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A DOUBLE INEQUALITY FOR THE APÉRY CONSTANT

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Abstract. A remarkable result which led to Apéry's proof of the irrationality of $\zeta(3)$ is given by the rapidly convergent series

$$\zeta(3)=\frac{5}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^3\binom{2k}{k}}=\frac{5}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(k!)^2}{k^3(2k)!}.$$

Let

$$R_n=\zeta(3)-\frac{5}{2}\sum_{k=1}^{n}\frac{(-1)^{k-1}(k!)^2}{k^3(2k)!}$$

denote the remainder of the series. In this paper, we obtain an asymptotic expansion of $(-1)^nR_n$. Based on the obtained result, we establish the upper and lower bounds of $(-1)^nR_n$. As an application of the obtained bounds, we give an approximate value of $\zeta(3)$.

1. Introduction

Euler's gamma function:

$$\Gamma(z)=\int_0^{\infty}t^{z-1}e^{-t}\,\mathrm{d}t,\qquad \Re(z)>0$$

is one of the most important functions in mathematical analysis and has applications in many diverse areas. The Riemann zeta function $\zeta(s)$ is defined by

$$\zeta(s):=\sum_{n=1}^{\infty}\frac{1}{n^s},\qquad \Re(s)>1.$$

This function plays a central role in the applications of complex analysis to number theory. The number-theoretic properties of $\zeta(s)$ are exhibited by the following result known as Euler's formula, which gives a relationship between the set of primes and the set of positive integers:

$$\zeta(s)=\prod_{p}\left(1-p^{-s}\right)^{-1},\qquad \Re(s)>1,$$

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where the product is taken over all primes. It is readily seen that $\zeta(s) \neq 0$ when $\Re(s) \geqslant 1$, and the Riemann's functional equation for $\zeta(s)$:

$$\zeta(s) = 2(2\pi)^{s-1}\Gamma(1-s)\sin\left(\frac{1}{2}\pi s\right)\zeta(1-s) \tag{1.1}$$

shows that $\zeta(s) \neq 0$ when $\Re(s) \leqslant 0$ except for the trivial zeros in

$$\zeta(-2n) = 0, \qquad n \in \mathbb{N} := \{1,2,\ldots\}.$$

Furthermore, in view of the following known relation:

$$\zeta(s) = \frac{1}{1-2^{1-s}}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}, \qquad \Re(s) > 0 \quad \text{and} \quad s \neq 1,$$

we find that $\zeta(s) < 0$ for $0 < s < 1, s \in \mathbb{R}$. The assertion that all the non-trivial zeros of $\zeta(s)$ have real part $\frac{1}{2}$ is popularly known as the Riemann hypothesis which was conjectured (but not proven) in the memoir of Riemann [16]. This hypothesis is still one of the most challenging mathematical problems today (see Edwards [9]), which was unanimously chosen to be one of the seven greatest unsolved mathematical puzzles of our time, so-called the millennium problems (see Devlin [8]).

Leonhard Euler (1707–1783), in 1735, considered the *Basel problem*:

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots = \zeta(2) = \frac{\pi^2}{6} \tag{1.2}$$

to 20 decimal places with only a few terms of his powerful summation formula discovered in the early 1730s, now called the Euler-Maclaurin summation formula. This probably convinced him that the sum in (1.2) equals $\pi^2/6$, which he proved in the same year 1735 (see [15]). Euler also proved

$$\zeta(2n) = (-1)^{n+1}\frac{(2\pi)^{2n}}{2\,(2n)!}B_{2n}, \quad n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \tag{1.3}$$

where B_n $(n \in \mathbb{N}_0)$ are the Bernoulli numbers defined by the following generating function:

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}, \qquad |z| < 2\pi$$

(see [18, Section 1.6]; see also [19, Section 1.7]). Subsequently, many authors have proved the Basel problem (1.2) and Equation (1.3) in various ways (see, e.g., [20]).

We get no information about $\zeta(2n+1)$ $(n \in \mathbb{N})$ from Riemann's functional equation, since both members of (1.1) vanish upon setting $s = 2n+1$ $(n \in \mathbb{N})$. In fact, until now no simple formula analogous to (1.3) is known for $\zeta(2n+1)$ or even for any special case such as $\zeta(3)$. It is not even known whether $\zeta(2n+1)$ is rational or irrational, except that the irrationality of $\zeta(3)$ was proved recently by Apéry [3]. But it is known that there are infinitely many $\zeta(2n+1)$ which are irrational (see [17] and [21]). For various series representations for $\zeta(2n+1)$, see [7] and also see [18, Chapter 4] and [19, Chapter 4].

A remarkable result which led to Apéry's proof of the irrationality of $\zeta(3)$ is given by the rapidly convergent series

$$\zeta(3) = \frac{5}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^3\binom{2k}{k}} = \frac{5}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(k!)^2}{k^3(2k)!}. \tag{1.4}$$

Chen and Srivastava [6, pp. 180–181] pointed out that the series representation (1.4) was proven independently by (among others) Hjörtnaes [12], Gosper [11, pp. 121–151], and Apéry [3].

