

THE MOLECULAR DECOMPOSITION OF ANISOTROPIC MIXED-NORM HARDY SPACES AND THEIR APPLICATION

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Abstract. Let $\vec{p} \in (0, \infty)^n$, A be an expansive dilation on \mathbb{R}^n , and $H_A^{\vec{p}}(\mathbb{R}^n)$ be the anisotropic mixed-norm Hardy space defined via the non-tangential grand maximal function. In this paper, the authors establish its molecular decomposition. As an application, the authors obtain the boundedness of a class of singular integral operators from $H_A^{\vec{p}}(\mathbb{R}^n)$ to $H_A^{\vec{p}}(\mathbb{R}^n)$. These results are still new even in the classical isotropic setting (in the case $A := 2I_{n \times n}$).

1. Introduction

As is well known, Hardy space $H^p(\mathbb{R}^n)$ is a good substitute for the Lebesgue space $L^p(\mathbb{R}^n)$ when $p \in (0, 1]$, and it plays an important role in various fields of analysis and partial differential equation; see, for examples, [4, 5, 6, 7, 14, 16, 15, 17]. On the other hand, the mixed-norm Lebesgue space $L^{\vec{p}}(\mathbb{R}^n)$, with the exponent vector $\vec{p} \in (0, \infty]^n$, is a natural generalization of the classical Lebesgue space $L^p(\mathbb{R}^n)$, via replacing the constant exponent p by an exponent vector \vec{p} . The study of mixed-norm Lebesgue spaces originated from Benedek and Panzone [2].

Let $\vec{p} \in (0, \infty]^n$. Recently, Cleanthous et al. [3] introduced the anisotropic mixed-norm Hardy space $H_a^{\vec{p}}(\mathbb{R}^n)$, via the non-tangential grand maximal function, and then obtained its maximal function characterization. Not long afterward, Huang et al. [8] further completed some real-variable thier characterization, such as the characterization in terms of the atomic characterization and the Littlewood-Paley characterization. Moreover, they obtained the boundedness of δ -type Calderón-Zygmund operators from $H^{\vec{p}}(\mathbb{R}^n)$ to $L^{\vec{p}}(\mathbb{R}^n)$ or from $H^{\vec{p}}(\mathbb{R}^n)$ to itself.

Very recently, Huang et al. [9] also introduced the new anisotropic mixed-norm Hardy space $H_A^{\vec{p}}(\mathbb{R}^n)$ associated with a general expansive matrix A , via the non-tangential grand maximal function, and then established its various real-variable characterizations of $H_A^{\vec{p}}$, respectively, in terms of the atomic characterization and the Littlewood-Paley characterization. Nevertheless, the molecular decompositions of $H_A^{\vec{p}}(\mathbb{R}^n)$ has not been established until now. Once its molecular decomposition is established, it can be conveniently used to prove the boundedness of many important operators on the

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space $H_A^{\vec{p}}(\mathbb{R}^n)$, for example, one of the most famous operator in harmonic analysis, Calderón-Zygmund operator. To complete the theory of the new anisotropic mixed-norm Hardy space $H_A^{\vec{p}}(\mathbb{R}^n)$, in this article, we establish the molecular decomposition of $H_A^{\vec{p}}(\mathbb{R}^n)$. Then, as application, we further obtain the boundedness of anisotropic Calderón-Zygmund operators from $H_A^{\vec{p}}(\mathbb{R}^n)$ to $H_A^{\vec{p}}(\mathbb{R}^n)$.

Precisely, this article is organized as follows.

In Section 2, we first recall some notation and definitions concerning expansive dilations, the mixed-norm Lebesgue space $L^{\vec{p}}(\mathbb{R}^n)$ and the anisotropic mixed-norm Hardy space $H_A^{\vec{p}}(\mathbb{R}^n)$, via the non-tangential grand maximal function. Then, motivated by Liu et al. [10, 11] and Huang et al. [9], we introduce the anisotropic mixed-norm molecular Hardy space $H_{A,\text{mol}}^{\vec{p},q,s,\varepsilon}(\mathbb{R}^n)$ and establish its equivalence with $H_A^{\vec{p}}(\mathbb{R}^n)$ in Theorem 2.11. When it comes back to the isotropic setting, i.e., $A := 2I_{n \times n}$, this result is still new, see Remark 2.12 for more details.

Section 3 is devoted to the proof of Theorem 2.11 via the atomic characterization of $H_A^{\vec{p}}(\mathbb{R}^n)$ established in [9, Theorem 4.7] (see also Lemma 3.3 below). It is worth pointing out that some of the proof methods of the molecular characterization of $H_A^{\vec{p}}(\mathbb{R}^n) = H_A^{p,p}(\mathbb{R}^n)$ ([11, Theorem 3.9]) don't work anymore in the present setting. For example, we search out some estimates related to $L^{\vec{p}}(\mathbb{R}^n)$ norms for some series of functions which can be reduced into dealing with the $L^q(\mathbb{R}^n)$ norms of the corresponding functions (see Lemma 3.4 below). Then, by using this key lemma and the Fefferman-Stein vector-valued inequality of the Hardy-Littlewood maximal operator M_{HL} on $L^{\vec{p}}(\mathbb{R}^n)$ (see Lemma 3.5 below), we prove their equivalences with $H_A^{\vec{p}}(\mathbb{R}^n)$ and $H_{A,\text{mol}}^{\vec{p},q,s,\varepsilon}(\mathbb{R}^n)$.

In Section 4, we first recall the definition of anisotropic Calderón-Zygmund operator of Bownik [1]. Then, as an application of the molecular characterization of $H_A^{\vec{p}}(\mathbb{R}^n)$, we obtain the boundedness of anisotropic Calderón-Zygmund operator from $H_A^{\vec{p}}$ to $H_A^{\vec{p}}$ (see Theorem 4.5 below). Particularly, when $A := 2I_{n \times n}$, this result is also new.

Finally, we make some conventions on notation. Let $\mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$. For any $\alpha := (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n := (\mathbb{Z}_+)^n$, let $|\alpha| := \alpha_1 + \dots + \alpha_n$ and

$$\partial^\alpha := \left(\frac{\partial}{\partial x_1} \right)^{\alpha_1} \dots \left(\frac{\partial}{\partial x_n} \right)^{\alpha_n}.$$

Throughout the whole paper, we denote by C a positive constant which is independent of the main parameters, but it may vary from line to line. For any $q \in [1, \infty]$, we denote by q' its conjugate index, namely, $1/q + 1/q' = 1$. For any $a \in \mathbb{R}$, $[a]$ denotes the maximal integer not larger than a . The symbol $D \lesssim F$ means that $D \leq CF$. If $D \lesssim F$ and $F \lesssim D$, we then write $D \sim F$. If E is a subset of \mathbb{R}^n , we denote by χ_E its characteristic function. If there are no special instructions, any space $\mathcal{X}(\mathbb{R}^n)$ is denoted simply by \mathcal{X} . For instance, $L^2(\mathbb{R}^n)$ is simply denoted by L^2 . Denote by \mathcal{S} the space of all Schwartz functions and \mathcal{S}' its dual space (namely, the space of all tempered distributions).

2. Molecular decomposition of $H_A^{\vec{p}}$

In this section, we first recall the notion of anisotropic mixed-norm Hardy space $H_A^{\vec{p}}$, via the non-tangential grand maximal function $M_N(f)$, and then given its molecular decomposition.

