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LYAPUNOV–TYPE INEQUALITIES FOR FRACTIONAL LANGEVIN DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we establish some new Lyapunov-type inequalities for Langevin equations involving derivatives of fractional orders with two classes of two-point boundary conditions, which have generalized some previous results.

1. Introduction

The first result in this field is due to the Russian mathematician A. M. Lyapunov in 1907 [23], which can be formulated as follows:

THEOREM 1. ([23]) *Let $q:[a,b]\to\mathbb{R}$ is continuous function. If a nontrivial continuous solution to the boundary value problem*

$$u''(t)+q(t)u(t)=0,\ a<t<b, \tag{1.1}$$

$$u(a)=u(b)=0, \tag{1.2}$$

exists. Then

$$\int_a^b |q(s)|ds \geqslant \frac{4}{b-a}. \tag{1.3}$$

Since Lyapunov-type inequality has been shown to be useful in studying various properties of differential equations, which include bounds for eigenvalues, stability criteria for periodic differential equations and estimates for disconjugation intervals, etc, there have been a lot of generalizations and extensions of Lyapunov-type inequality in the literature (see [8, 27, 29, 33, 28, 5, 32, 30] and the references therein).

Recently, many articles on Lyapunov-type inequality for fractional order differential equations have been published. The first work in this direction belongs to Ferreira in [9], where he established a Lyapunov-type inequality for the differential equations dependent to the fractional derivative of Riemann-Liouville type

$${}^{\mathfrak{RL}}\mathfrak{D}_{a^+}^{\eta}u(t)+q(t)u(t)=0,\ a<t<b,\ 1<\eta\leqslant 2, \tag{1.4}$$

$$u(a)=u(b)=0, \tag{1.5}$$

and he reached to the following result:

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THEOREM 2. ([9]) *If a nontrivial continuous solution to the fractional boundary value problem* (1.4)–(1.5) *exists. Then*

$$\int_a^b |q(s)|ds \geqslant \frac{\eta^\eta \Gamma(\eta)}{[(\eta-1)(b-a)]^{\eta-1}}. \tag{1.6}$$

Other noteworthy papers have been published on Lyapunov type inequalities for boundary value problems involving fractional derivative, for example see [12, 36, 1, 24, 17, 26, 25] and references therein.

In 2016, Ferreira [10] discussed a Lyapunov-type inequality for the following sequential fractional boundary value problem with Liouville-Caputo derivative:

$$\left({}^{\mathfrak{LC}}\mathfrak{D}^{\eta}_{a^+}\ {}^{\mathfrak{LC}}\mathfrak{D}^{\mu}_{a^+}\right)u(t) = -q(t)u(t),\ a<t<b, \tag{1.7}$$

$$u(a) = 0 = u(b), \tag{1.8}$$

where $0<\eta,\ \mu \leqslant 1,\ (1<\eta+\mu \leqslant 2)$. Accordingly, he has reached the following result:

THEOREM 3. ([10]) *If a nontrivial continuous solution to the fractional boundary value problem* (1.7)–(1.8) *exists. Then*

$$\int_a^b |q(s)|ds \geqslant \frac{(\eta+2\mu-1)^{\eta+2\mu-1}\Gamma(\eta+\mu+1)}{[(\eta+\mu-1)(b-a)]^{\eta+\mu-1}\mu^\mu}. \tag{1.9}$$

In 2019, in [11], Ferreira discussed a Lyapunov-type inequality for the problem (1.8) with the following fractional boundary conditions:

$$u(a) = 0 = {}^{\mathfrak{LC}}\mathfrak{D}^{\gamma}_{a^+}u(b),\ \ 0<\gamma \leqslant 1, \tag{1.10}$$

and he got the following result:

THEOREM 4. ([11]) *If a nontrivial continuous solution to the fractional boundary value problem* (1.7)–(1.10) *exists. Then*

$$\int_a^b (b-s)^{\eta+\mu-\gamma-1}|q(s)|ds \geqslant \frac{1}{C}, \tag{1.11}$$

where

$$C = (b-a)^\gamma \max\left\{\frac{\Gamma(\mu+1-\gamma)}{\Gamma(\eta+\mu-\gamma)\Gamma(\mu+1)},\right.$$
$$\left.\frac{(1-\eta)}{\mu\Gamma(\eta+\mu)}\left(\frac{\Gamma(\mu+1-\gamma)\Gamma(\eta+\mu-\gamma)}{\Gamma(\eta+\mu-\gamma)\Gamma(\mu)}\right)^{\frac{\eta+\mu-1}{\eta-1}},\ \textit{with}\ \eta<1\right\}. \tag{1.12}$$