Consider the identity (1.4) and let

$$S_n = \frac{5}{2}\sum_{k=1}^{n}\frac{(-1)^{k-1}(k!)^2}{k^3(2k)!}, \tag{1.5}$$

be the partial sums of the series (1.4). We now consider the remainder R_n defined as

$$\begin{aligned} R_n = \zeta(3) - S_n &= \frac{5}{2}\sum_{k=n+1}^{\infty}\frac{(-1)^{k-1}(k!)^2}{k^3(2k)!} = \frac{5}{2}\sum_{k=n+1}^{\infty}\frac{(-1)^{k-1}(2k+1)(\Gamma(k+1))^2}{k^3\Gamma(2(k+1))} \\ &= \frac{5}{2}\sum_{k=n+1}^{\infty}(-1)^{k-1}\frac{\sqrt{\pi}}{2^{2k}k^3}\frac{\Gamma(k+1)}{\Gamma(k+\frac{1}{2})}, \end{aligned} \tag{1.6}$$

by using the recurrence formula

$$\Gamma(x+1) = x\Gamma(x) \tag{1.7}$$

and duplication formula (see, [2, p. 256, Eq. (6.1.18)] and also [19, p. 6, Eq. (29)])

$$\Gamma(2x) = (2\pi)^{-\frac{1}{2}}2^{2x-\frac{1}{2}}\Gamma(x)\Gamma\left(x+\frac{1}{2}\right). \tag{1.8}$$

In this paper, we obtain the following asymptotic expansion:

$$(-1)^n R_n \sim \frac{1}{2^{2n+1}n^2}\sqrt{\frac{\pi}{n}}\left(1 - \frac{15}{8n} + \frac{225}{128n^2} + \frac{235}{1024n^3} - \frac{130261}{32768n^4} + \dots\right) \tag{1.9}$$

as $n \to \infty$. Moreover, we give a formula for determining the coefficients in expansion (1.9) (Theorem 3.1). Then we establish the upper and lower bounds of $(-1)^n R_n$ (Theorem 3.2). As an application of the obtained bounds, we give an approximate value of $\zeta(3)$ (Remark 3.3).

We end this section with the remark that all the numerical calculations presented in this study are performed by using the Maple software for symbolic computations.

2. Lemmas

LEMMA 2.1. (see [14, p. 141]) *The following asymptotic expansion holds:*

$$\frac{\Gamma(x+t)}{\Gamma(x+s)} \sim x^{t-s} \sum_{k=0}^{\infty} \binom{t-s}{k} B_k^{(t-s+1)}(t) x^{-k}, \qquad x \to \infty, \tag{2.1}$$

where $B_k^{(a)}(x)$ $(k \in \mathbb{N}_0)$ *denote the generalized Bernoulli polynomials defined by*

$$\left(\frac{t}{e^t - 1}\right)^a e^{xt} = \sum_{k=0}^{\infty} B_k^{(a)}(x) \frac{t^k}{k!}, \qquad |t| < 2\pi. \tag{2.2}$$

REMARK 2.1. The expansion (2.1) is analyzed in [1]. Burić and Elezović [4, Theorem 6.1] gave a recursive relation for successively determining the coefficients in expansion (2.1).

LEMMA 2.2. (see [5, Corollary 1]) *Let* $m \in \mathbb{N}_0$. *Then for* $x > 0$,

$$\sqrt{x}\exp\left(\sum_{j=1}^{2m}\left(1-\frac{1}{2^{2j}}\right)\frac{B_{2j}}{j(2j-1)x^{2j-1}}\right) < \frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})}$$
$$< \sqrt{x}\exp\left(\sum_{j=1}^{2m+1}\left(1-\frac{1}{2^{2j}}\right)\frac{B_{2j}}{j(2j-1)x^{2j-1}}\right), \tag{2.3}$$

where B_n *are the Bernoulli numbers.*

The choice $m = 2$ on the left hand side of (2.3) and the choice $m = 1$ on the right hand side of (2.3) then yields, for $x > 0$,

$$\sqrt{x}\exp\left(\frac{1}{8x} - \frac{1}{192x^3} + \frac{1}{640x^5} - \frac{17}{14336x^7}\right)$$
$$< \frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})} < \sqrt{x}\exp\left(\frac{1}{8x} - \frac{1}{192x^3} + \frac{1}{640x^5}\right). \tag{2.4}$$

LEMMA 2.3. *The following double inequality holds:*

$$\sqrt{x}\left(1 + \frac{1}{8x} + \frac{1}{128x^2} - \frac{5}{1024x^3} - \frac{21}{32768x^4}\right)$$
$$< \frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})} < \sqrt{x}\left(1 + \frac{1}{8x} + \frac{1}{128x^2}\right). \tag{2.5}$$

The left hand side of (2.5) *holds for* $x \geqslant 2$, *while the right hand side of* (2.5) *is valid for* $x \geqslant 1$.

Proof. By (2.4), it suffices to show that

$$f(x) > 0 \quad \text{for} \quad x \geqslant 2 \quad \text{and} \quad g(x) < 0 \quad \text{for} \quad x \geqslant 1,$$

where

$$f(x) = \frac{1}{8x} - \frac{1}{192x^3} + \frac{1}{640x^5} - \frac{17}{14336x^7} - \ln\left(1 + \frac{1}{8x} + \frac{1}{128x^2} - \frac{5}{1024x^3} - \frac{21}{32768x^4}\right),$$

$$g(x) = \frac{1}{8x} - \frac{1}{192x^3} + \frac{1}{640x^5} - \ln\left(1 + \frac{1}{8x} + \frac{1}{128x^2}\right).$$

Differentiation yields

$$f'(x) = -\frac{f_1(x-2)}{2048x^8 f_2(x-2)},$$

where

$$\begin{aligned} f_1(x) =& 25519973 + 85155168x + 114532528x^2 + 80050944x^3 \\ &+ 30797472x^4 + 6199296x^5 + 510720x^6 \end{aligned}$$

and

$$f_2(x) = 557739 + 1098592x + 811264x^2 + 266240x^3 + 32768x^4.$$

We then obtain $f'(x) < 0$ for $x \geqslant 2$. Hence, $f(x)$ is strictly decreasing for $x \geqslant 2$, and we have

$$f(x) > \lim_{t \to \infty} f(t) = 0 \quad \text{for} \quad x \geqslant 2.$$

Differentiation yields

$$g'(x) = \frac{129 + 788(x-1) + 1410(x-1)^2 + 992(x-1)^3 + 240(x-1)^4}{128x^6(128x^2 + 16x + 1)} > 0.$$