We begin with recalling the notion of expansive dilations on \mathbb{R}^n ; see [1, p. 5]. A real $n \times n$ matrix A is called an *expansive dilation*, shortly a *dilation*, if $\min_{\lambda \in \sigma(A)} |\lambda| > 1$, where $\sigma(A)$ denotes the set of all *eigenvalues* of A . Let λ_- and λ_+ be two *positive numbers* such that

$$1 < \lambda_- < \min\{|\lambda| : \lambda \in \sigma(A)\} \leq \max\{|\lambda| : \lambda \in \sigma(A)\} < \lambda_+.$$

It was proved in [1, p.5, Lemma 2.2] that, for a given dilation A , there exist a number $r \in (1, \infty)$ and a set $\Delta := \{x \in \mathbb{R}^n : |Px| < 1\}$, where P is some non-degenerate $n \times n$ matrix, such that $\Delta \subset r\Delta \subset A\Delta$, and one can additionally assume that $|\Delta| = 1$, where $|\Delta|$ denotes the n -dimensional Lebesgue measure of the set Δ . Let $B_k := A^k \Delta$ for $k \in \mathbb{Z}$. Then B_k is open, $B_k \subset rB_k \subset B_{k+1}$ and $|B_k| = b^k$, here and hereafter, $b := |\det A|$. An ellipsoid $x + B_k$ for some $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$ is called a *dilated ball*. Denote by \mathfrak{B} the set of all such dilated balls, namely,

$$\mathfrak{B} := \{x + B_k : x \in \mathbb{R}^n, k \in \mathbb{Z}\}. \quad (2.1)$$

Throughout the whole paper, let σ be the *smallest integer* such that $2B_0 \subset A^\sigma B_0$ and, for any subset E of \mathbb{R}^n , let $E^{\mathbb{C}} := \mathbb{R}^n \setminus E$. Then, for all $k, j \in \mathbb{Z}$ with $k \leq j$, it holds true that

$$B_k + B_j \subset B_{j+\sigma}, \quad (2.2)$$

$$B_k + (B_{k+\sigma})^{\mathbb{C}} \subset (B_k)^{\mathbb{C}}, \quad (2.3)$$

where $E + F$ denotes the *algebraic sum* $\{x + y : x \in E, y \in F\}$ of sets $E, F \subset \mathbb{R}^n$.

DEFINITION 2.1. A *quasi-norm*, associated with dilation A , is a Borel measurable mapping $\rho_A : \mathbb{R}^n \rightarrow [0, \infty)$, for simplicity, denoted by ρ satisfying

- (i) $\rho(x) > 0$ for all $x \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, here and hereafter, $\mathbf{0}$ denotes the origin of \mathbb{R}^n ;
- (ii) $\rho(Ax) = b\rho(x)$ for all $x \in \mathbb{R}^n$, where as above $b := |\det A|$;
- (iii) $\rho(x + y) \leq H[\rho(x) + \rho(y)]$ for all $x, y \in \mathbb{R}^n$, where $H \in [1, \infty)$ is a constant independent of x and y .

In the standard dyadic case $A := 2I_{n \times n}$, $\rho(x) := |x|^n$ for all $x \in \mathbb{R}^n$ is an example of homogeneous quasi-norms associated with A , here and hereafter, $I_{n \times n}$ denotes the $n \times n$ *unit matrix*, $|\cdot|$ always denotes the *Euclidean norm* in \mathbb{R}^n .

It was proved, in [1, p. 6, Lemma 2.4], that all homogeneous quasi-norms associated with a given dilation A are equivalent. Therefore, for a given dilation A , in what

follows, for simplicity, we always use the *step homogeneous quasi-norm* ρ defined by setting, for all $x \in \mathbb{R}^n$,

$$\rho(x) := \sum_{k \in \mathbb{Z}} b^k \chi_{B_{k+1} \setminus B_k}(x) \text{ if } x \neq \mathbf{0}, \text{ or else } \rho(\mathbf{0}) := 0.$$

By (2.2), we know that, for all $x, y \in \mathbb{R}^n$,

$$\rho(x+y) \leq b^\sigma (\max\{\rho(x), \rho(y)\}) \leq b^\sigma [\rho(x) + \rho(y)].$$

Now we recall the definition of mixed-norm Lebesgue space. Let $\vec{p} := (p_1, \dots, p_n) \in (0, \infty]^n$. The *mixed-norm Lebesgue space* $L^{\vec{p}}$ is defined to be the set of all measurable functions f such that

$$\|f\|_{L^{\vec{p}}} := \left\{ \int_{\mathbb{R}} \dots \left[\int_{\mathbb{R}} |f(x_1, \dots, x_n)|^{p_1} dx_1 \right]^{\frac{p_2}{p_1}} \dots dx_n \right\}^{\frac{1}{p_n}} < \infty$$

with the usual modifications made when $p_i = \infty$ for some $i \in \{1, \dots, n\}$.

For any $\vec{p} := (p_1, \dots, p_n) \in (0, \infty]^n$, let

$$p_- := \min\{p_1, \dots, p_n\} \quad \text{and} \quad p_+ := \max\{p_1, \dots, p_n\}. \quad (2.4)$$

LEMMA 2.2. [9, Lemma 3.4] *Let $\vec{p} \in (0, \infty]^n$. Then, for any $r \in (0, \infty)$ and $f \in L^{\vec{p}}$,*

$$\| |f|^r \|_{L^{\vec{p}}} = \|f\|_{L^{\vec{p}}}^r.$$

In addition, for any $\mu \in \mathbb{C}$, $\gamma \in [0, \min\{1, p_-\}]$ and $f, g \in L^{\vec{p}}$, $\|\mu f\|_{L^{\vec{p}}} = |\mu| \|f\|_{L^{\vec{p}}}$ and

$$\|f + g\|_{L^{\vec{p}}}^\gamma \leq \|f\|_{L^{\vec{p}}}^\gamma + \|g\|_{L^{\vec{p}}}^\gamma,$$

here and hereafter, for any $\alpha \in \mathbb{R}$, $\alpha \vec{p} := (\alpha p_1, \dots, \alpha p_n)$ and

$$p := \min\{p_-, 1\} \quad (2.5)$$

with p_- as in (2.4).

A C^∞ function φ is said to belong to the Schwartz class \mathcal{S} if for every integer $\ell \in \mathbb{Z}_+$ and multi-index α , $\|\varphi\|_{\alpha, \ell} := \sup_{x \in \mathbb{R}^n} [\rho(x)]^\ell |\partial^\alpha \varphi(x)| < \infty$. The dual space of \mathcal{S} , namely, the space of all tempered distributions on \mathbb{R}^n equipped with the weak- $*$ topology, is denoted by \mathcal{S}' . For any $N \in \mathbb{Z}_+$, let

$$\mathcal{S}_N := \{\varphi \in \mathcal{S} : \|\varphi\|_{\alpha, \ell} \leq 1, |\alpha| \leq N, \ell \leq N\}.$$

In what follows, for $\varphi \in \mathcal{S}$, $k \in \mathbb{Z}$ and $x \in \mathbb{R}^n$, let $\varphi_k(x) := b^{-k} \varphi(A^{-k}x)$.

DEFINITION 2.3. Let $\varphi \in \mathcal{S}$ and $f \in \mathcal{S}'$. The *non-tangential maximal function* $M_\varphi(f)$ with respect to φ is defined by setting, for any $x \in \mathbb{R}^n$,

$$M_\varphi(f)(x) := \sup_{y \in x + B_k, k \in \mathbb{Z}} |f * \varphi_k(y)|.$$

Moreover, for any given $N \in \mathbb{N}$, the *non-tangential grand maximal function* $M_N(f)$ of $f \in \mathcal{S}'$ is defined by setting, for any $x \in \mathbb{R}^n$,

$$M_N(f)(x) := \sup_{\varphi \in \mathcal{S}_N} M_\varphi(f)(x).$$

The following anisotropic mixed-norm Hardy space $H_A^{\vec{p}}$ was introduced in [9, Definition 2.5].

DEFINITION 2.4. Let $\vec{p} \in (0, \infty)^n$, A be a dilation and $N \in [\lfloor (1/\underline{p} - 1) \ln b / \ln \lambda_- \rfloor + 2, \infty)$, where \underline{p} is as in (2.5). The *anisotropic mixed-norm Hardy space* $H_A^{\vec{p}}$ is defined as

$$H_A^{\vec{p}} := \left\{ f \in \mathcal{S}' : M_N(f) \in L^{\vec{p}} \right\}$$

and for any $f \in H_A^{\vec{p}}$, let $\|f\|_{H_A^{\vec{p}}} := \|M_N(f)\|_{L^{\vec{p}}}$.

REMARK 2.5. Let $\vec{p} \in (0, \infty)^n$.

- (i) The quasi-norm of $H_A^{\vec{p}}$ in Definition 2.4 depends on N , however, by [9, Theorem 4.7], we know that the $H_A^{\vec{p}}$ is independent of the choice of N as long as $N \in [\lfloor (1/\underline{p} - 1) \ln b / \ln \lambda_- \rfloor + 2, \infty)$.
- (ii) When $\vec{p} := \{p, \dots, p\}$, where $p \in (0, \infty)$, the space $H_A^{\vec{p}}$ is reduced to the anisotropic Hardy H_A^p studied in [1, Definition 3.11].

