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A WEIGHTED QUANTITATIVE ISOPERIMETRIC INEQUALITY FOR KORÁNYI SPHERE IN HEISENBERG GROUP $\mathbb{H}^n$

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Abstract. It is well known that the Korányi sphere *w.r.t.* the Korányi distance is not an isoperimetric set in Heisenbeg group $\mathbb{H}^n$. In this paper, we investigate Korányi sphere in a Heisenberg group associated with a density $|z|^{-(2n+1)}e^{-\frac{\alpha}{|z|}}$ $(\alpha > 0)$, and derive a weighted isoperimetric inequality and a weighted quantitative isoperimetric inequality for Korányi spheres in half-cylinders. This note also shows that the Korányi sphere is the weighted isoperimetric set in the weighted Heisenberg group $\mathbb{H}^n$.

1. Introduction

The study of isoperimetric problems in Carnot-Carathéodory spaces has been an active field over the past few decades. But even in Carnot groups, there is very few known about the optimal constant in the isoperimetric inequality except for the fact that isoperimetric sets exist and have at least some very weak regularity properties [16]. With the only exception of the Grushin plane [21], isoperimetric sets have been only partially characterized in the Sub-Riemannian Heisenberg group $\mathbb{H}^n$ and are not known at all in more general Carnot groups.

The isoperimetric problem in the Heisenberg group consists in minimizing H-perimeter of sets with a given fixed volume. One could expect that the natural isoperimetric candidates sets in $\mathbb{H}^n$ are Carnot-Carathéodory sphere associated with the Carnot-Carathéodory metric, as they are the counterparts of the Euclidean sphere in the Euclidean space $\mathbb{R}^n$. However, Carnot-Carathéodory sphere are not isoperimetric [19, 20].

In 1983 Pansu [25] conjectured that, up to a left translation and a dilation, the isoperimetric set is

$$E_{\text{isop}} = \{(z,t) \in \mathbb{H}^n : |t| < \arccos|z| + |z|\sqrt{1-|z|^2}, |z| < 1\}. \tag{1.1}$$

The conjecture was made for dimension $n = 1$. This set is obtained by rotating a Carnot-Carathéodory geodesic around the center of the group. Now the set E_{isop} is

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called Pansu sphere, bubble set or Heisenberg bubble. In $\mathbb{H}^1$, Pansu's conjecture is proved assuming either the C^2 regularity of the minimizer [27] or its convexity [23]. In $\mathbb{H}^n$ with $n \geqslant 1$, the conjecture is proved assuming the axial symmetry of the minimizer [17, 22] or assuming a suitable cylindrical structure [28]. About the isoperimetric problems and related celebrated geometric inequalities in Heisenberg groups, one can see the book [2] or the works [3, 11, 18, 24] for details.

Recently, isoperimetric problems with density in sub-Riemannain manifolds have been also studied. He and Zhao [12] proved a weighted isoperimetric-type inequality for hypersurfaces in Carnot groups with smooth density. The weighted x-spherically symmetric isoperimetric problem was studied in Grushin space with density $|x|^p$ in [13]. And the weighted isoperimetric problem in Heisenberg group $\mathbb{H}^n$ with density $|x|^p$ was solved in [15].

On the other hand, the quantitative isoperimetric inequality in the Euclidean space and in Riemannian manifolds, which describes the stability of the isoperimetric inequality, has been object of intensive studies in recent years. The sharp quantitative isoperimetric inequality in the Euclidean space $\mathbb{R}^n$ states that there exists a constant $C(n) > 0$ depending only on the dimension n, such that for any Borel set $F \subset \mathbb{R}^n$ with $\mathscr{L}^n(F) = \mathscr{L}^n(B_1)$, the Lebesgue measure of a unit sphere B_1, one has the following estimate for the difference of perimeters

$$P(F) - P(B_1) \geqslant C(n) \inf_{x\in\mathbb{R}^n} \mathscr{L}^n(F \triangle (x+B_1))^2.$$

The related results, the techniques and the main ideas about this inequality have been presented in the paper [10]. Several generalization have been recently obtained in Riemannian manifolds with density, like the Gauss space [1, 5].

Recently, Franceschi, Leonardi and Monti [8] obtained quantitative isoperimetric inequalities for the Pansu sphere E_{isop} in half-cylinders by the construction of sub-calibrations. He and Zhao [14] also proved quantitative isoperimetric inequalities for the isoperimetric set in Grushin space with density $|x|^p$. Franceschi *et al.* [9] have studied a family of spheres with constant mean curvature in the 3-dimensional Riemannain Heisenberg group and also obtained quantitative isoperimetric inequalities for these CMC spheres in half-cylinders.

The Korányi metric on the Heisenberg group $\mathbb{H}^n$, which is equivalent to the Carnot-Carathéodory distance, is defined by

$$d_{\mathbb{H}}(p,q) = ||q^{-1} * p||_{\mathbb{H}}, \forall p,q \in \mathbb{H}^n$$

where the Korányi gauge $||\cdot||_{\mathbb{H}}$ is given by

$$||p||_{\mathbb{H}} = (|z|^4 + t^2)^{\frac{1}{4}}, \forall p = (z,t) \in \mathbb{H}^n.$$

For any $p = (z,t) \in \mathbb{H}^n$ and $R > 0$, the Korányi sphere is defined by

$$B_p(R) = \{q \in \mathbb{H}^n : d_{\mathbb{H}}(p,q) < R\}.$$

Let $B = B_o(1) = \{(z,t) : |z|^4 + t^2 < 1\}$ be the Korányi unit sphere where $o = (0,\cdots,0)$ is the origin. By a left translation and a dilation, any Korányi sphere can be turned into the Korányi unit sphere. Obviously, the Korányi unit sphere $\partial B = \{(z,t) : |z|^4 + t^2 = 1\}$ is compact and of class C^2. It is also of class C^3. By a computation, we know that the horizontal mean curvature of the Korányi unit sphere is given by $\mathscr{H}_H = \frac{2n+1}{2n}|z|$, away from the characteristic set $\{(0,0,\pm 1)\}$. The first variation formulas for area and volume imply that the boundary of any C^2 solution of the isoperimetric problem is a compact hypersurface with constant mean curvature in sub-Riemannian sense, see [4] and [26]. So the Korányi sphere are not isoperimetric in $\mathbb{H}^n$.