We then obtain $g'(x) > 0$ for $x \geqslant 1$. Hence, $g(x)$ is strictly increasing for $x \geqslant 1$, and we have

$$g(x) < \lim_{t \to \infty} g(t) = 0 \quad \text{for} \quad x \geqslant 1.$$

The proof of Lemma 2.3 is complete. $\square$

3. Main results

THEOREM 3.1. *Let R_n be defined by (1.6). As $n \to \infty$, we have*

$$(-1)^n R_n \sim \frac{1}{2^{2n+1}n^2}\sqrt{\frac{\pi}{n}}\left(\sum_{k=0}^{\infty}\frac{r_k}{n^k}\right), \tag{3.1}$$

with the coefficients r_k given by

$$r_k = \frac{5}{4}\sum_{j=1}^{\infty}P_k(j), \qquad k \in \mathbb{N}_0, \tag{3.2}$$

where

$$P_k(j) = \frac{(-1)^{j-1}}{2^{2(j-1)}}\sum_{\ell=0}^{k}\binom{1/2}{\ell}B_\ell^{(3/2)}(j+1)\frac{(-j)^{k-\ell}(k-\ell+1)(k-\ell+2)}{2}, \quad k \in \mathbb{N}_0, \tag{3.3}$$

where $B_k^{(a)}(x)$ denote the generalized Bernoulli polynomials defined by (2.2).

Proof. It follows from (1.6) that

$$(-1)^n R_n = \frac{5}{2}\sum_{j=1}^{\infty}(-1)^{j-1}u_{n+j},$$

where

$$u_k = \frac{\sqrt{\pi}}{2^{2k}k^3}\frac{\Gamma(k+1)}{\Gamma(k+\frac{1}{2})}.$$

The choice $(t,s) = (j+1, j+\frac{1}{2})$ in (2.1) yields

$$\frac{\Gamma(x+j+1)}{\sqrt{x}\,\Gamma(x+j+\frac{1}{2})} \sim \sum_{k=0}^{\infty}\binom{1/2}{k}B_k^{(3/2)}(j+1)x^{-k}, \qquad x \to \infty. \tag{3.4}$$

By using (3.4), we find, as $n \to \infty$,

$$\begin{aligned}
2^{2n+2}n^2\sqrt{\frac{n}{\pi}}(-1)^{j-1}u_{n+j} &= \frac{(-1)^{j-1}}{2^{2(j-1)}}\left(1+\frac{j}{n}\right)^{-3}\frac{\Gamma(n+j+1)}{\sqrt{n}\,\Gamma(n+j+\frac{1}{2})}\\
&\sim \frac{(-1)^{j-1}}{2^{2(j-1)}}\sum_{k=0}^{\infty}\frac{(-1)^k(k+1)(k+2)}{2}\left(\frac{j}{n}\right)^k\sum_{k=0}^{\infty}\binom{1/2}{k}B_k^{(3/2)}(j+1)\frac{1}{n^k}\\
&= \frac{(-1)^{j-1}}{2^{2(j-1)}}\sum_{k=0}^{\infty}\left(\sum_{\ell=0}^{k}\binom{1/2}{\ell}B_\ell^{(3/2)}(j+1)\frac{(-j)^{k-\ell}(k-\ell+1)(k-\ell+2)}{2}\right)\frac{1}{n^k}\\
&= \sum_{k=0}^{\infty}\frac{P_k(j)}{n^k},
\end{aligned} \tag{3.5}$$

where

$$P_k(j) = \frac{(-1)^{j-1}}{2^{2(j-1)}} \sum_{\ell=0}^{k} \binom{1/2}{\ell} B_\ell^{(3/2)}(j+1) \frac{(-j)^{k-\ell}(k-\ell+1)(k-\ell+2)}{2}, \quad k \in \mathbb{N}_0.$$

Summing the expansion (3.5) side by side, we obtain

$$2^{2n+2} n^2 \sqrt{\frac{n}{\pi}} \sum_{j=1}^{\infty} (-1)^{j-1} u_{n+j} \sim \sum_{j=1}^{\infty} \left(\sum_{k=0}^{\infty} P_k(j) \right) \frac{1}{n^k} = \sum_{k=0}^{\infty} \left(\sum_{j=1}^{\infty} P_k(j) \right) \frac{1}{n^k},$$

which can be written as

$$\begin{aligned}(-1)^n R_n &= \frac{5}{2} \sum_{j=1}^{\infty} (-1)^{j-1} u_{n+j} \sim \frac{1}{2^{2n+1} n^2} \sqrt{\frac{\pi}{n}} \sum_{k=0}^{\infty} \left(\sum_{j=1}^{\infty} \frac{5}{4} P_k(j) \right) \frac{1}{n^k} \\ &= \frac{1}{2^{2n+1} n^2} \sqrt{\frac{\pi}{n}} \sum_{k=0}^{\infty} \frac{r_k}{n^k}.\end{aligned}$$