LEMMA 2.6. [9, Lemma 4.6] Let $\vec{p} \in (0, \infty)^n$ and $N \in \mathbb{N} \cap [\lfloor (\frac{1}{\min\{1, p_- \}} - 1) \frac{\ln b}{\ln \lambda_-} \rfloor + 2, \infty)$ with p_- as in (2.4). Then $H_A^{\vec{p}} \cap L^{\vec{p}/p_-}$ is dense in $H_A^{\vec{p}}$.

LEMMA 2.7. [9, Proposition 1] Let $\vec{p} \in (0, \infty)^n$ and $N \in \mathbb{N} \cap [\lfloor (\frac{1}{\min\{1, p_- \}} - 1) \frac{\ln b}{\ln \lambda_-} \rfloor + 2, \infty)$ with p_- as in (2.4). Then $H_A^{\vec{p}}$ is complete.

Now we introduce the definition of anisotropic mixed-norm molecules as follows.

DEFINITION 2.8. Let $\vec{p} \in (0, \infty)^n$, $q \in (1, \infty]$,

$$s \in \left[\left[\left(\frac{1}{p_-} - 1 \right) \frac{\ln b}{\ln \lambda_-} \right], \infty \right) \cap \mathbb{Z}_+ \quad (2.6)$$

and $\varepsilon \in (0, \infty)$. A measurable function \mathfrak{M} is called an *anisotropic mixed-norm* $(\vec{p}, q, s, \varepsilon)$ -*molecule* associated with a dilated ball $x_0 + B_i \in \mathfrak{B}$ if

- (i) for each $j \in \mathbb{Z}_+$, $\|\mathfrak{M}\|_{L^q(U_j(x_0+B_i))} \leq \frac{b^{-j\varepsilon}|B_i|^{1/q}}{\|\chi_{x_0+B_i}\|_{L^{\vec{p}}}}$, where $U_0(x_0+B_i) := x_0+B_i$ and for each $j \in \mathbb{N}$, $U_j(x_0+B_i) := x_0+(A^j B_i) \setminus (A^{j-1} B_i)$,
- (ii) for all $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leq s$, $\int_{\mathbb{R}^n} \mathfrak{M}(x)x^\alpha dx = 0$.

REMARK 2.9. Let $\vec{p} \in (0, \infty)^n$.

- (i) When $\vec{p} := \{p, \dots, p\}$, where $p \in (0, 1)$, the definition of the molecule in Definition 2.8 is reduced to the molecule in [11, Definition 3.7].
- (ii) When it comes back to the isotropic setting, i.e., $A := 2I_{n \times n}$, and $\rho(x) := |x|^n$ for all $x \in \mathbb{R}^n$, the definition of the molecule in Definition 2.4 is also new.

In what follows, we call an anisotropic mixed-norm $(\vec{p}, q, s, \varepsilon)$ -molecule simply by $(\vec{p}, q, s, \varepsilon)$ -molecule. Via $(\vec{p}, q, s, \varepsilon)$ -molecules, we introduce the following anisotropic mixed-norm molecular Hardy space $H_{A, \text{mol}}^{\vec{p}, q, s, \varepsilon}$.

DEFINITION 2.10. Let $\vec{p} \in (0, \infty)^n$, $q \in (1, \infty]$, A be a dilation and let s be as in (2.6). The *anisotropic mixed-norm molecular Hardy space* $H_{A, \text{mol}}^{\vec{p}, q, s, \varepsilon}$ is defined to be the set of all distributions $f \in \mathcal{S}'$ satisfying that there exist $\{\lambda_i\}_{i \in \mathbb{N}} \subset \mathbb{C}$ and a sequence of $(\vec{p}, q, s, \varepsilon)$ -molecules $\{\mathfrak{M}_i\}_{i \in \mathbb{N}}$ associated, respectively, with $\{B^{(i)}\}_{i \in \mathbb{N}} \subset \mathfrak{B}$ such that

$$f = \sum_{i \in \mathbb{N}} \lambda_i \mathfrak{M}_i \text{ in } \mathcal{S}'.$$

Moreover, for any $f \in H_{A, \text{mol}}^{\vec{p}, q, s, \varepsilon}$, let

$$\|f\|_{H_{A, \text{mol}}^{\vec{p}, q, s, \varepsilon}} := \inf \left\| \left\{ \sum_{i \in \mathbb{N}} \left[\frac{|\lambda_i| \chi_{B^{(i)}}}{\|\chi_{B^{(i)}}\|_{L^{\vec{p}}}} \right]^p \right\}^{1/p} \right\|_{L^{\vec{p}}},$$

where the infimum is taken over all the decompositions of f as above.

The following Theorem 2.11 shows the molecular characterization of $H_A^{\vec{p}}$, whose proof is given in the next section.

THEOREM 2.11. Let $\vec{p} \in (0, 1]^n$ and $q \in (1, \infty] \cap (p_+, \infty]$ with p_+ as in (2.4), s be as in (2.6), $\varepsilon \in (\max\{1, (s+1) \log_b(\lambda_+)\}, \infty)$ and $N \in \mathbb{N} \cap [(1/\underline{p} - 1) \ln b / \ln \lambda_- + 2, \infty)$ with \underline{p} as in (2.5). Then

$$H_A^{\vec{p}} = H_{A, \text{mol}}^{\vec{p}, q, s, \varepsilon}$$

with equivalent quasi-norm.

REMARK 2.12. Let $\vec{p} \in (0, 1]^n$.

- (i) Liu et al. [12] introduced the anisotropic Hardy-Lorentz space $H_A^{p,q}$, where $p \in (0, 1]$ and $q \in (0, \infty]$. When $\vec{p} := \{p, \dots, p\}$ with $p \in (0, 1]$, the molecular characterization of $H_A^{\vec{p}}$ in Theorem 2.11 is reduced to the molecular characterization of anisotropic Hardy spaces $H_A^p = H_A^{p,p}$ in [11, Theorem 3.9].
- (ii) When it comes back to the isotropic setting, i.e., $A := 2I_{n \times n}$, the molecular characterization of $H_A^{\vec{p}}$ in Theorem 2.11 is still new.

3. Proof of Theorem 2.11

To show Theorem 2.11, we recall the following notion of anisotropic mixed-norm (\vec{p}, q, s) -atoms introduced in [9, Definition 4.1].

DEFINITION 3.1. Let $\vec{p} \in (0, \infty)^n$, $q \in (1, \infty]$ and $s \in [\lfloor (1/p_- - 1) \ln b / \ln \lambda_- \rfloor, \infty) \cap \mathbb{Z}_+$ with p_- as in (2.4). An *anisotropic mixed-norm (\vec{p}, q, s) -atom* is a measurable function a on \mathbb{R}^n satisfying

- (i) $\text{supp } a \subset B$, where $B \in \mathfrak{B}$ and \mathfrak{B} is as in (2.1);
- (ii) $\|a\|_{L^q} \leq \frac{|B|^{1/q}}{\|\chi_B\|_{L^{\vec{p}}}}$;
- (iii) $\int_{\mathbb{R}^n} a(x)x^\alpha dx = 0$ for any $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leq s$.

Throughout this article, we call an anisotropic mixed-norm (\vec{p}, q, s) -atom simply by a (\vec{p}, q, s) -atom. The following anisotropic mixed-norm atomic Hardy space was introduced in [9, Definition 4.2]

DEFINITION 3.2. Let $\vec{p} \in (0, \infty)^n$, $q \in (1, \infty]$, A be a dilation and s be as in (2.6). The *anisotropic mixed-norm atomic Hardy space $H_A^{\vec{p}, q, s}$* is defined to be the set of all distributions $f \in \mathcal{S}'$ satisfying that there exist $\{\lambda_i\}_{i \in \mathbb{N}} \subset \mathbb{C}$ and a sequence of (\vec{p}, q, s) -atoms $\{a_i\}_{i \in \mathbb{N}}$ supported, respectively, on $\{B^{(i)}\}_{i \in \mathbb{N}} \subset \mathfrak{B}$ such that

$$f = \sum_{i \in \mathbb{N}} \lambda_i a_i \text{ in } \mathcal{S}'.$$

Moreover, for any $f \in H_A^{\vec{p}, q, s}$, let

$$\|f\|_{H_A^{\vec{p}, q, s}} := \inf \left\| \left\{ \sum_{i \in \mathbb{N}} \left[\frac{|\lambda_i| \chi_{B^{(i)}}}{\|\chi_{B^{(i)}}\|_{L^{\vec{p}}}} \right]^p \right\}^{1/p} \right\|_{L^{\vec{p}}},$$

where the infimum is taken over all the decompositions of f as above.