In this paper, we endow the Heisenberg group with the density $|z|^{-(2n+1)}e^{-\frac{\alpha}{|z|}}$ where $\alpha > 0$ is constant. With this density, the weighted horizontal mean curvature of Korányi spheres ∂B is constant. This suggests the Korányi sphere perhaps is exactly the weighted isoperimetric set in Heisenberg group with the density $|z|^{-(2n+1)}e^{-\frac{\alpha}{|z|}}$.

For any $0 \leqslant \varepsilon < 1$, we define the half-cyclinder

$$C_\varepsilon = \{(z,t) \in \mathbb{H}^n : |z| < 1 \ \text{ and } \ t > t_\varepsilon\},$$

where $t_\varepsilon = f(1-\varepsilon)$ with $f(r) = \sqrt{1-r^4}$. Let $V_\phi(F)$ and $P_{H,\phi}(F)$ be weighted volume and weighted H-perimeter of a measurable set $F \subset \mathbb{H}^n$, respectively. Their definitions are seen in Section 2. We will prove a weighted quantitative isoperimetric inequality for the Korányi sphere B with respect to compact perturbations in half-cylinders.

THEOREM 1.1. *Let F be a measurable set in the Heisenberg group $\mathbb{H}^n$ with the density $|z|^{-(2n+1)}e^{-\frac{\alpha}{|z|}}$ $(\alpha > 0)$, where F satisfies $V_\phi(F) = V_\phi(B)$.*

(i) If $F \triangle B \subset\subset C_0$, then we have

$$P_{H,\phi}(F) - P_{H,\phi}(B) \geqslant \frac{\alpha a_\alpha}{3(2+\sqrt{2})\omega_{2n}^2} V_\phi(B \triangle F)^3$$

where ω_{2n} is the Euclidean volume of the $2n$-dimensional unit sphere and a_α is the minimum of the funtion $r^{2(2n+1)}e^{\frac{2\alpha}{r}}$ on $(0,1)$.

(ii) If $F \triangle B \subset\subset C_\varepsilon$ with $0 < \varepsilon < 1$, then we have

$$P_{H,\phi}(F) - P_{H,\phi}(B) \geqslant \frac{\alpha b_{\alpha,\varepsilon}\sqrt{1-(1-\varepsilon)^4}}{\left[1+(1-\varepsilon)^4+\sqrt{1+(1-\varepsilon)^4}\right]\omega_{2n}} V_\phi(B \triangle F)^2$$

where ω_{2n} is the Euclidean volume of the $2n$-dimensional unit sphere and $b_{\alpha,\varepsilon}$ is the minimum of the function $r^{2n+1}e^{\frac{\alpha}{r}}$ on $(0, 1-\varepsilon)$.

COROLLARY 1.2. *Let F be a measurable set in the Heisenberg group $\mathbb{H}^n$ with the density $|z|^{-(2n+1)}e^{-\frac{\alpha}{|z|}}$ $(\alpha > 0)$, where F satisfies $V_\phi(F) = V_\phi(B)$ and $F \triangle B \subset C_\varepsilon$. Then we have*

$$P_{H,\phi}(F) \geqslant P_{H,\phi}(B).$$

This corollary states that the Korányi sphere are minimizers of the weighted H-perimeter in the class of sets with the same weighted volume and with the symmetric difference contained in half-cyclinders. In other words, we confirm that Korányi sphere can be regarded as an isoperimetric set in weighted Heisenberg group $\mathbb{H}^n$.

2. Preliminaries

The $(2n+1)$-dimensional Heisenberg group is the manifold $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R} \cong \mathbb{R}^{2n+1}$, $n \in \mathbb{N}$, endowed with the group product

$$(z,t) * (z',t') = (z+z', t+t'+2I_m z\overline{z'}),$$

where $t,t' \in \mathbb{R}$, $z = x+iy, z' = x'+iy' \in \mathbb{C}^n$ with $x,y,x',y' \in \mathbb{R}^n$ and $z\overline{z'} = \sum_{j=1}^n z_j \overline{z'_j}$. The Lie algebra of the Heisenberg group is spanned by the following left-invariant vector fields:

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad j = 1, \cdots, n; \quad T = \frac{\partial}{\partial t}.$$

The horizontal distribution at a point $p \in \mathbb{H}^n$ is defined by

$$H_p = \text{span}\{X_i(p), Y_i(p) : i = 1, \cdots, n\}.$$

The horizontal distribution is nonintegrable. In fact there holds $[X_i, Y_i] = -4T \neq 0$ for any $i = 1, \cdots, n$. All other commutators vanish.

Let $g = \langle \cdot, \cdot \rangle$ be the (left-invariant) Riemannian metric which makes the basis $\{X_i, Y_i, T : i = 1, \cdots, n\}$ an orthornormal frame. The natural volume in $\mathbb{H}^n$ is the Haar measure, which coincides with Lebesgue measure in $\mathbb{R}^{2n+1}$. The horizontal gradient ∇_H of a smooth function u is defined by $\nabla_H u = (X_1 u, \cdots, X_n u, Y_1 u, \cdots, Y_n u)$. Let $\Omega \subset \mathbb{H}^n$ be an open set. The horizontal divergence of a vector field $\varphi \in C^1(\Omega; \mathbb{R}^{2n})$ is defined by

$$\text{div}_H \varphi = \sum_{i=1}^n (X_i \varphi_i + Y_i \varphi_{n+i}).$$

The horizontal perimeter of a Lebesgue measurable set $E \subset \mathbb{H}^n$ in Ω is given by

$$P_H(E;\Omega) = \sup\Big\{ \int_E \text{div}_H \varphi dz dt : \varphi \in C_c^1(\Omega; \mathbb{R}^{2n}), ||\varphi||_\infty \leqslant 1 \Big\}.$$

DEFINITION 2.1. Let the Heisenberg group $\mathbb{H}^n$ be endowed with a density e^ϕ. Then the weighted volume of a measurable set $E \subset H^n$ is defined by

$$V_\phi(E) = \int_E e^\phi dz dt$$

and the weighted H-perimeter of E in an open set $\Omega \subset \mathbb{H}^n$ is defined by

$$P_{H,\phi}(E;\Omega) = \sup\Big\{ \int_E (\text{div}_{H,\phi} \varphi) e^\phi dz dt : \varphi \in C_c^1(\Omega; \mathbb{R}^{2n}), ||\varphi||_\infty \leqslant 1 \Big\}, \tag{2.1}$$

where $\text{div}_{H,\phi} \varphi = e^{-\phi} \text{div}_H (e^\phi \varphi)$ is called the weighted horizontal divergence of φ.