The proof of Theorem 3.1 is completed. □

Here we give explicit numerical values of the first few terms of r_k by using the formula (3.2). This shows how easily we can determine the coefficients r_k in (3.1). Noting that (see [10] and [13, p. 19])

$$\begin{aligned}
B_0^{(a)}(x) &= 1, \\
B_1^{(a)}(x) &= x - \frac{a}{2}, \\
B_2^{(a)}(x) &= x^2 - ax + \frac{a(3a-1)}{12}, \\
B_3^{(a)}(x) &= x^3 - \frac{3a}{2} x^2 + \frac{a(3a-1)}{4} x - \frac{a^2(a-1)}{8}, \\
B_4^{(a)}(x) &= x^4 - 2ax^3 + \frac{a(3a-1)}{2} x^2 - \frac{a^2(a-1)}{2} x + \frac{a(15a^3 - 30a^2 + 5a + 2)}{240}, \\
B_5^{(a)}(x) &= x^5 - \frac{5a}{2} x^4 + \frac{5a(3a-1)}{6} x^3 - \frac{5a^2(a-1)}{4} x^2 \\
&\quad + \frac{a(15a^3 - 30a^2 + 5a + 2}{48} x - \frac{a^2(a-1)(3a^2 - 7a - 2)}{96},
\end{aligned}$$

we obtain

$$\begin{aligned}
B_0^{(3/2)}(j+1) &= 1, \\
B_1^{(3/2)}(j+1) &= j + \frac{1}{4}, \\
B_2^{(3/2)}(j+1) &= j^2 + \frac{1}{2} j - \frac{1}{16}, \\
B_3^{(3/2)}(j+1) &= j^3 + \frac{3}{4} j^2 - \frac{3}{16} j - \frac{5}{64},
\end{aligned}$$

$$B_4^{(3/2)}(j+1) = j^4 + j^3 - \frac{3}{8}j^2 - \frac{5}{16}j + \frac{21}{1280},$$
$$B_5^{(3/2)}(j+1) = j^5 + \frac{5}{4}j^4 - \frac{5}{8}j^3 - \frac{25}{32}j^2 + \frac{21}{256}j + \frac{57}{1024}.$$

Thus, we obtain from (3.3) that

$$P_0(j) = (-1)^{j-1}\frac{1}{2^{2(j-1)}},$$
$$P_1(j) = (-1)^{j}\frac{20j-1}{2^{2j+1}},$$
$$P_2(j) = (-1)^{j-1}\frac{560j^2-56j+1}{2^{2j+5}},$$
$$P_3(j) = (-1)^{j}\frac{6720j^3-1008j^2+36j+5}{2^{2j+8}},$$
$$P_4(j) = (-1)^{j-1}\frac{295680j^4-59136j^3+3168j^2+880j-21}{2^{2j+13}},$$
$$P_5(j) = (-1)^{j}\frac{3075072j^5-768768j^4+54912j^3+22880j^2-1092j-399}{2^{2j+16}}.$$

By using (3.2), we give the first few coefficients r_k as follows:

$$r_0 = \frac{5}{4}\sum_{j=1}^{\infty}P_0(j) = \frac{5}{4}\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{2^{2(j-1)}} = 1,$$
$$r_1 = \frac{5}{4}\sum_{j=1}^{\infty}P_1(j) = \frac{5}{4}\sum_{j=1}^{\infty}(-1)^{j}\frac{20j-1}{2^{2j+1}} = -\frac{15}{8},$$
$$r_2 = \frac{5}{4}\sum_{j=1}^{\infty}P_2(j) = \frac{5}{4}\sum_{j=1}^{\infty}(-1)^{j-1}\frac{560j^2-56j+1}{2^{2j+5}} = \frac{225}{128},$$
$$r_3 = \frac{5}{4}\sum_{j=1}^{\infty}P_3(j) = \frac{5}{4}\sum_{j=1}^{\infty}(-1)^{j}\frac{6720j^3-1008j^2+36j+5}{2^{2j+8}} = \frac{235}{1024},$$
$$r_4 = \frac{5}{4}\sum_{j=1}^{\infty}P_4(j) = \frac{5}{4}\sum_{j=1}^{\infty}(-1)^{j-1}\frac{295680j^4-59136j^3+3168j^2+880j-21}{2^{2j+13}}$$
$$= -\frac{130261}{32768},$$
$$r_5 = \frac{5}{4}\sum_{j=1}^{\infty}P_5(j)$$
$$= \frac{5}{4}\sum_{j=1}^{\infty}(-1)^{j}\frac{3075072j^5-768768j^4+54912j^3+22880j^2-1092j-399}{2^{2j+16}}$$
$$= \frac{1439967}{262144}.$$

We note that the values of r_k (for $k = 0,1,2,3,4$) above are equal to the coefficients appearing in (1.9).

THEOREM 3.2. *For $n \geqslant 1$, we have*

$$L_n < (-1)^n R_n < U_n, \tag{3.6}$$

where

$$L_n = \frac{1}{2^{2n+1}n^2}\sqrt{\frac{\pi}{n}}\left(1-\frac{15}{8n}\right) \quad \textit{and} \quad U_n = \frac{1}{2^{2n+1}n^2}\sqrt{\frac{\pi}{n}}. \tag{3.7}$$

Proof. First of all, we prove the left hand side of (3.6). We consider two cases to prove the left hand side of (3.6).

Case 1. $n = 2m, \quad m \in \mathbb{N}$.

The left hand side of (3.6) becomes

$$L_{2m} < R_{2m}, \quad m \in \mathbb{N}. \tag{3.8}$$

For $m \in \mathbb{N}$, let

$$\xi_m = R_{2m} - L_{2m}.$$

We have

$$\lim_{m\to\infty} \xi_m = 0.$$

In order to prove (3.8), it suffices to show that the sequence $\{\xi_m\}$ is strictly decreasing for $m \geqslant 1$. Direct computation yields

$$\begin{aligned}
\xi_m - \xi_{m+1} &= \frac{5}{2}\left(\frac{((2m+1)!)^2}{(2m+1)^3(4m+2)!} - \frac{((2m+2)!)^2}{(2m+2)^3(4m+4)!}\right) - L_{2m} + L_{2m+2} \\
&= \frac{5}{2}\left(\frac{(\Gamma(2m+2))^2}{(2m+1)^3\Gamma(4m+3)} - \frac{(\Gamma(2m+3))^2}{(2m+2)^3\Gamma(4m+5)}\right) - L_{2m} + L_{2m+2} \\
&= \frac{5\sqrt{\pi}}{(4m+1)4^{2m+1}}\left(\frac{1}{(2m+1)^2} - \frac{2m+1}{8(m+1)^2(4m+3)}\right)\frac{\Gamma(2m+1)}{\Gamma(2m+\frac{1}{2})} - L_{2m} + L_{2m+2} \\
&= \frac{5\sqrt{\pi}}{(4m+1)4^{2m+1}}\frac{24m^3+76m^2+74m+23}{8(2m+1)^2(m+1)^2(4m+3)}\frac{\Gamma(2m+1)}{\Gamma(2m+\frac{1}{2})} - L_{2m} + L_{2m+2}
\end{aligned} \tag{3.9}$$

by using (1.7) and (1.8).