LEMMA 3.3. [9, Theorem 4.7] Let $\vec{p} \in (0, 1]^n$, $q \in (\max\{p_+, 1\}, \infty]$ with p_+ as in (2.4), $s \in [\lfloor (1/p_- - 1)\ln b/\ln \lambda_- \rfloor, \infty) \cap \mathbb{Z}_+$ with p_- as in (2.4) and $N \in \mathbb{N} \cap [\lfloor (1/\underline{p} - 1)\ln b/\ln \lambda_- \rfloor + 2, \infty)$. Then

$$H_A^{\vec{p}} = H_A^{\vec{p}, q, s}$$

with equivalent quasi-norms.

We also need the following useful technical lemma, whose proof is similar to [13, Lemma 4.1] that the details being omitted.

LEMMA 3.4. Let $\vec{p} \in (0, 1)^n$, $t \in (0, \underline{p}]$ with \underline{p} as in (2.5) and $r \in [1, \infty) \cap (p_+, \infty]$ with p_+ as in (2.4). Then there exists a positive constant C such that, for any sequence $\{B^{(k)}\}_{k \in \mathbb{N}} \subset \mathfrak{B}$ of dilated balls, numbers $\{\lambda_k\}_{k \in \mathbb{N}} \subset \mathbb{C}$ and measurable functions $\{a_k\}_{k \in \mathbb{N}}$ satisfying that, for each $k \in \mathbb{N}$, $\text{supp } a_k \subset B^{(k)}$ and $\|a_k\|_{L^r} \leq |B^{(k)}|^{1/r}$, it holds true that

$$\left\| \left(\sum_{k \in \mathbb{N}} |\lambda_k a_k|^t \right)^{1/t} \right\|_{L^{\vec{p}}} \leq C \left\| \left(\sum_{k \in \mathbb{N}} |\lambda_k \chi_{B^{(k)}}|^t \right)^{1/t} \right\|_{L^{\vec{p}}}.$$

We recall the definition of anisotropic Hardy-Littlewood maximal function $M_{HL}(f)$. For any $f \in L^1_{\text{loc}}$ and $x \in \mathbb{R}^n$,

$$M_{HL}(f)(x) := \sup_{x \in B \in \mathfrak{B}} \frac{1}{|B|} \int_B |f(z)| dz, \quad (3.1)$$

where \mathfrak{B} is as in (2.1).

LEMMA 3.5. [9, Lemma 4.4] Let $\vec{p} \in (0, \infty)^n$ and $u \in (1, \infty]$. Then there exists a positive constant C such that, for any sequence $\{f_k\}_{k \in \mathbb{N}}$ of measurable functions,

$$\left\| \left\{ \sum_{k \in \mathbb{N}} [M_{HL}(f_k)]^u \right\}^{1/u} \right\|_{L^{\vec{p}}} \leq C \left\| \left(\sum_{k \in \mathbb{N}} |f_k|^u \right)^{1/u} \right\|_{L^{\vec{p}}}$$

with the usual modification made when $u = \infty$, where M_{HL} denotes the anisotropic Hardy-Littlewood maximal operator as in (3.1).

Proof of Theorem 2.11. By the definitions of (\vec{p}, q, s) -atom and $(\vec{p}, q, s, \varepsilon)$ -molecule, we find that a (\vec{p}, ∞, s) -atom is also a $(\vec{p}, q, s, \varepsilon)$ -molecule, which implies that

$$H_A^{\vec{p}, \infty, s} \subset H_{A, \text{mol}}^{\vec{p}, q, s, \varepsilon}.$$

This combined with Lemma 3.3 further implies that to prove Theorem 2.11, it suffices to show $H_{A, \text{mol}}^{\vec{p}, q, s, \varepsilon} \subset H_A^{\vec{p}}$.

To show this, for any $f \in H_{A, \text{mol}}^{\vec{p}, q, s, \varepsilon}$, by Definition 2.10, we deduce that there exists a sequence of $(\vec{p}, q, s, \varepsilon)$ -molecules, $\{\mathfrak{M}_i\}_{i \in \mathbb{N}}$, associated with dilated balls $\{B^{(i)}\}_{i \in \mathbb{N}} \subset \mathfrak{B}$ where $B^{(i)} := x_i + B_{\ell_i}$ with $x_i \in \mathbb{R}^n$ and $\ell_i \in \mathbb{Z}$ such that

$$f = \sum_{i \in \mathbb{N}} \lambda_i \mathfrak{M}_i \text{ in } \mathcal{S}'$$

and

$$\|f\|_{H_{A, \text{mol}}^{\vec{p}, q, s, \varepsilon}} \sim \left\| \left\{ \sum_{i \in \mathbb{N}} \left[\frac{|\lambda_i| \chi_{B^{(i)}}}{\|\chi_{B^{(i)}}\|_{L^{\vec{p}}}} \right]^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{L^{\vec{p}}}. \quad (3.2)$$

To prove $f \in H_A^{\vec{p}}$, it is easy to see that, for all $N \in \mathbb{N} \cap [(1/\underline{p} - 1)/\ln \lambda_- + 2, \infty)$,

$$\begin{aligned} \|M_N(f)\|_{L^{\vec{p}}}^{\underline{p}} &= \left\| M_N \left(\sum_{i \in \mathbb{N}} \lambda_i \mathfrak{M}_i \right) \right\|_{L^{\vec{p}}}^{\underline{p}} \leq \left\| \sum_{i \in \mathbb{N}} |\lambda_i| M_N(\mathfrak{M}_i) \right\|_{L^{\vec{p}}}^{\underline{p}} \\ &\leq \left\| \sum_{i \in \mathbb{N}} |\lambda_i| M_N(\mathfrak{M}_i) \chi_{A^{2\sigma} B^{(i)}} \right\|_{L^{\vec{p}}}^{\underline{p}} + \left\| \sum_{i \in \mathbb{N}} |\lambda_i| M_N(\mathfrak{M}_i) \chi_{(A^{2\sigma} B^{(i)})^c} \right\|_{L^{\vec{p}}}^{\underline{p}} \\ &\leq \left\| \left\{ \sum_{i \in \mathbb{N}} [|\lambda_i| M_N(\mathfrak{M}_i) \chi_{A^{2\sigma} B^{(i)}}]^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{L^{\vec{p}}}^{\underline{p}} + \left\| \sum_{i \in \mathbb{N}} |\lambda_i| M_N(\mathfrak{M}_i) \chi_{(A^{2\sigma} B^{(i)})^c} \right\|_{L^{\vec{p}}}^{\underline{p}} \\ &=: \mathbf{I}_1 + \mathbf{I}_2, \end{aligned}$$

where $A^{2\sigma} B^{(i)}$ is the $A^{2\sigma}$ concentric expanse on $B^{(i)}$ that is $A^{2\sigma} B^{(i)} := x_i + A^{2\sigma} B_{\ell_i}$.

To estimate \mathbf{I}_1 , for any $\tilde{q} \in ((\max\{p_+, 1\}, q)$, by the boundedness of M_N on L^r for all $r \in (1, \infty)$ and Hölder's inequality, we have

$$\begin{aligned} &\left\| \|\chi_{B^{(i)}}\|_{L^{\vec{p}}} M_N(\mathfrak{M}_i) \chi_{A^{2\sigma} B^{(i)}} \right\|_{L^{\tilde{q}}} \quad (3.3) \\ &\leq \|\chi_{B^{(i)}}\|_{L^{\vec{p}}} \|M_N(\mathfrak{M}_i)\|_{L^{\tilde{q}}} \lesssim \|\chi_{B^{(i)}}\|_{L^{\vec{p}}} \|\mathfrak{M}_i\|_{L^{\tilde{q}}} \\ &\sim \|\chi_{B^{(i)}}\|_{L^{\vec{p}}} \sup_{\|g\|_{L^{\tilde{q}'}}=1} \left| \int_{\mathbb{R}^n} \mathfrak{M}_i(x) g(x) dx \right| \\ &\lesssim \|\chi_{B^{(i)}}\|_{L^{\vec{p}}} \sup_{\|g\|_{L^{\tilde{q}'}}=1} \sum_{j \in \mathbb{Z}_+} \int_{U_j(B^{(i)})} |\mathfrak{M}_i(x)| |g(x)| dx \\ &\lesssim \|\chi_{B^{(i)}}\|_{L^{\vec{p}}} \sup_{\|g\|_{L^{\tilde{q}'}}=1} \sum_{j \in \mathbb{Z}_+} \|\mathfrak{M}_i\|_{L^q(U_j(B^{(i)}))} \left[\int_{U_j(B^{(i)})} |g(x)|^{q'} dx \right]^{1/q'}, \end{aligned}$$

where for all $i \in \mathbb{N}$, $U_0(B^{(i)}) := B^{(i)}$ and for any $j \in \mathbb{N}$,

$$U_j(B^{(i)}) := x_i + (A^j B_{\ell_i}) \setminus (A^{j-1} B_{\ell_i}). \quad (3.4)$$