If $P_{H,\phi}(E;\Omega) < \infty$, we say that E has finite weighted H-perimeter in Ω. Obviously, (2.1) can also be rewritten as

$$P_{H,\phi}(E;\Omega) = \sup\{\int_E \mathrm{div}_H(e^{\phi}\varphi)dzdt : \varphi \in C_c^1(\Omega;\mathbb{R}^{2n}), ||\varphi||_\infty \leqslant 1\}. \tag{2.2}$$

If $P_{H,\phi}(E;\Omega) < \infty$, by Proposition 2.3 [24] we have

$$\int_E \mathrm{div}_{H,\phi}\varphi dV_\phi = -\int_\Omega \langle \varphi, v_E\rangle d\mu_{E,\phi}, \tag{2.3}$$

where v_E is the measure theoretic inner horizontal normal of E, $dV_\phi = e^{\phi}dzdt$ is the weighted volume measure and $d\mu_{E,\phi} = e^{\phi}d\mu_E$ is called the weighted H-perimeter measure where μ_E is H-perimeter measure. For any open set $\Omega \subset \mathbb{H}^{2n+1}$, we have $P_{H,\phi}(E;\Omega) = \mu_{E,\phi}(\Omega)$. In the case of $\Omega = \mathbb{H}^n$, we set $P_{H,\phi}(E) = P_{H,\phi}(E;\mathbb{H}^n)$.

Let $\Sigma \subset \mathbb{H}^n$ be a C^2 regular hypersurface that can be locally given by the zero set of a function $u \in C^1$. The characteristic locus Σ_0 of Σ is defined as $\{p \in \Sigma : |\nabla_H u(p)| = 0\}$. Then the horizontal mean curvature of Σ at the noncharacteristic point (z,t) is given by

$$H_\Sigma = \frac{1}{2n}\mathrm{div}_H \frac{\nabla_H u(z,t)}{|\nabla_H u(z,t)|}. \tag{2.4}$$

The definition depends on a choice of sign. We shall choose orientable embedded hypersufaces such that $H_\Sigma \geqslant 0$. A C^2 regular hypersurface $\Sigma \in \mathbb{H}^n$ is called a horizontal constant curvature surface if H_Σ is constant along the noncharacteristic locus. For a set $E = \{(z,t) \in \mathbb{H}^n : u(z,t) > 0\}$, the inner unit horizontal normal in the formula (2.3) is given on $\Sigma = \partial E$ by the vector

$$v_E = v_\Sigma = \frac{\nabla_H u(z,t)}{|\nabla_H u(z,t)|}.$$

Let the Heisenberg group $\mathbb{H}^n$ be endowed with a density e^{ϕ}. The weighted horizontal mean curvature of Σ is defined by

$$H_{\Sigma,\phi} = -\frac{1}{2n}\mathrm{div}_{H,\phi} v_\Sigma = -\frac{1}{2n}(\mathrm{div}_H v_\Sigma + \langle v_\Sigma, \nabla_H \phi\rangle). \tag{2.5}$$

By (2.4), (2.5) can also be written as

$$H_{\Sigma,\phi} = H_\Sigma - \frac{1}{2n}\langle v_\Sigma, \nabla_H \phi\rangle. \tag{2.6}$$

REMARK 2.1. The definition (2.5) of the weighted horizontal mean curvature formula can be obtained from the first variation formula for the weighted H-perimeter, see [12, 15]. In fact, the first variation formula for the weighted H-perimeter of the set E is given by

$$P'_{H,\phi}(0) = -2n\int_\Sigma (\mathrm{div}_H v_\Sigma + \langle v_\Sigma, \nabla_H \phi\rangle)\langle U, v\rangle \phi d\Sigma - \int_\Sigma \mathrm{div}_\Sigma(\langle U, v\rangle \phi v_H^\top)d\Sigma,$$

where U is a C^2 vector field with compact support on $\Sigma = \partial E$ and associated with one-parameter family of diffeomorphisms $\{\varphi_t\}_{t\in\mathbb{R}}$, v is a unit vector field normal to Σ and $d\Sigma$ is the Riemannian area element.

Up to a left translation and a dilation, any Korányi sphere can be described as the following

$$B = \{(z,t) \in \mathbb{H}^n : |t| < \sqrt{1-|z|^4}, |z| < 1\}.$$

From the equation of ∂B we deduce that the unit horizontal normal vectors of ∂B^+ and ∂B^-, respectively, are the following:

$$v_{\partial B^+} = \sum_{i=1}^{n} \Big[(-x_i|z| - \frac{\sqrt{1-|z|^4}}{|z|} y_i)X_i + (-y_i|z| + \frac{\sqrt{1-|z|^4}}{|z|} x_i)Y_i\Big],$$

$$v_{\partial B^-} = \sum_{i=1}^{n} \Big[(-x_i|z| + \frac{\sqrt{1-|z|^4}}{|z|} y_i)X_i + (-y_i|z| - \frac{\sqrt{1-|z|^4}}{|z|} x_i)Y_i\Big].$$

Then it follows that

$$X_i\Big(-x_i|z| - \frac{\sqrt{1-|z|^4}}{|z|} y_i\Big) = -|z| - \frac{x_i^2}{|z|} + \frac{2x_iy_i}{\sqrt{1-|z|^4}},$$

$$Y_i\Big(-y_i|z| + \frac{\sqrt{1-|z|^4}}{|z|} x_i\Big) = -|z| - \frac{y_i^2}{|z|} - \frac{2x_iy_i}{\sqrt{1-|z|^4}}$$

and

$$X_i\Big(-x_i|z| + \frac{\sqrt{1-|z|^4}}{|z|} y_i\Big) = -|z| - \frac{x_i^2}{|z|} - \frac{2x_iy_i}{\sqrt{1-|z|^4}},$$

$$Y_i\Big(-y_i|z| - \frac{\sqrt{1-|z|^4}}{|z|} x_i\Big) = -|z| - \frac{y_i^2}{|z|} + \frac{2x_iy_i}{\sqrt{1-|z|^4}},$$

respectively. So the horizontal mean curvature of ∂B is

$$H_{\partial B} = -\frac{1}{2n}\mathrm{div}_H v_{\partial B} = \frac{2n+1}{2n}|z|.$$

This shows the Korányi sphere is not the isoperimetric set in Heisenberg group $\mathbb{H}^n$.