By the left hand side of (2.5), we obtain, for $m \geqslant 1$,

$$\begin{aligned}
\xi_m - \xi_{m+1} > &\frac{5\sqrt{\pi}}{(4m+1)4^{2m+1}}\frac{24m^3+76m^2+74m+23}{8(2m+1)^2(m+1)^2(4m+3)} \\
&\times \sqrt{2m}\left(1+\frac{1}{8(2m)}+\frac{1}{128(2m)^2}-\frac{5}{1024(2m)^3}-\frac{21}{32768(2m)^4}\right) \\
&-\frac{1}{2m^2 4^{2m+1}}\sqrt{\frac{\pi}{2m}}\left(1-\frac{15}{16m}\right) \\
&+\frac{1}{2(m+1)^2 4^{2(m+1)+1}}\sqrt{\frac{\pi}{2(m+1)}}\left(1-\frac{15}{16(m+1)}\right),
\end{aligned}$$

which can be written for $m \geqslant 1$ as

$$\frac{4^{2m+1}}{\sqrt{2\pi m}}(\xi_m - \xi_{m+1}) > P(m) + Q(m)\sqrt{\frac{1}{m(m+1)}}, \tag{3.10}$$

where

$$Q(m) = \frac{1}{64(m+1)^2}\left(1 - \frac{15}{16(m+1)}\right)$$

and

$$\begin{aligned} P(m) &= \frac{5}{(4m+1)}\frac{24m^3+76m^2+74m+23}{8(2m+1)^2(m+1)^2(4m+3)} \\ &\quad \times \left(1 + \frac{1}{8(2m)} + \frac{1}{128(2m)^2} - \frac{5}{1024(2m)^3} - \frac{21}{32768(2m)^4}\right) \\ &\quad - \frac{1}{2m^2}\frac{1}{2m}\left(1 - \frac{15}{16m}\right) \\ &= -\frac{V(m)}{4194304(2m+1)^2(m+1)^4(4m+3)m^4(4m+1)}, \end{aligned}$$

with

$$\begin{aligned} V(m) &= 2097152m^8 - 23846912m^7 - 173529600m^6 - 448968488m^5 - 593899172m^4 \\ &\quad - 435712438m^3 - 176191665m^2 - 36126472m - 2946705 \\ &= 1695547046633835 + 2376427301166092(m-18) \\ &\quad + 750330166097163(m-18)^2 + 112556116286954(m-18)^3 \\ &\quad + 9658557388108(m-18)^4 + 503468511448(m-18)^5 \\ &\quad + 15847122432(m-18)^6 + 278142976(m-18)^7 + 2097152(m-18)^8. \end{aligned}$$

Then, (3.10) can be written for $m \geqslant 1$ as

$$\begin{aligned} \frac{4^{2m+1}}{\sqrt{2\pi m}}(\xi_m - \xi_{m+1}) &> Q(m)\sqrt{\frac{1}{m(m+1)}} \\ &\quad - \frac{V(m)}{4194304(2m+1)^2(m+1)^4(4m+3)m^4(4m+1)}. \end{aligned} \tag{3.11}$$

We find, for $m \geqslant 18$,

$$\begin{aligned} &\left(Q(m)\sqrt{\frac{1}{m(m+1)}}\right)^2 - \left(\frac{V(m)}{4194304(2m+1)^2(m+1)^4(4m+3)m^4(4m+1)}\right)^2 \\ &\qquad = \frac{P_{18}(m-18)}{17592186044416(2m+1)^4(m+1)^8(4m+3)^2m^8(4m+1)^2}, \end{aligned}$$

where

$$P_{18}(x) = 17592186044416x^{18} + 5790028231868416x^{17} + \cdots + 915694272840257992103082188746472775$$

is a polynomial of the 18th degree, having all coefficients positive. We then obtain from (3.11) that

$$\xi_m > \xi_{m+1} \quad \text{for} \quad m \geqslant 18.$$

Direct computation yields

$$\begin{aligned}
&\xi_1 \approx 3.528 \times 10^{-3}, && \xi_2 \approx 1.117 \times 10^{-5}, && \xi_3 \approx 1.166 \times 10^{-7}, \\
&\xi_4 \approx 2.025 \times 10^{-9}, && \xi_5 \approx 4.668 \times 10^{-11}, && \xi_6 \approx 1.288 \times 10^{-12}, \\
&\xi_7 \approx 4.033 \times 10^{-14}, && \xi_8 \approx 1.383 \times 10^{-15}, && \xi_9 \approx 5.094 \times 10^{-17}, \\
&\xi_{10} \approx 1.982 \times 10^{-18}, && \xi_{11} \approx 8.071 \times 10^{-20}, && \xi_{12} \approx 3.411 \times 10^{-21}, \\
&\xi_{13} \approx 1.487 \times 10^{-22}, && \xi_{14} \approx 6.659 \times 10^{-24}, && \xi_{15} \approx 3.051 \times 10^{-25}, \\
&\xi_{16} \approx 1.426 \times 10^{-26}, && \xi_{17} \approx 6.786 \times 10^{-28}, && \xi_{18} \approx 3.279 \times 10^{-29}.
\end{aligned}$$