From (3.4) and Hölder's inequality, we have that, for any $i \in \mathbb{N}$ and $j \in \mathbb{Z}_+$,

$$\begin{aligned} \left[\int_{U_j(B^{(i)})} |g(x)|^{q'} dx \right]^{1/q'} &\leq |A^j B_{\ell_i}|^{1/q'} \left[\frac{1}{|A^j B_{\ell_i}|} \int_{x_i + A^j B_{\ell_i}} |g(x)|^{q'} dx \right]^{1/q'} \\ &\leq |A^j B_{\ell_i}|^{1/q'} \inf_{x \in x_i + B_{\ell_i}} \left[M_{HL}(|g|^{q'})(x) \right]^{1/q'} \\ &\leq |A^j B_{\ell_i}|^{1/q'} \left\{ \frac{1}{|B_{\ell_i}|} \int_{x_i + B_{\ell_i}} \left[M_{HL}(|g|^{q'})(x) \right]^{\tilde{q}'/q'} dx \right\}^{1/\tilde{q}'} . \end{aligned}$$

From this the size condition of \mathfrak{M}_i , $1/q' < 1 < \varepsilon$ and the fact that M_{HL} is bounded on L^r for all $r \in (1, \infty)$, we conclude that for any $\tilde{q} \in ((\max\{p_+, 1\}, q)$ and $i \in \mathbb{N}$,

$$\begin{aligned} &\| \chi_{B^{(i)}} \|_{L^{\tilde{p}}} M_N(\mathfrak{M}_i) \chi_{A^{2\sigma} B^{(i)}} \|_{L^{\tilde{q}}} \\ &\leq \| \chi_{B^{(i)}} \|_{L^{\tilde{p}}} \sup_{\|g\|_{L^{\tilde{q}'}}=1} \sum_{j \in \mathbb{Z}_+} b^{j(1/q' - \varepsilon)} \frac{|B^{(i)}|^{1/\tilde{q}}}{\| \chi_{B^{(i)}} \|_{L^{\tilde{p}}}} \\ &\quad \times \left\{ \int_{x_i + B_{\ell_i}} \left[M_{HL}(|g|^{q'})(x) \right]^{\tilde{q}'/q'} dx \right\}^{1/\tilde{q}'} \\ &\lesssim |B^{(i)}|^{1/\tilde{q}} \sup_{\|g\|_{L^{\tilde{q}'}}=1} \left\{ \int_{x_i + B_{\ell_i}} \left[M_{HL}(|g|^{q'})(x) \right]^{\tilde{q}'/q'} dx \right\}^{1/\tilde{q}'} \\ &\lesssim |B^{(i)}|^{1/\tilde{q}} \sup_{\|g\|_{L^{\tilde{q}'}}=1} \left[\int_{\mathbb{R}^n} |g(x)|^{\tilde{q}'} dx \right]^{1/\tilde{q}'} \sim |B^{(i)}|^{1/\tilde{q}} . \end{aligned}$$

By this Lemma 3.4, $\tilde{q} \in ((\max\{p_+, 1\}, q)$ and (3.2), we obtain

$$\begin{aligned} \mathbf{I}_1 &= \left\| \left\{ \sum_{i \in \mathbb{N}} \left[\frac{|\lambda_i|}{\| \chi_{B^{(i)}} \|_{L^{\tilde{p}}}} \| \chi_{B^{(i)}} \|_{L^{\tilde{p}}} M_N(\mathfrak{M}_i) \chi_{A^{\sigma} B^{(i)}} \right]^{\frac{p}{2}} \right\}^{1/p} \right\|_{L^{\tilde{p}}}^2 \\ &\lesssim \left\| \left\{ \sum_{i \in \mathbb{N}} \left[\frac{|\lambda_i|}{\| \chi_{B^{(i)}} \|_{L^{\tilde{p}}}} \chi_{B^{(i)}} \right]^{\frac{p}{2}} \right\}^{1/p} \right\|_{L^{\tilde{p}}}^2 \sim \|f\|_{H_A^{\tilde{p}, q, s, \varepsilon}}^2 . \end{aligned}$$

To deal with \mathbf{I}_2 , for any $i \in \mathbb{N}$ and $x \in (x_i + A^{2\sigma} B_{\ell_i})^c$, by $M_N(f)(x) \sim M_N^0(f)(x)$ [1, Proposition 3.10] and proceeding as in the proof of [11, (3.48)], we know that

$$M_N(\mathfrak{M}_i)(x) \lesssim \| \chi_{B^{(i)}} \|_{L^{\tilde{p}}}^{-1} \frac{|B^{(i)}|^\theta}{[\rho(x - x_i)]^\theta} \lesssim \| \chi_{B^{(i)}} \|_{L^{\tilde{p}}}^{-1} \left[M_{HL}(\chi_{B^{(i)}})(x) \right]^\theta , \quad (3.5)$$

where for any $i \in \mathbb{N}$, x_i denotes the centre of the dilated ball $B^{(i)}$ and

$$\theta := \left(\frac{\ln b}{\ln \lambda_-} + s + 1 \right) \frac{\ln \lambda_-}{\ln b} > \frac{1}{\tilde{p}} . \quad (3.6)$$

From this Remark 2.2(i), Lemma 3.5 and (3.2), we deduce that

$$\begin{aligned}
I_2 &\lesssim \left\| \sum_{i \in \mathbb{N}} \frac{|\lambda_i|}{\|\chi_{B^{(i)}}\|_{L^{\vec{p}}}} [M_{HL}(\chi_{B^{(i)}})]^\theta \right\|_{L^{\vec{p}}}^p & (3.7) \\
&\sim \left\| \left\{ \sum_{i \in \mathbb{N}} \frac{|\lambda_i|}{\|\chi_{B^{(i)}}\|_{L^{\vec{p}}}} [M_{HL}(\chi_{B^{(i)}})]^\theta \right\}^{1/\theta} \right\|_{L^{\theta \vec{p}}}^{\theta p} \\
&\lesssim \left\| \left\{ \sum_{i \in \mathbb{N}} \frac{|\lambda_i| \chi_{B^{(i)}}}{\|\chi_{B^{(i)}}\|_{L^{\vec{p}}}} \right\}^{1/\theta} \right\|_{L^{\theta \vec{p}}}^{\theta p} \sim \left\| \sum_{i \in \mathbb{N}} \frac{|\lambda_i| \chi_{B^{(i)}}}{\|\chi_{B^{(i)}}\|_{L^{\vec{p}}}} \right\|_{L^{\vec{p}}}^p \\
&\lesssim \left\| \left\{ \sum_{i \in \mathbb{N}} \left[\frac{|\lambda_i| \chi_{B^{(i)}}}{\|\chi_{B^{(i)}}\|_{L^{\vec{p}}}} \right]^p \right\}^{1/p} \right\|_{L^{\vec{p}}}^p \sim \|f\|_{H_A^{\vec{p}, q, s, \varepsilon}}^p.
\end{aligned}$$

This together with I_1 and I_2 , shows that

$$\|f\|_{H_A^{\vec{p}}} \sim \|M_N(f)\|_{L^{\vec{p}}} \lesssim \|f\|_{H_A^{\vec{p}, q, s, \varepsilon}}.$$

This implies that $f \in H_A^{\vec{p}}$ and hence $H_{A, \text{mol}}^{\vec{p}, q, s, \varepsilon} \subset H_A^{\vec{p}}$. This finishes the proof of Theorem 2.11. \square

4. Applications

In this section, as an application of the molecular characterization of $H_A^{\vec{p}}$ in Theorem 2.11, we obtain the boundedness of anisotropic Calderón-Zygmund operators from $H_A^{\vec{p}}$ to itself. Particularly, when $A := 2I_{n \times n}$, this result is still new. We recall that the definition of anisotropic Calderón-Zygmund operators associated with dilation A .