Now we endow the Heisenberg group $\mathbb{H}^n$ with density $e^{\phi} = |z|^{-(2n+1)}e^{-\frac{\alpha}{|z|}}$ $(\alpha > 0)$. We find $\phi = -\ln|z|^{2n+1} - \frac{\alpha}{|z|}$ and compute

$$\nabla_H \phi = \sum_{i=1}^{n} \Big[\frac{-(2n+1)|z| + \alpha}{|z|^3}(x_iX_i + y_iY_i)\Big].$$

So the weighted horizontal mean curvature of ∂B is

$$H_{\partial B,\phi} = H_{\partial B} - \frac{1}{2n}\langle v_{\partial B}, \nabla_H \phi\rangle = \frac{\alpha}{2n}.$$

3. Proof of Theorem 1.1

In this section, we prove the weighted quantitative inequality for the Korányi sphere by a sub-calibration argument. The proof of Theorem 1.1 is based on the foliation of the half-cylinder C_ε by a family of constant weighted horizontal mean curvature surfaces with quantitative estimates on the mean curvature. We give the following lemma first.

LEMMA 3.1. *Let the Heisenberg group $\mathbb{H}^n$ be endowed with density*

$$e^{\phi} = |z|^{-(2n+1)} e^{-\frac{\alpha}{|z|}}.$$

There exists a continuous function $u : C_\varepsilon \to \mathbb{R}$ with level sets $\Sigma_\varepsilon = \{(z,t) \in C_\varepsilon : u(z,t) = s\}$, $s \in \mathbb{R}$, such that:

(i) $u \in C^1(C_\varepsilon \cap B) \cap C^1(C_\varepsilon \setminus B)$ *and* $\frac{\nabla_H u}{|\nabla_H u|}$ *is continuously defined on* $C_\varepsilon \setminus \{z = 0\}$*;*

(ii) $\cup_{s>1} \Sigma_s = C_\varepsilon \cap B$ *and* $\cup_{s \leqslant 1} \Sigma_s = C_\varepsilon \setminus B$*;*

(iii) Each Σ_s is a hypersurface of class C^2 with constant weighted horizontal mean curvature, and namely,

$$H_{\Sigma_s,\phi} = \frac{\alpha}{2ns^2} \quad \text{for} \quad s > 1$$

and

$$H_{\Sigma_s,\phi} = \frac{\alpha}{2n} \quad \text{for} \quad s \leqslant 1;$$

(iv) For any point $(z, f(|z|) - t) \in \Sigma_s$ with $s > 1$, we have

$$1 - \frac{2n}{\alpha} H_{\Sigma_s,\phi}(z, f(|z|) - t) \geqslant \frac{1}{2+\sqrt{2}} t^2 \quad \text{when} \quad \varepsilon = 0 \tag{3.1}$$

and

$$1 - \frac{2n}{\alpha} H_{\Sigma_s,\phi}(z, f(|z|) - t) \geqslant \frac{2\sqrt{1-(1-\varepsilon)^4}}{1+(1-\varepsilon)^4+\sqrt{1+(1-\varepsilon)^4}} t \quad \text{when} \quad 0 < \varepsilon < 1. \tag{3.2}$$

Proof. The profile function of the set B is the function $f : [0,1] \to \mathbb{R}$

$$f(r) = \sqrt{1 - r^4}. \tag{3.3}$$

Its first and second derivatives are

$$f'(r) = -\frac{2r^3}{\sqrt{1-r^4}} \quad \text{and} \quad f''(r) = -\frac{2r^2(3-2r^4)}{(1-r^4)^{\frac{3}{2}}}, \quad 0 \leqslant r < 1. \tag{3.4}$$

We define the function $g:[0,1)\to\mathbb{R}$

$$g(r)=2f(r)-rf'(r)=\frac{2}{\sqrt{1-r^4}}. \tag{3.5}$$

Its derivative is

$$g'(r)=\frac{4r^3}{(1-r^4)^{\frac{3}{2}}}\geqslant 0. \tag{3.6}$$

Now we construct a foliation of C_ε. In $C_\varepsilon\setminus B$, the leaves Σ_s of the foliation are vertical translations of the top part of the boundary ∂B. In $C_\varepsilon\cap B$, the leaves Σ_s are constructed as follows: the surface ∂B is dilated by a factor larger than 1 where the dilation is given by $\delta_\lambda(z,t)=(\lambda z,\lambda^2 t)$ $(\forall\lambda>0)$, and then it is translated downwards in such a way that the surface $\{t=t_\varepsilon=f(1-\varepsilon)\}$ is also the leaf.

We construct a function u on the set $C_\varepsilon\setminus B$ as

$$u(z,t)=f(|z|)-t+1,\quad (z,t)\in C_\varepsilon\setminus B. \tag{3.7}$$

Then u satisfies $u(z,t)\leqslant 1$ for $t\geqslant f(|z|)$ and $u(z,t)=1$ for $t=f(|z|)$. Let $\Sigma_s=\{(z,t)\in C_\varepsilon\setminus B:u(z,t)=s\}$. Then we have $s\leqslant 1$ and $\Sigma_1=\partial B$. From (3.7), we know $u\in C^1(C_\varepsilon\setminus B)$ and $\cup_{s\leqslant 1}\Sigma_s=C_\varepsilon\setminus B$.