Hence, we have

$$\xi_m > \xi_{m+1} \quad \text{for all} \quad m \geqslant 1.$$

Case 2. $n = 2m - 1, \quad m \in \mathbb{N}.$
The left hand side of (3.6) becomes

$$L_{2m-1} < -R_{2m-1}, \quad m \in \mathbb{N}. \tag{3.12}$$

For $m \in \mathbb{N}$, let

$$\eta_m = -R_{2m-1} - L_{2m-1}.$$

We have

$$\lim_{m \to \infty} \eta_m = 0.$$

In order to prove (3.12), it suffices to show that the sequence $\{\eta_m\}$ is strictly decreasing for $m \geqslant 1$. Direct computation yields

$$\eta_m - \eta_{m+1} = \frac{5\sqrt{\pi}}{2^{4m+2}} \left(\frac{1}{4m^3} - \frac{1}{(2m+1)^2(4m+1)} \right) \frac{\Gamma(2m+1)}{\Gamma(2m+\frac{1}{2})} + L_{2m+1} - L_{2m-1}, \tag{3.13}$$

by using (1.7) and (1.8).

By the left hand side of (2.5), we obtain, for $m \geqslant 1$,

$$\begin{aligned}
\eta_m - \eta_{m+1} > \frac{5\sqrt{\pi}}{2^{4m+2}} &\left(\frac{1}{4m^3} - \frac{1}{(2m+1)^2(4m+1)}\right) \\
&\times \sqrt{2m}\left(1 + \frac{1}{8(2m)} + \frac{1}{128(2m)^2} - \frac{5}{1024(2m)^3} - \frac{21}{32768(2m)^4}\right) \\
&+ \frac{1}{2^{4m+3}(2m+1)^2}\sqrt{\frac{\pi}{2m+1}}\left(1 - \frac{15}{8(2m+1)}\right) \\
&- \frac{1}{2^{4m-1}(2m-1)^2}\sqrt{\frac{\pi}{2m-1}}\left(1 - \frac{15}{8(2m-1)}\right),
\end{aligned}$$

which can be written for $m \geqslant 1$ as

$$\frac{2^{4m-1}}{\sqrt{2\pi m}}(\eta_m - \eta_{m+1}) > A(m) + B(m) - C(m), \tag{3.14}$$

where

$$\begin{aligned}
A(m) = \frac{5}{8} &\left(\frac{1}{4m^3} - \frac{1}{(2m+1)^2(4m+1)}\right) \\
&\times \left(1 + \frac{1}{8(2m)} + \frac{1}{128(2m)^2} - \frac{5}{1024(2m)^3} - \frac{21}{32768(2m)^4}\right) \\
= &\frac{5(12m^3+20m^2+8m+1)(524288m^4+32768m^3+1024m^2-320m-21)}{16777216m^7(2m+1)^2(4m+1)},
\end{aligned}$$

$$\begin{aligned}
B(m) &= \frac{1}{16(2m+1)^2}\sqrt{\frac{1}{2m(2m+1)}}\left(1 - \frac{15}{8(2m+1)}\right) \\
&= \frac{16m-7}{128(2m+1)^3}\sqrt{\frac{1}{2m(2m+1)}},
\end{aligned}$$

$$C(m) = \frac{1}{(2m-1)^2}\sqrt{\frac{1}{2m(2m-1)}}\left(1 - \frac{15}{8(2m-1)}\right) = \frac{16m-23}{8(2m-1)^3}\sqrt{\frac{1}{2m(2m-1)}}.$$

For $m \geqslant 1$, we have

$$A(m) + B(m) > \frac{1}{8m^3} + \frac{5}{128m^4} - \frac{1105}{65536m^5}. \tag{3.15}$$

The proof of (3.15) is given in Appendix. We then obtain from (3.14) that for $m \geqslant 1$,

$$\frac{2^{4m-1}}{\sqrt{2\pi m}}(\eta_m - \eta_{m+1}) > \frac{1}{8m^3} + \frac{5}{128m^4} - \frac{1105}{65536m^5} - C(m). \tag{3.16}$$

Direct computation yields

$$\left(\frac{1}{8m^3}+\frac{5}{128m^4}-\frac{1105}{65536m^5}\right)^2-\big(C(m)\big)^2=\frac{P_8(m-2)}{4294967296m^{10}(2m-1)^7},$$

where

$$\begin{aligned}P_8(m)=&1567417733739+8282623009950m+19002234923324m^2\\&+24956756713368m^3+20739238319376m^4+11330108144544m^5\\&+4073998388544m^6+930332004480m^7+122457292800m^8\\&+7077888000m^9.\end{aligned}$$

We then obtain from (3.16)

$$\eta_m>\eta_{m+1}\quad\text{for}\quad m\geqslant 2.$$

Direct computation yields

$$\eta_1=0.24180523\ldots,\qquad \eta_2=0.00015627\ldots.$$

Hence, we have

$$\eta_m>\eta_{m+1}\quad\text{for all}\quad m\geqslant 1.$$

Now, we prove the right hand side of (3.6). We consider two cases to prove the right hand side of (3.6).