DEFINITION 4.1. A locally integrable function K on $\Omega := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y\}$ is called an *anisotropic Calderón-Zygmund kernel* (with respect to a dilation A and a quasi-norm ρ) if there exist positive constants C and δ such that

(i) $|K(x, y)| \leq \frac{C}{\rho(x-y)}$ for all $x \neq y$;

(ii) if $(x, y) \in \Omega$, $x' \in \mathbb{R}^n$ and $\rho(x' - x) \leq b^{-2\sigma} \rho(x - y)$, then

$$|K(x', y) - K(x, y)| \leq C \frac{[\rho(x' - x)]^\delta}{[\rho(x - y)]^{1+\delta}};$$

(iii) if $(x, y) \in \Omega$, $\tilde{y} \in \mathbb{R}^n$ and $\rho(\tilde{y} - y) \leq b^{-2\sigma} \rho(x - y)$, then

$$|K(x, \tilde{y}) - K(x, y)| \leq C \frac{[\rho(\tilde{y} - y)]^\delta}{[\rho(x - y)]^{1+\delta}}.$$

We call that T is an *anisotropic Calderón-Zygmund operator* if T is a continuous linear operator mapping \mathcal{S} into \mathcal{S}' that extends to a bounded linear operator on L^2 and there exists an anisotropic Calderón-Zygmund kernel K such that, for all $f \in C_c^\infty$ and $x \notin \text{supp}(f)$,

$$Tf(x) := \int_{\mathbb{R}^n} K(x, y)f(y)dy.$$

To obtain the boundedness of anisotropic Calderón-Zygmund operators on $H_A^{\vec{p}}$, we need to increase the smooth hypothesis on the kernel K . The following definition was introduced by Bownik in [1, Definition 9.2].

DEFINITION 4.2. Let $N \in \mathbb{Z}_+$. We say that T is an *anisotropic Calderón-Zygmund operator of order N* if T satisfies Definition 4.1 with the kernel K in the class C^N as a function of y . We also require that there exists a positive constant C such that for any $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leq N$ and $(x, y) \in \Omega$,

$$\left| \partial_y^\alpha \left[K \left(\cdot, A^\ell \cdot \right) \right] \left(x, A^{-\ell} y \right) \right| \leq C[\rho(x-y)]^{-1} = Cb^{-\ell}, \tag{4.1}$$

where $\ell \in \mathbb{Z}$ is the unique integer such that $\rho(x-y) = b^\ell$ with the implicit equivalent positive constants independent of x, y and ℓ . More formally,

$$\partial_y^\alpha \left[K \left(\cdot, A^\ell \cdot \right) \right] \left(x, A^{-\ell} y \right)$$

means $(\partial_y^\alpha \tilde{K})(x, A^{-\ell} y)$, where $\tilde{K}(x, y) := K(x, A^\ell y)$ for all $(x, y) \in \mathbb{R}^n$ and $x \neq A^\ell y$.

REMARK 4.3. In Definition 4.2, when $N \in \mathbb{Z}_+$, $A := 2I_{n \times n}$ and $\rho(x) := |x|^n$ for all $x \in \mathbb{R}^n$, then (4.1) becomes that for any $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leq N$ and $(x, y) \in \Omega$,

$$|\partial^\alpha K(x, y)| \leq C|x-y|^{-n-|\alpha|}, \tag{4.2}$$

which is standard and well known. More examples of anisotropic Calderón-Zygmund operator of order N as in Definition 4.1; see [1, p. 61].

To obtain the boundedness of anisotropic Calderón-Zygmund operators from $H_A^{\vec{p}}$ to $H_A^{\vec{p}}$, we need to prove that anisotropic Calderón-Zygmund operators T map atoms into harmless constant multiples of molecules. Generally, we cannot expect this unless we also assume that the considered operators preserve vanishing moments, which is given in the following definition introduced by Bownik ([1, Definition 9.4]).

DEFINITION 4.4. We say that an anisotropic Calderón-Zygmund operator of order N satisfies $T^*(x^\gamma) = 0$ for all $|\gamma| \leq s$, where $s < N \ln \lambda_- / \ln \lambda_+$, if for any $f \in L^q$ with compact support, $q \in [1, \infty) \cap (p_+, \infty]$ with p_+ as in (2.4) and

$$\int_{\mathbb{R}^n} x^\alpha f(x)dx = 0 \text{ for all } |\alpha| < N,$$

we also have

$$\int_{\mathbb{R}^n} x^\gamma T(f)(x)dx = 0 \text{ for all } |\gamma| \leq s.$$

THEOREM 4.5. *Let $\vec{p} \in (0, 1]^n$. If $N \in \mathbb{N}$ and T is an anisotropic Calderón-Zygmund operator of order N , then T can be extended to a bounded linear operator from $H_A^{\vec{p}}$ to $H_A^{\vec{p}}$, provided that $T^*(x^\alpha) = 0$ for all $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leq s$, where $s \in [(1/p_- - 1)\ln b/\ln \lambda_-], \infty) \cap \mathbb{Z}_+$ with p_- as in (2.4) and $s < N \ln \lambda_-/\ln \lambda_+$. Moreover, there exists a positive constant C such that for all $f \in H_A^{\vec{p}}$,*

$$\|T(f)\|_{H_A^{\vec{p}}} \leq C \|f\|_{H_A^{\vec{p}}}. \quad (4.3)$$

REMARK 4.6. (i) When $\vec{p} := \{p, \dots, p\}$ with $p \in (0, 1]$, the space $H_A^{\vec{p}}$ is reduced to the anisotropic Hardy space H_A^p and now Theorem 4.5 coincides with [1, Theorem 9.8] of Bownik.

(ii) When $A := 2I_{n \times n}$ and $\rho(x) := |x|^n$ for all $x \in \mathbb{R}^n$, the above result is still new.

To prove Theorem 4.5, we need some technical lemmas. The following conclusion is from [9, Theorem 4.7] and its proof, which is needed in the proof of Theorem 4.5.

LEMMA 4.7. *Let $\vec{p} \in (0, \infty)^n$ and $s \in [(1/p_- - 1)\ln b/\ln \lambda_-], \infty) \cap \mathbb{Z}_+$ with p_- as in (2.4). Then for any $f \in H_A^{\vec{p}} \cap L^{\vec{p}/p_-}$, there exist $\{\lambda_i\}_{i \in \mathbb{N}} \subset \mathbb{C}$, dilated balls $\{x_i + B_{\ell_i}\}_{i \in \mathbb{N}} \subset \mathfrak{B}$ and (\vec{p}, ∞, s) -atoms $\{a_i\}_{i \in \mathbb{N}}$ such that*

$$f = \sum_{i \in \mathbb{N}} \lambda_i a_i \text{ in } \mathcal{S}',$$

where the series also converges almost everywhere.

LEMMA 4.8. [9, Lemma 4.5] *Let $\vec{p} \in (0, \infty)^n$ and $q \in (1, \infty) \cap (p_+, \infty]$ with p_+ as in (2.4). Assume that $\{\lambda_i\}_{i \in \mathbb{N}} \subset \mathbb{C}$, $\{B^{(i)}\}_{i \in \mathbb{N}} \subset \mathfrak{B}$ and $\{a_i\}_{i \in \mathbb{N}} \in L^q$ satisfy for any $i \in \mathbb{N}$, $\text{supp } a_i \subset B^{(i)}$,*

$$\|a_i\|_{L^q} \leq \frac{|B^{(i)}|^{1/q}}{\|\chi_{B^{(i)}}\|_{L^{\vec{p}}}}$$

and

$$\left\| \left\{ \sum_{i \in \mathbb{N}} \left[\frac{|\lambda_i| \chi_{B^{(i)}}}{\|\chi_{B^{(i)}}\|_{L^{\vec{p}}}} \right]^p \right\}^{1/p} \right\|_{L^{\vec{p}}} < \infty.$$

Then

$$\left\| \left[\sum_{i \in \mathbb{N}} |\lambda_i a_i|^p \right]^{1/p} \right\|_{L^{\vec{p}}} \leq C \left\| \left\{ \sum_{i \in \mathbb{N}} \left[\frac{|\lambda_i| \chi_{B^{(i)}}}{\|\chi_{B^{(i)}}\|_{L^{\vec{p}}}} \right]^p \right\}^{1/p} \right\|_{L^{\vec{p}}},$$

where \underline{p} is as in (2.5) and C is a positive constant independent of $\{\lambda_i\}_{i \in \mathbb{N}}$, $\{B^{(i)}\}_{i \in \mathbb{N}}$ and $\{a_i\}_{i \in \mathbb{N}}$.