In the following we will define the function u on the set

$$D_\varepsilon=C_\varepsilon\cap B=\{(z,t)\in B:|z|<1-\varepsilon,t_\varepsilon<t<f(|z|)\}.$$

Setting $r=|z|$ and $r_\varepsilon=1-\varepsilon$. Let $F_\varepsilon:D_\varepsilon\times(1,\infty)\to\mathbb{R}$ be a function

$$F_\varepsilon(z,t,s)=s^2\left[f\left(\frac{r}{s}\right)-f\left(\frac{r_\varepsilon}{s}\right)\right]+t_\varepsilon-t. \tag{3.8}$$

For any $(z,t)\in D_\varepsilon$ we have

$$\lim_{s\to 1^+}F_\varepsilon(z,t,s)=f(r)-f(r_\varepsilon)+t_\varepsilon-t=f(r)-t>0,$$

$$\begin{aligned}\lim_{s\to\infty}F_\varepsilon(z,t,s)&=\lim_{s\to\infty}\left[\frac{f(\frac{r}{s})-f(\frac{r_\varepsilon}{s})}{(\frac{1}{s})^2}+t_\varepsilon-t\right]\\&=\lim_{l\to 0^+}\frac{rf'(rl)-r_\varepsilon f'(r_\varepsilon l)}{2l}+(t_\varepsilon-t)\\&=\lim_{l\to 0^+}\frac{r^2f''(rl)-r_\varepsilon^2f''(r_\varepsilon l)}{2}+(t_\varepsilon-t)\\&=t_\varepsilon-t<0.\end{aligned}$$

On the other hand, using (3.5) and (3.8) we have

$$\partial_sF_\varepsilon=s\left[g\left(\frac{r}{s}\right)-g\left(\frac{r_\varepsilon}{s}\right)\right]<0. \tag{3.9}$$

So there exists a unique $s>1$ such that $F_\varepsilon(z,t,s)=0$ for any $(z,t)\in D_\varepsilon$. Consequently, we can define a function $u:D_\varepsilon\to\mathbb{R},s=u(z,t)$ which is determined by the equation $F_\varepsilon(z,t,s)=0$.

Obviously we have $u \in C^1(C_\varepsilon \cap B)$ and $C_\varepsilon \cap B = \cup_{s>1}\Sigma_s$, where $\Sigma_s = \{(z,t) \in C_\varepsilon \cap B : s = u(z,t)$ is determined by the equation $F_\varepsilon(z,t,s) = 0\}$.

By (3.8), we find

$$\partial_{x_i} F_\varepsilon(z,t,s) = \frac{sx_i}{r} f'\left(\frac{r}{s}\right), \quad \partial_{y_i} F_\varepsilon(z,t,s) = \frac{sy_i}{r} f'\left(\frac{r}{s}\right), \quad i = 1,\cdots,n. \tag{3.10}$$

By the implicit function theorem, the derivatives of u can be computed from the partial derivatives of F_ε. Namely using (3.4), (3.9) and (3.10), we obtain

$$\partial_{x_i} u(z,t) = -\frac{\partial_{x_i} F_\varepsilon}{\partial_s F_\varepsilon} = \frac{2r^2 x_i}{s\sqrt{s^4 - r^4}\left[g(\frac{r}{s}) - g(\frac{r_\varepsilon}{s})\right]},$$

$$\partial_{y_i} u(z,t) = -\frac{\partial_{y_i} F_\varepsilon}{\partial_s F_\varepsilon} = \frac{2r^2 y_i}{s\sqrt{s^4 - r^4}\left[g(\frac{r}{s}) - g(\frac{r_\varepsilon}{s})\right]},$$

$$\partial_t u(z,t) = -\frac{\partial_t F_\varepsilon}{\partial_s F_\varepsilon} = \frac{1}{s[g(\frac{r}{s}) - g(\frac{r_\varepsilon}{s})]}. \tag{3.11}$$

Then we have

$$X_i u = (\partial_{x_i} + 2y_i\partial_t)u = \frac{2r^2 x_i + 2y_i\sqrt{s^4 - r^4}}{s\sqrt{s^4 - r^4}\left[g(\frac{r}{s}) - g(\frac{r_\varepsilon}{s})\right]},$$

$$Y_i u = (\partial_{y_i} - 2x_i\partial_t)u = \frac{2r^2 y_i - 2x_i\sqrt{s^4 - r^4}}{s\sqrt{s^4 - r^4}\left[g(\frac{r}{s}) - g(\frac{r_\varepsilon}{s})\right]}.$$

So the squared length of the horizontal gradient of u on D_ε is

$$|\nabla_H u|^2 = \sum_{i=1}^n (X_i u)^2 + (Y_i u)^2 = \frac{4r^2 s^2}{(s^4 - r^4)\left[g(\frac{r}{s}) - g(\frac{r_\varepsilon}{s})\right]^2}.$$

Note that $|\nabla_H u(z,t)| = 0$ if and only if $z = 0$. So for any $(z,t) \in D_\varepsilon$ with $z \neq 0$, we have

$$\frac{X_i u}{|\nabla_H u|} = -\frac{r^2 x_i + \sqrt{s^4 - r^4}\, y_i}{s^2 r}, \quad \frac{Y_i u}{|\nabla_H u|} = -\frac{r^2 y_i - \sqrt{s^4 - r^4}\, x_i}{s^2 r}, \quad i = 1,\cdots,n. \tag{3.12}$$

If $(z,t) \in D_\varepsilon$ tends to $(\bar{z},\bar{t}) \in \partial B$ with $\bar{z} \neq 0$ and $\bar{t} > 0$, then $s = u(z,t)$ converges to 1. From (3.12), we have

$$\begin{aligned}\lim_{(z,t)\to(\bar{z},\bar{t})} \frac{\nabla_H u(z,t)}{|\nabla_H u(z,t)|} &= \sum_{i=1}^n \Big(\frac{-r^2\bar{x}_i - \sqrt{1 - r^4}\,\bar{y}_i}{r} X_i + \frac{-r^2\bar{y}_i + \sqrt{1 - r^4}\,\bar{x}_i}{r} Y_i \Big) \\ &= \frac{\nabla_H u(\bar{z},\bar{t})}{|\nabla_H u(\bar{z},\bar{t})|},\end{aligned}$$

where the last equality is computed by the definition (3.7) of u. The above equality shows that $\frac{\nabla_H u}{|\nabla_H u|}$ is continuous on $C_\varepsilon \setminus \{z = 0\}$. We complete the proof of claims (i) and (ii).