Case 1. $n=2m,\quad m\in\mathbb{N}.$
The right hand side of (3.6) becomes

$$R_{2m}<U_{2m},\quad m\in\mathbb{N}. \tag{3.17}$$

For $m\in\mathbb{N}$, let

$$x_m=R_{2m}-U_{2m}.$$

We have

$$\lim_{m\to\infty}x_m=0.$$

In order to prove (3.17), it suffices to show that the sequence $\{x_m\}$ is strictly increasing for $m\geqslant 1$. In view of (3.9), we have

$$x_m-x_{m+1}=\frac{5\sqrt{\pi}}{(4m+1)4^{2m+1}}\,\frac{24m^3+76m^2+74m+23}{8(2m+1)^2(m+1)^2(4m+3)}\,\frac{\Gamma(2m+1)}{\Gamma(2m+\frac{1}{2})}-U_{2m}+U_{2m+2}.$$

By the right hand side of (2.5), we obtain, for $m \geqslant 1$,

$$x_m - x_{m+1} < \frac{5\sqrt{\pi}}{(4m+1)4^{2m+1}} \frac{24m^3+76m^2+74m+23}{8(2m+1)^2(m+1)^2(4m+3)} \times \sqrt{2m}\left(1+\frac{1}{8(2m)}+\frac{1}{128(2m)^2}\right) - \frac{1}{2m^2 4^{2m+1}}\sqrt{\frac{\pi}{2m}} + \frac{1}{2(m+1)^2 4^{2(m+1)+1}}\sqrt{\frac{\pi}{2(m+1)}},$$

which can be written as

$$\frac{4^{2m+1}}{\sqrt{2\pi m}}(x_m - x_{m+1}) < \frac{5}{(4m+1)} \frac{24m^3+76m^2+74m+23}{8(2m+1)^2(m+1)^2(4m+3)} \times \left(1+\frac{1}{8(2m)}+\frac{1}{128(2m)^2}\right) - \frac{1}{4m^3} + \frac{1}{64(m+1)^2}\sqrt{\frac{1}{m(m+1)}} = -I(m)+J(m) = -(I(m)-J(m)),$$

where

$$I(m) = \frac{4096m^6+63744m^5+220168m^4+277060m^3+150574m^2+34701m+3072}{4096m^3(4m+1)(4m+3)(m+1)^2(2m+1)^2},$$

$$J(m) = \frac{1}{64(m+1)^2}\sqrt{\frac{1}{m(m+1)}}.$$

We find that

$$I^2(m) - J^2(m) = \frac{P_{12}(m)}{16777216m^6(m+1)^5(4m+1)^2(2m+1)^4(4m+3)^2},$$

where

$$P_{12}(m) = 471859200m^{12} + 6273761280m^{11} + 36094218240m^{10} + 115302127680m^9 + 226485168256m^8 + 288989191984m^7 + 246621642800m^6 + 142361530476m^5 + 55406153176m^4 + 14281679445m^3 + 2342489001m^2 + 222640128m + 9437184.$$

Hence, we have, for $m \geqslant 1$,

$$I^2(m) > J^2(m) \Longrightarrow I(m) > J(m) \Longrightarrow x_m < x_{m+1}.$$

Case 2. $n = 2m - 1, \quad m \in \mathbb{N}.$
The right hand side of (3.6) becomes

$$-R_{2m-1} < U_{2m-1}, \quad m \in \mathbb{N}. \tag{3.18}$$

For $m \in \mathbb{N}$, let

$$y_m = -R_{2m-1} - U_{2m-1}.$$

We have

$$\lim_{m\to\infty} y_m = 0.$$

In order to prove (3.18), it suffices to show that the sequence $\{y_m\}$ is strictly increasing for $m \geqslant 1$. In view of (3.13), we have

$$y_m - y_{m+1} = \frac{5\sqrt{\pi}}{2^{4m+2}}\left(\frac{1}{4m^3} - \frac{1}{(2m+1)^2(4m+1)}\right)\frac{\Gamma(2m+1)}{\Gamma(2m+\frac{1}{2})} + U_{2m+1} - U_{2m-1}.$$

By the right hand side of (2.5), we obtain, for $m \geqslant 1$,

$$y_m - y_{m+1} < \frac{5\sqrt{\pi}}{2^{4m+2}}\left(\frac{1}{4m^3} - \frac{1}{(2m+1)^2(4m+1)}\right)\sqrt{2m}\left(1 + \frac{1}{8(2m)} + \frac{1}{128(2m)^2}\right) + \frac{1}{2^{4m+3}(2m+1)^2}\sqrt{\frac{\pi}{2m+1}} - \frac{1}{2^{4m-1}(2m-1)^2}\sqrt{\frac{\pi}{2m-1}},$$

which can be written for $m \geqslant 1$ as

$$\frac{2^{4m-1}}{\sqrt{2\pi m}}(y_{m+1} - y_m) > -\frac{5}{8}\left(\frac{1}{4m^3} - \frac{1}{(2m+1)^2(4m+1)}\right)\left(1 + \frac{1}{8(2m)} + \frac{1}{128(2m)^2}\right) - \frac{1}{16(2m+1)^2}\frac{1}{\sqrt{2m(2m+1)}} + \frac{1}{(2m-1)^2}\frac{1}{\sqrt{2m(2m-1)}}.$$

It is easy to show that, for $m \geqslant 1$,

$$\frac{1}{\sqrt{2m(2m+1)}} < \frac{1}{2m} - \frac{1}{8m^2} + \frac{3}{64m^3},$$

$$\frac{1}{\sqrt{2m(2m-1)}} > \frac{1}{2m} + \frac{1}{8m^2} + \frac{3}{64m^3}.$$

We then obtain

$$\frac{2^{4m-1}}{\sqrt{2\pi m}}(y_{m+1} - y_m) > -\frac{5}{8}\left(\frac{1}{4m^3} - \frac{1}{(2m+1)^2(4m+1)}\right)\left(1 + \frac{1}{8(2m)} + \frac{1}{128(2m)^2}\right) - \frac{1}{16(2m+1)^2}\left(\frac{1}{2m} - \frac{1}{8m^2} + \frac{3}{64m^3}\right)$$

$$+\frac{1}{(2m-1)^2}\left(\frac{1}{2m}+\frac{1}{8m^2}+\frac{3}{64m^3}\right)$$
$$=\frac{P_6(m-1)}{16384m^5(2m+1)^2(4m+1)},$$

where

$$P_6(m)=72595+500796m+1310056m^2+1721668m^3$$
$$+1218688m^4+444416m^5+65536m^6.$$