LEMMA 4.9. *Let $\vec{p} \in (0, 1]^n$, $q \in (1, \infty]$ and $s \in [(1/p_- - 1)\ln b / \ln \lambda_-], \infty) \cap \mathbb{Z}_+$ with p_- as in (2.4). Suppose that T is an anisotropic Calderón-Zygmund operator of order $N \in \mathbb{N}$ satisfying $T^*(x^\alpha) = 0$, for all $|\alpha| \leq s$ with $s < N \ln \lambda_- / \ln \lambda_+$. Then for any (\vec{p}, q, s) -atom a supported on some $x_0 + B_{j_0}$ with $x_0 \in \mathbb{R}^n$ and $j_0 \in \mathbb{Z}$, $T(a)$ is a harmless constant multiple of a $(\vec{p}, q, s, \varepsilon)$ -molecule associated with $x_0 + B_{j_0}$ where $\varepsilon := N \ln \lambda_- / \ln b + 1/q'$.*

Proof. Let a be a (\vec{p}, q, s) -atom with $\text{supp } a \subset B_{j_0}$, $j_0 \in \mathbb{N}$. Then for the anisotropic Calderón-Zygmund operator T of order N satisfying $T^*(x^\alpha) = 0$ for all $|\alpha| \leq s$, by Definition 4.4 and the vanishing moments of a , we obtain that $T(a)$ has the vanishing moments up to order s .

Let $U_0 := B_{j_0+\sigma}$ and for any $j \in \mathbb{Z}_+$, $U_j := B_{j_0+j+\sigma+1} \setminus B_{j_0+j+\sigma}$. To prove that $T(a)$ is a harmless constant multiple of a $(\vec{p}, q, s, \varepsilon)$ -molecule, we only need to show that

$$\|T(a)\|_{L^q(U_0)} \leq \frac{|B_{j_0}|^{1/q}}{\|\chi_{B_{j_0}}\|_{L^{\vec{p}}}} \quad (4.4)$$

and for any $j \in \mathbb{N}$ and $x \in U_j$,

$$\|T(a)\|_{L^q(U_j)} \lesssim \frac{b^{-j\varepsilon}|B_{j_0}|^{1/q}}{\|\chi_{B_{j_0}}\|_{L^{\vec{p}}}}. \quad (4.5)$$

Indeed, for any $x \in U_0$, applying the fact that T is bounded on L^q for all $q \in (1, \infty)$, $\text{supp } a \subset B_{j_0}$ and the size condition of a , we obtain that

$$\|T(a)\|_{L^q(U_0)} \leq \|a\|_{L^r(B_{j_0})} \leq \frac{|B_{j_0}|^{1/q}}{\|\chi_{B_{j_0}}\|_{L^{\vec{p}}}}$$

and hence (4.4) holds true.

For any (\vec{p}, q, s) -atom a , $j \in \mathbb{N}$, $x \in B_{j_0+j+\sigma+1} \setminus B_{j_0+j+\sigma}$ and $y \in B_{j_0}$, by (2.2) and (2.3), we have $x - y \in B_{j_0+j+2\sigma+1} \setminus B_{j_0+j}$ and hence $\rho(x - y) \sim b^{j_0+j}$. From this and (4.1), we deduce that for all $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leq N$,

$$|\partial^\alpha [K(\cdot, A^{j_0+j}\cdot)](x, A^{-j_0-j}y)| \lesssim [\rho(x - y)]^{-1} \sim b^{-j_0-j}. \quad (4.6)$$

By the condition that $\text{supp } a \subset B_{j_0}$, we have

$$T(a)(x) = \int_{B_{j_0}} K(x, y)a(y)dy = \int_{B_{j_0}} \tilde{K}(x, A^{-j_0-j}y)a(y)dy, \quad (4.7)$$

where $\tilde{K}(x, y) := K(x, A^{j_0+j}y)$ for all $x, y \in \mathbb{R}^n$ with $x \neq A^{j_0+j}y$. Now we expand $\tilde{K}(x, y)$ into the Taylor polynomial of degree $N - 1$ (only in y variable) at the point $(x, \mathbf{0})$ that is

$$\tilde{K}(x, \tilde{y}) = \sum_{|\alpha| \leq N-1} \frac{\partial_y^\alpha \tilde{K}(x, \mathbf{0})}{\alpha!} (\tilde{y})^\alpha + R_N(\tilde{y}), \quad (4.8)$$

where $\tilde{y} := A^{-j_0-j}y$ with $y \in B_{j_0}$, and hence $\tilde{y} \in B_{-j}$. Then applying (4.8) for $\tilde{y} \in B_{-j}$, using (4.6), we have

$$|R_N(\tilde{y})| \lesssim \sup_{z \in B_{-j}} \sup_{|\alpha|=N} \left| \partial_y^\alpha \tilde{K}(x, z) \right| |\tilde{y}|^N \lesssim b^{-(j_0+j)} \sup_{z \in B_{-j}} |z|^N. \quad (4.9)$$

Notice that $z \in B_{-j}$ and hence $\rho(z) < b^{-j} \leq 1$. From this and $|z| \lesssim [\rho(z)]^{\log_b(\lambda_-)}$ (see [1, p. 11, (3.3)]), we conclude that

$$\sup_{z \in B_{-j}} |z|^N \lesssim \sup_{z \in B_{-j}} [\rho(z)]^{N \log_b(\lambda_-)} \lesssim b^{-jN \log_b(\lambda_-)}.$$

By this and (4.9), we obtain

$$|R_N(\tilde{y})| \lesssim b^{-(j_0+j+jN \log_b(\lambda_-))}. \quad (4.10)$$

From (4.7), (4.8), (4.10) and the Hölder inequality, we deduce that

$$\begin{aligned} |T(a)(x)| &\leq \int_{B_{j_0}} |R_N(A^{-j_0-j}y) a(y)| dy \\ &\lesssim b^{-(j_0+j+jN \log_b(\lambda_-))} \int_{B_{j_0}} |a(y)| dy \\ &\lesssim b^{-j[1+N \log_b(\lambda_-)+j_0/q]} \|a\|_{L^q(B_{j_0})}, \end{aligned}$$

which together with the size condition of a and $\varepsilon := N \log_b(\lambda_-) + 1/q'$, implies that

$$\begin{aligned} \|T(a)\|_{L^q(U_j)} &\lesssim b^{-j(1+N \log_b(\lambda_-))} b^{-j_0/q} \|a\|_{L^q(B_{j_0})} |B_{j_0+j+\sigma+1}|^{1/q} \\ &\lesssim b^{-j(1+N \log_b(\lambda_-))} b^{j/q} \frac{|B_{j_0}|^{1/q}}{\|\chi_{B_{j_0}}\|_{L^{\bar{p}}}} \\ &\sim \frac{b^{-j\varepsilon} |B_{j_0}|^{1/q}}{\|\chi_{x_0+B_{j_0}}\|_{L^{\bar{p}}}}. \end{aligned}$$

Thus for any $j \in \mathbb{N}$, (4.5) holds true. This completes the proof of Lemma 4.9. \square

Proof of Theorem 4.5. First, we show that (4.3) holds true for any $f \in H_A^{\vec{p}} \cap L^{\vec{p}/p-}$ with $r \in (1, \infty) \cap (p_+, \infty]$. For any $f \in H_A^{\vec{p}} \cap L^{\vec{p}/p-}$, by Lemma 4.7, we know that there exist $\{\lambda_i\}_{i \in \mathbb{N}} \subset \mathbb{C}$ and a sequence of (\vec{p}, q, s) -atoms, $\{a_i\}_{i \in \mathbb{N}}$, supported, respectively, on $\{B^{(i)}\}_{i \in \mathbb{N}} \subset \mathfrak{B}$, where $B^{(i)} := x_i + B_{\ell_i}$ with $x_i \in \mathbb{R}^n$ and $\ell_i \in \mathbb{Z}$, such that

$$f = \sum_{i \in \mathbb{N}} \lambda_i a_i \text{ in } \mathcal{S}' \text{ and almost everywhere,}$$

and

$$\|f\|_{H_A^{\vec{p}, q, s}} \sim \left\| \left\{ \sum_{i \in \mathbb{N}} \left[\frac{|\lambda_i| \chi_{B^{(i)}}}{\|\chi_{B^{(i)}}\|_{L^{\bar{p}}}} \right]^p \right\}^{1/p} \right\|_{L^{\bar{p}}}. \quad (4.11)$$