In the case of $e^{\phi} = |z|^{-(2n+1)}e^{-\frac{\alpha}{|z|}}$, we get $\phi = -(2n+1)\ln|z| - \frac{\alpha}{|z|}$ and

$$\nabla_H \phi = \sum_{i=1}^{n} \frac{-(2n+1)r+\alpha}{r^3}(x_iX_i + y_iY_i), \quad z \neq 0.$$

From (3.7), we know that the inner unit horizontal normal of Σ_s with $s \leqslant 1$ is

$$v_{\Sigma_s} = \sum_{i=1}^{n}\Big(-\frac{r^2x_i+\sqrt{1-r^4}y_i}{r}X_i - \frac{r^2y_i-\sqrt{1-r^4}x_i}{r}Y_i\Big).$$

So the weighted horizontal mean curvature $H_{\Sigma_s,\phi}$ of Σ_s with $s \leqslant 1$ is

$$H_{\Sigma_s,\phi} = \frac{1}{2n}(-\mathrm{div}_H v_{\Sigma_s} - \langle v_{\Sigma_s}, \nabla_\alpha \phi\rangle) = \frac{\alpha}{2n}.$$

From (3.12) we compute the inner unit horizontal normal of Σ_s with $s > 1$ as

$$v_{\Sigma_s} = \sum_{i=1}^{n}\Big(-\frac{r^2x_i+\sqrt{s^4-r^4}y_i}{sr}X_i - \frac{r^2y_i-\sqrt{s^4-r^4}x_i}{sr}Y_i\Big).$$

So the weighted horizontal mean curvature $H_{\Sigma_s,\phi}$ of Σ_s with $s > 1$ is

$$H_{\Sigma_s,\phi} = \frac{\alpha}{2ns^2}.$$

Thus we prove claim (iii). At last we prove claim (iv).

Fixing a point z with $|z| < 1-\varepsilon$ and for $0 \leqslant t < f(|z|) - t_\varepsilon$, we define the function

$$h_z(t) = u(z, f(|z|) - t) = s = \frac{1}{\sqrt{\frac{2n}{\alpha}H_{\Sigma_s,\phi}}}, \tag{3.13}$$

where $s \geqslant 1$ is uniquely determined by $(z, f(|z|)-t) \in \Sigma_s$. Then the function $t \to h_z(t)$ is increasing and $h_z(0) = u(z, f(|z|) = 1$.

From (3.11), for all $0 \leqslant t < f(|z|) - t_\varepsilon$, we know

$$h_z'(t) = -\partial_t u(z, f(|z|) - t) = \frac{1}{h_z(t)\big[g(\frac{r_\varepsilon}{h_z(t)}) - g(\frac{r}{h_z(t)})\big]}.$$

Since g is strictly increasing, $h_z(t)$ satisfies

$$h_z'(t) \geqslant \frac{1}{h_z(t)\big[g(\frac{r_\varepsilon}{h_z(t)}) - g(0)\big]} = \frac{1}{h_z(t)\big[g(\frac{r_\varepsilon}{h_z(t)}) - 2\big]}.$$

This is equivalent to

$$2h_z(t)h_z'(t)\Big[\frac{h_z(t)^2}{\sqrt{h_z(t)^4 - r_\varepsilon^4}} - 1\Big] \geqslant 1. \tag{3.14}$$

Integrating both sides of (3.14) on the interval $[0,t]$, we obtain

$$\sqrt{h_z(t)^4 - r_\varepsilon^4} - \sqrt{1 - r_\varepsilon^4} - h_z(t)^2 + 1 \geqslant t.$$

It follows that

$$\begin{aligned} h_z(t)^2 &\geqslant \sqrt{[t + \sqrt{1 - r_\varepsilon^4} + h_z(t)^2 - 1]^2 + r_\varepsilon^4} \\ &\geqslant \sqrt{t^2 + 1 + 2t\sqrt{1 - r_\varepsilon^4}}. \end{aligned} \tag{3.15}$$

From (3.13) and (3.15), we have

$$\begin{aligned} 1 - \frac{2n}{\alpha} H_{\Sigma_s,\phi}(z, f(|z|) - t) &= 1 - \frac{1}{h_z(t)^2} \\ &\geqslant 1 - \frac{1}{\sqrt{t^2 + 1 + 2t\sqrt{1 - r_\varepsilon^4}}}. \end{aligned} \tag{3.16}$$

When $\varepsilon = 0$, we have $r_\varepsilon = 1$. So (3.16) turns into

$$\begin{aligned} 1 - \frac{2n}{\alpha} H_{\Sigma_s,\phi}(z, f(|z|) - t) &\geqslant 1 - \frac{1}{\sqrt{t^2 + 1}} \\ &= \frac{t^2}{\sqrt{t^2 + 1}(\sqrt{t^2 + 1} + 1)} \\ &\geqslant \frac{t^2}{2 + \sqrt{2}}. \end{aligned}$$

When $0 < \varepsilon < 1$, we have $r_\varepsilon = 1 - \varepsilon$ and $0 \leqslant t < 1 - \sqrt{1 - r_\varepsilon^4}$. So (3.16) turns into

$$\begin{aligned} 1 - \frac{2n}{\alpha} H_{\Sigma_s,\phi}(z, f(|z|) - t) &\geqslant \frac{t(t + 2\sqrt{1 - r_\varepsilon^4})}{\sqrt{t^2 + 1 + 2t\sqrt{1 - r_\varepsilon^4}}(\sqrt{t^2 + 1 + 2t\sqrt{1 - r_\varepsilon^4}} + 1)} \\ &\geqslant \frac{2t\sqrt{1 - r_\varepsilon^4}}{\sqrt{t^2 + 1 + 2t\sqrt{1 - r_\varepsilon^4}}(\sqrt{t^2 + 1 + 2t\sqrt{1 - r_\varepsilon^4}} + 1)} \\ &\geqslant \frac{2\sqrt{1 - r_\varepsilon^4}}{\sqrt{1 + r_\varepsilon^4}(\sqrt{1 + r_\varepsilon^4} + 1)} t \\ &= \frac{2\sqrt{1 - (1 - \varepsilon)^4}}{1 + (1 - \varepsilon)^4 + \sqrt{1 + (1 - \varepsilon)^4}} t. \quad \square \end{aligned}$$

