum_{m=1}^{\infty}m^{p(1-\lambda_1)-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}n^{q(1-\lambda_2)-1}b_n^q\right]^{\frac{1}{q}}. \tag{27}$$

$$\left\{\sum_{n=1}^{\infty}n^{p\lambda_2-1}\left[\sum_{m=1}^{\infty}\frac{a_m}{(\max\{m,n\})^{\lambda}}\right]^p\right\}^{\frac{1}{p}} < \frac{\lambda}{\lambda_1\lambda_2}\left[\sum_{m=1}^{\infty}m^{p(1-\lambda_1)-1}a_m^p\right]^{\frac{1}{p}}. \tag{28}$$

4. Operator expressions

We set functions $\varphi(m):=(m-\alpha)^{p(1-\widehat{\lambda}_1)-1}$, $\psi(n):=(n-\alpha)^{q(1-\widehat{\lambda}_2)-1}$, then,

$$\psi^{1-p}(n)=(n-\alpha)^{p\widehat{\lambda}_2-1}\ \ (m,n\in\mathbf{N}).$$

Define the following real normed spaces:

$$l_{p,\varphi}:=\left\{a=\{a_m\}_{m=1}^{\infty};||a||_{p,\varphi}:=\left(\sum_{m=1}^{\infty}\varphi(m)|a_m|^p\right)^{\frac{1}{p}}<\infty\right\},$$

$$l_{q,\psi}:=\left\{b=\{b_n\}_{n=1}^{\infty};||b||_{q,\psi}:=\left(\sum_{n=1}^{\infty}\psi(n)|b_n|^q\right)^{\frac{1}{q}}<\infty\right\},$$

$$l_{p,\psi^{1-p}}:=\left\{c=\{c_n\}_{n=1}^{\infty};||c||_{p,\psi^{1-p}}:=\left(\sum_{n=1}^{\infty}\psi^{1-p}(n)|c_n|^p\right)^{\frac{1}{p}}<\infty\right\}.$$

Assuming that $a\in l_{p,\varphi}$, setting

$$c=\{c_n\}_{n=1}^{\infty},c_n:=\sum_{m=1}^{\infty}\frac{a_m}{(\max\{m,n\}-\alpha)^{\lambda}},\ \ n\in\mathbf{N},$$

we can rewrite (17) as follows:

$$||c||_{p,\psi^{1-p}} < k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)||a||_{p,\varphi} < \infty,$$

namely, $c \in l_{p,\psi^{1-p}}$.

DEFINITION 1. Define an operator $T : l_{p,\varphi} \to l_{p,\psi^{1-p}}$ as follows: For any $a \in l_{p,\varphi}$, there exists a unique representation $c \in l_{p,\psi^{1-p}}$, satisfying for any $n \in \mathbf{N}$, $Ta(n) = c_n$. Define the dual pair Ta and $b \in l_{q,\psi}$, and the norm of T as follows:

$$(Ta,b) := \sum_{n=1}^{\infty}\left[\sum_{m=1}^{\infty}\frac{a_m}{(\max\{m,n\}-\alpha)^\lambda}\right]b_n,$$

$$||T|| := \sup_{a(\neq 0)\in l_{p,\varphi}}\frac{||Ta||_{p,\psi^{1-p}}}{||a||_{p,\varphi}}.$$

By Theorem 1 and 2, we have

THEOREM 3. *If* $a \in l_{p,\varphi}$, $b \in l_{q,\psi}$, $||a||_{p,\varphi}$, $||b||_{q,\psi} > 0$, *then we have the following equivalent inequalities:*

$$(Ta,b) < k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)||a||_{p,\varphi}||b||_{q,\psi}, \tag{29}$$

$$||Ta||_{p,\psi^{1-p}} < k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)||a||_{p,\varphi}. \tag{30}$$

Moreover, if $\lambda_1+\lambda_2=\lambda$, *then the constant factor* $k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)$ *in* (29) *and* (30) *is the best possible, namely,*

$$||T|| = k_\lambda(\lambda_1) = \frac{\lambda}{\lambda_1\lambda_2}. \tag{31}$$

On the other hand, if the constant factor $k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)$ in (29) and (30) is the best possible, then for $\lambda-\lambda_i \leqslant 1$ $(i=1,2)$, we have $\lambda_1+\lambda_2=\lambda$.

5. Conclusions

In this paper, following the way of [17] and [8], by means of the weight coefficients, the idea of introduced parameters and the Euler-Maclaurin summation formula, a refined Hardy-Littlewood-Polya inequality as well as the equivalent forms are given in Lemma 2 and Theorem 1. The equivalent statements of the best possible constant factor related to a few parameters and some particular cases are considered in Theorem 2 and Remark 3. The operator expressions are given in Theorem 3. The lemmas and theorems provide an extensive account of this type of inequalities.

Acknowledgements. This work is supported by the National Natural Science Foundation (Nos. 62166011, 12001113), and the Characteristic Innovation Project of Guangdong Provincial Colleges and Universities in 2020 (No. 2020KTSCX088). We are grateful for this help.

The authors thank the referee for his useful propose to reform the paper.

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(Received March 10, 2022)

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