It is easy to see that, for all $N \in \mathbb{N} \cap [(1/\underline{p}-1)/\ln \lambda_-] + 2, \infty$ and $x \in \mathbb{R}^n$,

$$\begin{aligned}
\|M_N(T(f))\|_{L^{\underline{p}}}^{\underline{p}} &= \left\| M_N \left(T \left(\sum_{i \in \mathbb{N}} \lambda_i a_i \right) \right) \right\|_{L^{\underline{p}}}^{\underline{p}} \leq \left\| \sum_{i \in \mathbb{N}} |\lambda_i| M_N(T(a_i)) \right\|_{L^{\underline{p}}}^{\underline{p}} \quad (4.12) \\
&\leq \left\| \sum_{i \in \mathbb{N}} |\lambda_i| M_N(T(a_i)) \chi_{A^{2\sigma} B^{(i)}} \right\|_{L^{\underline{p}}}^{\underline{p}} + \left\| \sum_{i \in \mathbb{N}} |\lambda_i| M_N(T(a_i)) \chi_{(A^{2\sigma} B^{(i)})^c} \right\|_{L^{\underline{p}}}^{\underline{p}} \\
&\lesssim \left\| \left\{ \sum_{i \in \mathbb{N}} [|\lambda_i| M_N(T(a_i)) \chi_{A^{2\sigma} B^{(i)}}]^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{L^{\underline{p}}}^{\underline{p}} + \left\| \sum_{i \in \mathbb{N}} |\lambda_i| M_N(T(a_i)) \chi_{(A^{2\sigma} B^{(i)})^c} \right\|_{L^{\underline{p}}}^{\underline{p}} \\
&=: K_1 + K_2,
\end{aligned}$$

where $A^{2\sigma} B^{(i)}$ is the $A^{2\sigma}$ concentric expansion on $B^{(i)}$ that is $A^{2\sigma} B^{(i)} := x_i + A^{2\sigma} B_{\ell_i}$ and \underline{p} as in (2.5).

For K_1 , from the fact that M_N and T are bounded on L^q for all $q \in (1, \infty)$, we know that

$$\|M_N(T(a_i)) \chi_{A^{2\sigma} B^{(i)}}\|_{L^q} \lesssim \|a_i \chi_{A^{2\sigma} B^{(i)}}\|_{L^q} \lesssim \frac{|B^{(i)}|^{1/q}}{\|\chi_{B^{(i)}}\|_{L^{\underline{p}}}}.$$

From this Lemma 4.8 and (4.11), we further deduce that

$$K_1 \lesssim \left\| \left\{ \sum_{i \in \mathbb{N}} \left[\frac{|\lambda_i| \chi_{B^{(i)}}}{\|\chi_{B^{(i)}}\|_{L^{\underline{p}}}} \right]^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{L^{\underline{p}}}^{\underline{p}} \sim \|f\|_{H_A^{\underline{p}, q, s}}^{\underline{p}}.$$

For K_2 , for any $i \in \mathbb{N}$ and $x \in (A^{2\sigma} B^{(i)})^c$, by Lemma 4.9, for any (\bar{p}, q, s) -atom $a_i(x)$ supported on a ball $B^{(i)}$, we see that $T(a_i)$ is a harmless constant multiple of a $(\bar{p}, q, s, \varepsilon)$ -molecule associated with $B^{(i)}$, where $\varepsilon := N \log_b(\lambda_-) + 1/q'$. From this and an argument similar to that used in the proof of (3.5), we know that

$$M_N(T(a_i))(x) \lesssim \|\chi_{B^{(i)}}\|_{L^{\bar{p}}}^{-1} [M_{HL}(\chi_{B^{(i)}})(x)]^{\theta}, \quad (4.13)$$

where, for any $i \in \mathbb{N}$, x_i denotes the centre of the dilated ball $B^{(i)}$ and θ as in (3.6). By (4.13) and an argument same as that used in the proof of (3.7), we obtain

$$K_2 \lesssim \left\| \left\{ \sum_{i \in \mathbb{N}} \left[\frac{|\lambda_i| \chi_{B^{(i)}}}{\|\chi_{B^{(i)}}\|_{L^{\bar{p}}}} \right]^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{L^{\bar{p}}}^{\underline{p}} \sim \|f\|_{H_A^{\bar{p}, q, s}}^{\underline{p}}.$$

Combining (4.12) and the estimates of K_1 and K_2 , we further conclude that

$$\|T(f)\|_{H_A^{\bar{p}}} \lesssim \|f\|_{H_A^{\bar{p}, q, s}} \sim \|f\|_{H_A^{\bar{p}}}.$$

Next, we prove that (4.3) also holds true for any $f \in H_A^{\bar{p}}$. Let $f \in H_A^{\bar{p}}$, by Lemma 2.6, we know that there exists a sequence $\{f_j\}_{j \in \mathbb{Z}_+} \subset H_A^{\bar{p}} \cap L^{\bar{p}/p_-}$ with p_- as in (2.4),

such that $f_j \rightarrow f$ as $j \rightarrow \infty$ in $H_A^{\vec{p}}$. Therefore, $\{f_j\}_{j \in \mathbb{Z}_+}$ is a Cauchy sequence in $H_A^{\vec{p}}$. By this, we see that for any $j, k \in \mathbb{Z}_+$,

$$\|T(f_j) - T(f_k)\|_{H_A^{\vec{p}}} = \|T(f_j - f_k)\|_{H_A^{\vec{p}}} \lesssim \|f_j - f_k\|_{H_A^{\vec{p}}}.$$

Notice that $\{T(f_j)\}_{j \in \mathbb{Z}_+}$ is also a Cauchy sequence in $H_A^{\vec{p}}$. Applying Lemma 2.7, we conclude that there exist a $g \in H_A^{\vec{p}}$ such that $T(f_j) \rightarrow g$ as $j \rightarrow \infty$ in $H_A^{\vec{p}}$. Let $T(f) := g$. We claim that $T(f)$ is well defined. Indeed, for any other sequence $\{h_j\}_{j \in \mathbb{Z}_+} \subset H_A^{\vec{p}} \cap L^{\vec{p}/p_-}$ with p_- as in (2.4) satisfying $h_j \rightarrow f$ as $j \rightarrow \infty$ in $H_A^{\vec{p}}$, by Remark 2.2(i), we have

$$\begin{aligned} \|T(h_j) - T(f)\|_{H_A^{\vec{p}}}^{\vec{p}} &\leq \|T(h_j) - T(f_j)\|_{H_A^{\vec{p}}}^{\vec{p}} + \|T(f_j) - g\|_{H_A^{\vec{p}}}^{\vec{p}} \\ &\lesssim \|h_j - f_j\|_{H_A^{\vec{p}}}^{\vec{p}} + \|T(f_j) - g\|_{H_A^{\vec{p}}}^{\vec{p}} \\ &\lesssim \|h_j - f\|_{H_A^{\vec{p}}}^{\vec{p}} + \|f - f_j\|_{H_A^{\vec{p}}}^{\vec{p}} + \|T(f_j) - g\|_{H_A^{\vec{p}}}^{\vec{p}} \rightarrow 0 \text{ as } j \rightarrow \infty, \end{aligned}$$

which is wished.

From this, we see that, for any $f \in H_A^{\vec{p}}$,

$$\|T(f)\|_{H_A^{\vec{p}}} = \|g\|_{H_A^{\vec{p}}} = \lim_{j \rightarrow \infty} \|T(f_j)\|_{H_A^{\vec{p}}} \lesssim \lim_{j \rightarrow \infty} \|f_j\|_{H_A^{\vec{p}}} \sim \|f\|_{H_A^{\vec{p}}},$$

which implies that (4.3) also holds true for any $f \in H_A^{\vec{p}}$ and hence completes the proof of Theorem 4.5. \square

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