In 1908, Langevin formulated his famous equation, with a derivative from the enteger order, which describes the evolution of certain physical phenomena, see [18]. The Langevin equation has been recently developed to the fractional order, where Lim et al. [19] introduced a new form of Langevin equations involving two different fractional orders. Subsequent works on a fractional Langevin equation were developed. For example see [4, 20, 35, 6, 22, 21]. In addition, other studies have reported outstanding results on the existence of the solutions for the Langevin differential equations of fractional order by using fixed point theorems, see [3, 34, 15, 16, 3, 2, 13, 37] and references therein. The usefulness of the fractional Langevin equation is demonstrated in the description of the viscoelastic anomalous diffusion in complex liquids, herein the reader may refer to (Subsection 2.1 in [31]) and the references cited therein for more details.

Motivated by the aforementioned works, especialy papers [10, 11] and [3, 2], in this paper, we discuss Lyapunov-type inequalities for the following class of Langevin equations involving two fractional orders:

$$ {}^{\mathfrak{LC}}\mathfrak{D}^{\eta}_{a^+}\left({}^{\mathfrak{LC}}\mathfrak{D}^{\mu}_{a^+}+\lambda\right)u(t)=-q(t)u(t), \qquad a<t<b, \tag{1.13} $$

with one of the two-point boundary conditions:

$$ u(a)=0=u(b), \tag{1.14} $$

$$ u(a)=0={}^{\mathfrak{LC}}\mathfrak{D}^{\gamma}_{a^+}u(b), \tag{1.15} $$

where (either $0<\mu\leqslant 1$ or $1<\mu\leqslant 2$), $0<\eta,\gamma\leqslant 1$, $\lambda\in\mathbb{R}$, such that $1<\eta+\mu\leqslant 2$ and $\gamma<\mu$; ${}^{\mathfrak{LC}}\mathfrak{D}^{\eta}_{a^+}$ and ${}^{\mathfrak{LC}}\mathfrak{D}^{\mu}_{a^+}$ denotes the Liouville-Caputo fractional derivative of order η and μ, respectively; $q:[a,b]\to\mathbb{R}$ is a continuous function.

To the best of the authors' knowledge, there are no generalizations of Lyapunov-type inequality for fractional Langevin equations and our results also have generalized the conclusions on [10, 11].

2. Preliminaries

In this section, we give some basic concepts of fractional calculus.

DEFINITION 1. ([14]) The Riemann-Liouville fractional integral of order $\eta>0$ for a function $f\in L^1[a,b]$ is defined by

$$ \mathfrak{I}^{\eta}_{a^+}f(t)=\frac{1}{\Gamma(\eta)}\int_a^t(t-s)^{\eta-1}f(s)\,ds, \tag{2.1} $$

where $t\in[a,b]$ and Γ is Gamma Euler function.

DEFINITION 2. ([14]) The Liouville-Caputo fractional derivative of order $\eta>0$ for a function $f\in C^n[a,b]$ is defined by

$$ {}^{\mathfrak{LC}}\mathfrak{D}^{\eta}_{a^+}f(t)=\mathfrak{I}^{n-\eta}_{a^+}f^{(n)}(t)=\frac{1}{\Gamma(n-\eta)}\int_a^t(t-s)^{n-\eta-1}f^{(n)}(s)\,ds, \tag{2.2} $$

where $n-1<\eta\leqslant n,\ n\in\mathbb{N}$.

PROPOSITION 1. ([14]) Let $\mu > 0$ and $\eta > 0$ with $n-1 < \eta \leqslant n$, for all $f \in L^1[a,b]$, we have the following properties

$$\mathfrak{I}^{\eta}_{a^+}\,\mathfrak{I}^{\mu}_{a^+} f(t) = \mathfrak{I}^{\mu}_{a^+}\mathfrak{I}^{\eta}_{a^+} f(t) = \mathfrak{I}^{\eta+\mu}_{a^+} f(t); \tag{2.3}$$

$$^{\mathfrak{LC}}\mathfrak{D}^{\eta}_{a^+}\,\mathfrak{I}^{\eta}_{a^+} f(t) = f(t); \tag{2.4}$$

$$\mathfrak{I}^{\eta}_{a^+}\,{}^{\mathfrak{LC}}\mathfrak{D}^{\eta}_{a^+} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)(t-a)^k}{k!}; \tag{2.5}$$

In particular, when $0 < \eta \leqslant 1$, we have

$$\mathfrak{I}^{\eta}_{a^+}\,{}^{\mathfrak{LC}}\mathfrak{D}^{\eta}_{a^+} f(t) = f(t) - f(a); \tag{2.6}$$