We then obtain

$$y_{m+1}>y_m \quad \text{for} \quad m\geqslant 1.$$

The proof of Theorem 3.2 is complete. □

REMARK 3.1. Some computer experiments indicate that the following inequality holds:

$$(-1)^nR_n<\frac{1}{2^{2n+1}n^2}\sqrt{\frac{\pi}{n}}\left(1-\frac{15}{8n}+\frac{225}{128n^2}+\frac{235}{1024n^3}\right) \quad \text{for} \quad n\geqslant 1.$$

REMARK 3.2. Write (1.6) as

$$(-1)^nR_n=\left(\frac{5}{2}\frac{((n+1)!)^2}{(n+1)^3(2(n+1))!}-\frac{5}{2}\frac{((n+2)!)^2}{(n+2)^3(2(n+2))!}\right)$$
$$+\left(\frac{5}{2}\frac{((n+3)!)^2}{(n+3)^3(2(n+3))!}-\frac{5}{2}\frac{((n+4)!)^2}{(n+4)^3(2(n+4))!}\right)+\cdots$$

and

$$(-1)^nR_n=\frac{5}{2}\frac{((n+1)!)^2}{(n+1)^3(2(n+1))!}-\left(\frac{5}{2}\frac{((n+2)!)^2}{(n+2)^3(2(n+2))!}-\frac{5}{2}\frac{((n+3)!)^2}{(n+3)^3(2(n+3))!}\right)$$
$$-\left(\frac{5}{2}\frac{((n+4)!)^2}{(n+4)^3(2(n+4))!}-\frac{5}{2}\frac{((n+5)!)^2}{(n+5)^3(2(n+5))!}\right)-\cdots,$$

respectively. Noting that the sequence

$$\left\{\frac{((n+1)!)^2}{(n+1)^3(2(n+1))!}\right\}_{n=1}^{\infty}$$

is strictly decreasing, we obtain, for $n\geqslant 1$,

$$\frac{5}{2}\left(\frac{((n+1)!)^2}{(n+1)^3(2(n+1))!}-\frac{((n+2)!)^2}{(n+2)^3(2(n+2))!}\right)<(-1)^nR_n<\frac{5}{2}\frac{((n+1)!)^2}{(n+1)^3(2(n+1))!}. \tag{3.19}$$

The lower bound in (3.6) is for $n\geqslant 8$ sharper than the lower bound in (3.19), the upper bound in (3.6) is for $n\geqslant 11$ sharper than the upper bound in (3.19) and, moreover, (3.6) has a simple form.

REMARK 3.3. We now apply (3.6) to give an approximate value of $\zeta(3)$. Write (3.6) as

$$(-1)^n S_n + L_n < (-1)^n \zeta(3) < (-1)^n S_n + U_n, \tag{3.20}$$

where S_n are given in (1.5), and L_n and U_n are given in (3.7). The choice $n = 2m$ in (3.20) yields

$$p_m < \zeta(3) < q_m, \tag{3.21}$$

where

$$p_n = S_{2m} + L_{2m} \quad \text{and} \quad q_n = S_{2m} + U_{2m}.$$

For $m = 10$ in (3.21), we have

$$p_{10} = 1.2020569031595942\ldots,$$

$$q_{10} = 1.2020569031595943\ldots,$$

We then get an approximate value of $\zeta(3)$,

$$\zeta(3) \approx 1.202056903159594.$$

The choice $m = 100$ in (3.21) gives

$$\begin{aligned}\zeta(3) \approx\, &1.20205690315959428539973816151144999076498629234049\\ &88817922715553418382057863130901864558736093352581\\ &4619915779526071941849199.\end{aligned}$$

Appendix: Proof of (3.15)

Direct computation yields

$$A(1) + B(1) = \frac{2540811}{16777216} + \frac{\sqrt{6}}{2304} = 0.1525\ldots,$$
$$\left[\frac{1}{8m^3} + \frac{5}{128m^4} - \frac{1105}{65536m^5}\right]_{m=1} = \frac{9647}{65536} = 0.1472\ldots.$$

Hence, (3.15) is valid for $m = 1$.

We now prove that (3.15) holds for $m \geqslant 2$. It suffices to show that

$$B(m) > \frac{1}{8m^3} + \frac{5}{128m^4} - \frac{1105}{65536m^5} - A(m) \tag{A.1}$$

for $m \geqslant 2$. We find

$$\frac{1}{8m^3} + \frac{5}{128m^4} - \frac{1105}{65536m^5} - A(m) = \frac{P_7(m-2)}{16777216m^7(2m+1)^2(4m+1)},$$

$$\big(B(m)\big)^2-\left(\frac{1}{8m^3}+\frac{5}{128m^4}-\frac{1105}{65536m^5}-A(m)\right)^2$$
$$=\frac{P_{14}(m-2)}{281474976710656m^{14}(2m+1)^7(4m+1)^2},$$

where

$$P_7(m)=123512745+548828520m+954374332m^2+881332716m^3$$
$$+477390592m^4+153616384m^5+27394048m^6+2097152m^7$$

and

$$P_{14}(x)=58626303590400x^{14}+1745037830389760x^{13}$$
$$+\cdots+16554929018148618775$$

is a polynomial of the 14th degree, having all coefficients positive. We see that (A.1) holds for $m \geqslant 2$. Hence, (3.15) holds for all $m \geqslant 1$. □

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