Proof of Theorem 1.1. Let $u : C_\varepsilon \to \mathbb{R}$ be the function given by Lemma 3.1 and let $\Sigma_s = \{(x,y) \in C_\varepsilon : u(z,t) = s\}$ be the leaves of the foliation, $s \in \mathbb{R}$. We define the vector field $X : C_\varepsilon \setminus \{z = 0\} \to \mathbb{H}^n$ by

$$X = -\frac{\nabla_H u}{|\nabla_H u|}.$$

Then X satisfies the following properties:

i) $|X| = 1$;

ii) For $(z,t) \in \partial B \cap C_\varepsilon$, we have $X(z,t) = -\nu_B(z,t)$, where $\nu_B(z,t)$ is the unit inner horizontal horizontal normal to ∂B;

iii) For any point $(z,t) \in \Sigma_s$ with $s \leqslant 1$, we have

$$\mathrm{div}_{H,\phi} X(z,t) = \alpha. \tag{3.17}$$

For any point $(z,t) \in \Sigma_s$ with $s > 1$, we have

$$\mathrm{div}_{H,\phi} X(z,t) = \frac{\alpha}{s^2} < \alpha. \tag{3.18}$$

Let $F \subset \mathbb{H}^{2n+1}$ be a set with the finite weighted H-perimeter such that $V_\phi(F) = V_\phi(B)$ and $F \triangle B \subset\subset C_\varepsilon$. By Theorem 2.2.2 in [7], without loss of generality we can assume that the boundary ∂F of F is C^∞.

For $\delta > 0$, let $B^\delta = \{(z,t) \in B : |z| > \delta\}$. By (3.18) and the generalized Gauss-Green formula (2.3), we have

$$\begin{aligned}
V_\phi(B^\delta \setminus F) &= \int_{B^\delta \setminus F} e^\phi \, dzdt \\
&\geqslant \int_{B^\delta \setminus F} \frac{\mathrm{div}_{H,\phi} X}{\alpha} e^\phi \, dzdt \\
&= \frac{1}{\alpha} \Big\{ \int_{\partial F \cap B^\delta} \langle X, \nu_F \rangle d\mu_{F,\phi} - \int_{\partial B^\delta \setminus F} \langle X, \nu_{B^\delta} \rangle d\mu_{B^\delta,\phi} \Big\}.
\end{aligned}$$

Letting $\delta \to 0^+$ and using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
V_\phi(B \setminus F) &= \int_{B \setminus F} e^\phi \, dzdt \\
&\geqslant \int_{B \setminus F} \frac{\mathrm{div}_{H,\phi} X}{\alpha} e^\phi \, dzdt \\
&= \frac{1}{\alpha} \Big\{ \int_{\partial F \cap B} \langle X, \nu_F \rangle d\mu_{F,\phi} - \int_{\partial B \setminus F} \langle X, \nu_B \rangle d\mu_{B,\phi} \Big\} \qquad (3.19) \\
&\geqslant \frac{1}{\alpha} \Big\{ \int_{\partial B \setminus F} d\mu_{B,\phi} - \int_{\partial F \cap B} d\mu_{F,\phi} \Big\} \\
&= \frac{1}{\alpha} \Big\{ P_{H,\phi}(B; C_\varepsilon \setminus F) - P_{H,\phi}(F; B) \Big\}.
\end{aligned}$$

By a similar computation, we also have

$$\begin{aligned}
V_\phi(F \setminus B) &= \int_{F \setminus B} e^\phi \, dzdt \\
&= \int_{F \setminus B} \frac{\mathrm{div}_{H,\phi} X}{\alpha} e^\phi \, dzdt
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\alpha}\Big\{ -\int_{\partial F\setminus B}\langle X,\nu_F\rangle d\mu_{F,\phi} + \int_{\partial B\cap F}\langle X,\nu_B\rangle d\mu_{B,\phi}\Big\}\\
&\leqslant \frac{1}{\alpha}\Big\{ \int_{\partial F\setminus B} d\mu_{F,\phi} - \int_{\partial B\cap F} d\mu_{B,\phi}\Big\}\\
&= \frac{1}{\alpha}\Big\{ P_{H,\phi}(F;C_\varepsilon\setminus B) - P_{H,\phi}(B;F)\Big\}. \qquad (3.20)
\end{aligned}
$$

On the other hand , we have

$$
\begin{aligned}
\int_{B\setminus F}\frac{\mathrm{div}_{H,\phi}X}{\alpha}e^{\phi}dzdt &= \int_{B\setminus F}\Big[1+\Big(\frac{\mathrm{div}_{H,\phi}X}{\alpha}-1\Big)\Big]e^{\phi}dzdt\\
&= V_\phi(B\setminus F) - \int_{B\setminus F}\Big(1-\frac{\mathrm{div}_{H,\phi}X}{\alpha}\Big)e^{\phi}dzdt.
\end{aligned} \qquad (3.21)
$$

From (3.19), (3.20) and (3.21), we obtain

$$
\begin{aligned}
&\frac{1}{\alpha}\{P_{H,\phi}(B;C_\varepsilon\setminus F) - P_{H,\phi}(F;B)\}\\
&\leqslant \int_{B\setminus F}\frac{\mathrm{div}_{H,\phi}X}{\alpha}e^{\phi}dzdt\\
&= V_\phi(B\setminus F) - \int_{B\setminus F}\Big(1-\frac{\mathrm{div}_{H,\phi}X}{\alpha}\Big)e^{\phi}dzdt\\
&= V_\phi(F\setminus B) - \int_{B\setminus F}\Big(1-\frac{\mathrm{div}_{H,\phi}X}{\alpha}\Big)e^{\phi}dzdt\\
&\leqslant \frac{1}{\alpha}\{P_{H,\phi}(F;C_\varepsilon\setminus B) - P_{H,\phi}(B;F)\} - \int_{B\setminus F}\Big(1-\frac{\mathrm{div}_{H,\phi}X}{\alpha}\Big)e^{\phi}dzdt.
\end{aligned}
$$

This is equivalent to

$$
P_{H,\phi}(F) - P_{H,\phi}(B) \geqslant \alpha\int_{B\setminus F}\Big(1-\frac{\mathrm{div}_{H,\phi}X}{\alpha}\Big)e^{\phi}dzdt. \qquad (3.22)
$$

For any z with $|z|<1-\varepsilon$, we define the vertical sections $B^z=\{t:(z,t)\in B\}$ and $F^z=\{t:(z,t)\in F\}$. By Fubini theorem, we have

$$
\int_{B\setminus F}\Big(1-\frac{\mathrm{div}_{H,\phi}X}{\alpha}\Big)e^{\phi}dzdt = \int_{\{|z|<1-\varepsilon\}}\int_{B^z\setminus F^z}\Big(1-\frac{\mathrm{div}_{H,\phi}X}{\alpha}\Big)|z|^{-(2n+1)}e^{-\frac{\alpha}{|z|}}dtdz.
$$

Let $m(z)=\mathscr{L}^1(B^z\setminus F^z)$, where $\mathscr{L}^1$ denotes the 1-dimensional Lebesgue measure, then we obtain

$$
\begin{aligned}
&\int_{B\setminus F}\Big(1-\frac{\mathrm{div}_{H,\phi}X}{\alpha}\Big)|z|^{-(2n+1)}e^{-\frac{\alpha}{|z|}}dzdt\\
&= \int_{\{|z|<1-\varepsilon\}}\int_{f(|z|)-m(z)}^{f(|z|)}\Big(1-\frac{\mathrm{div}_{H,\phi}X}{\alpha}\Big)dt|z|^{-(2n+1)}e^{-\frac{\alpha}{|z|}}dz \qquad (3.23)\\
&= \int_{\{|z|<1-\varepsilon\}}\int_0^{m(z)}\Big(1-\frac{1}{h_z(t)^2}\Big)dt|z|^{-(2n+1)}e^{-\frac{\alpha}{|z|}}dz,
\end{aligned}
$$

where $h_z(t) = u(z, f(|z|) - t)$ is the function introduced in (3.13).

So from (3.22) and (3.23) we have

$$P_{H,\phi}(F) - P_{H,\phi}(B) \geqslant \alpha \int_{\{|z|<1-\varepsilon\}} \int_0^{m(z)} \Big(1 - \frac{1}{h_z(t)^2}\Big) dt |z|^{-(2n+1)} e^{-\frac{\alpha}{|z|}} dz. \tag{3.24}$$

When $\varepsilon = 0$, by (3.1), (3.24) and Hölder inequality, we have

$$\begin{aligned}
&P_{H,\phi}(F) - P_{H,\phi}(B) \\
&\geqslant \alpha \int_{\{|z|<1\}} \int_0^{m(z)} \frac{1}{2+\sqrt{2}} t^2 dt |z|^{-(2n+1)} e^{-\frac{\alpha}{|z|}} dz \\
&= \frac{\alpha}{3(2+\sqrt{2})} \int_{\{|z|<1\}} m(z)^3 |z|^{-(2n+1)} e^{-\frac{\alpha}{|z|}} dz \\
&\geqslant \frac{\alpha a_\alpha}{3(2+\sqrt{2})} \int_{\{|z|<1\}} (m(z)|z|^{-(2n+1)} e^{-\frac{\alpha}{|z|}})^3 dz \\
&\geqslant \frac{\alpha a_\alpha}{3(2+\sqrt{2})\omega_{2n}^2} \Big(\int_{\{|z|<1\}} m(z)|z|^{-(2n+1)} e^{-\frac{\alpha}{|z|}} dz \Big)^3, \\
&= \frac{\alpha a_\alpha}{3(2+\sqrt{2})\omega_{2n}^2} V_\phi(B \triangle F)^3
\end{aligned}$$

where ω_{2n} is the Euclidean volume of the $2n$-dimensional unit sphere and a_α is the minimum of $|z|^{2(2n+1)} e^{\frac{2\alpha}{|z|}}$ on $(0,1)$. In fact, when $\alpha \geqslant 2n+1$, we have $a_\alpha = e^{2\alpha}$; when $\alpha < 2n+1$, we have $a_\alpha = (\frac{\alpha e}{2n+1})^{2(2n+1)}$.

When $0 < \varepsilon < 1$, by (3.2), (3.24) and Hölder inequality, we have

$$\begin{aligned}
&P_{H,\phi}(F) - P_{H,\phi}(B) \\
&\geqslant \alpha \int_{\{|z|<1-\varepsilon\}} \int_0^{m(z)} \frac{2\sqrt{1-(1-\varepsilon)^4}}{1+(1-\varepsilon)^4+\sqrt{1+(1-\varepsilon)^4}} t dt |z|^{-(2n+1)} e^{-\frac{\alpha}{|z|}} dz \\
&= \frac{\alpha\sqrt{1-(1-\varepsilon)^4}}{1+(1-\varepsilon)^4+\sqrt{1+(1-\varepsilon)^4}} \int_{\{|z|<1-\varepsilon\}} m(z)^2 |z|^{-(2n+1)} e^{-\frac{\alpha}{|z|}} dz \\
&\geqslant \frac{\alpha b_{\alpha,\varepsilon}\sqrt{1-(1-\varepsilon)^4}}{1+(1-\varepsilon)^4+\sqrt{1+(1-\varepsilon)^4}} \int_{\{|z|<1-\varepsilon\}} (m(z)|z|^{-(2n+1)} e^{-\frac{\alpha}{|z|}})^2 dz \\
&\geqslant \frac{\alpha b_{\alpha,\varepsilon}\sqrt{1-(1-\varepsilon)^4}}{\big[1+(1-\varepsilon)^4+\sqrt{1+(1-\varepsilon)^4}\big]\omega_{2n}} \Big(\int_{\{|z|<1-\varepsilon\}} m(z)|z|^{-(2n+1)} e^{-\frac{\alpha}{|z|}}) dz \Big)^2 \\
&= \frac{\alpha b_{\alpha,\varepsilon}\sqrt{1-(1-\varepsilon)^4}}{\big[1+(1-\varepsilon)^4+\sqrt{1+(1-\varepsilon)^4}\big]\omega_{2n}} V_\phi(B \triangle F)^2,
\end{aligned}$$

where $b_{\alpha,\varepsilon}$ is the minimum of $|z|^{2n+1} e^{\frac{\alpha}{|z|}}$ on $(0, 1-\varepsilon)$. In fact, when $\alpha \geqslant (2n+1)(1-\varepsilon)$, we have $b_{\alpha,\varepsilon} = (1-\varepsilon)^{2n+1} e^{\frac{\alpha}{1-\varepsilon}}$; when $\alpha < (2n+1)(1-\varepsilon)$, we have $b_{\alpha,\varepsilon} = (\frac{\alpha e}{2n+1})^{2n+1}$. □

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