In case $1 < \eta \leqslant 2$, we have

$$\mathfrak{I}^{\eta}_{a^+}\,{}^{\mathfrak{LC}}\mathfrak{D}^{\eta}_{a^+} f(t) = f(t) - f(a) - f'(a)(t-a). \tag{2.7}$$

LEMMA 1. ([14]) *For $\eta > 0$ with $n-1 < \eta \leqslant n$, $n \in \mathbb{N}$, the general solution of fractional differential equation ${}^{\mathfrak{LC}}\mathfrak{D}^{\eta}_{a^+} u(t) = 0$ is given by*

$$u(t) = c_0 + c_1(t-a) + c_2(t-a)^2 + \ldots + c_{n-1}(t-a)^{n-1}, \tag{2.8}$$

where $c_i \in \mathbb{R}$, $i = 0,1,2,\ldots,n-1$.

LEMMA 2. *Let $0 < \gamma \leqslant 1$, and $\mu > \gamma$. Then*

$$^{\mathfrak{LC}}\mathfrak{D}^{\gamma}_{a^+}(t-a)^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+1-\gamma)}(t-a)^{\mu-\gamma}. \tag{2.9}$$

LEMMA 3. ([7]) *Suppose that $f \in C^k([a,b],\mathbb{R})$ for some $a < b$ and some $k \in \mathbb{N}$. Moreover let $\eta,\mu > 0$ be such that there exist some $m \in \mathbb{N}$ with $m \leqslant k$ and $\eta+\mu, \mu \in [m-1,m]$. Then*

$$^{\mathfrak{LC}}\mathfrak{D}^{\eta}_{a^+}\,{}^{\mathfrak{LC}}\mathfrak{D}^{\mu}_{a^+} f(t) = {}^{\mathfrak{LC}}\mathfrak{D}^{\eta+\mu}_{a^+} f(t).$$

3. Main results

This section is divided into two Subsections according to the value of μ (Case $0 < \mu \leqslant 1$ and Case $1 < \mu \leqslant 2$). In each case, we discuss two problems: (1.13)–(1.14) and (1.13)–(1.15). By using the Green's function and its properties for each problem, we obtain the Lyapunov-type inequality associated.

3.1. Case $0 < \mu \leqslant 1$

3.1.1. Discussion of problem (1.13)–(1.14)

LEMMA 4. *Assume that* $0 < \mu \leqslant 1$. *And let* $u(t) \in C([a,b],\mathbb{R})$, *the problem* (1.13)–(1.14) *has equivalent to the fractional integral equation*

$$u(t) = \int_a^b \lambda G_\mu(t,s)u(s)\,ds + \int_a^b G_{\eta+\mu}(t,s)q(s)u(s)\,ds, \tag{3.1}$$

where

$$G_\delta(t,s) = \frac{1}{\Gamma(\delta)}\begin{cases} \left(\frac{t-a}{b-a}\right)^{\mu}(b-s)^{\delta-1} - (t-s)^{\delta-1}, & a \leqslant s < t \leqslant b, \\ \left(\frac{t-a}{b-a}\right)^{\mu}(b-s)^{\delta-1}, & a \leqslant t \leqslant s \leqslant b, \end{cases} \tag{3.2}$$

with $\delta \in \{\mu, \eta+\mu\}$.

Proof. Applying the operator $\mathfrak{I}^{\eta}_{a^+}$ on the equation ${}^{\mathfrak{LC}}\mathfrak{D}^{\eta}_{a^+}\left({}^{\mathfrak{LC}}\mathfrak{D}^{\mu}_{a^+} + \lambda\right)u(t) = -q(t)u(t)$, we get

$${}^{\mathfrak{LC}}\mathfrak{D}^{\mu}_{a^+}u(t) = -\lambda u(t) - \mathfrak{I}^{\eta}_{a^+}(qu)(t) + c_0, \quad c_0 \in \mathbb{R}. \tag{3.3}$$

Next, Applying the operator $\mathfrak{I}^{\mu}_{a^+}$ on the equation (3.3), we obtain

$$u(t) = -\lambda\mathfrak{I}^{\mu}_{a^+}u(t) - \mathfrak{I}^{\eta+\mu}_{a^+}(qu)(t) + c_0\frac{(t-a)^{\mu}}{\Gamma(\mu+1)} + c_1, \quad c_1 \in \mathbb{R}, \tag{3.4}$$

by the boundary conditions $u(a) = 0 = u(b)$, we get $c_1 = 0$, and

$$c_0 = \frac{\Gamma(\mu+1)}{(b-a)^{\mu}}\left[\lambda\mathfrak{I}^{\mu}_{a^+}u(b) + \mathfrak{I}^{\eta+\mu}_{a^+}(qu)(b)\right].$$

Substituting the value of c_0 and c_1 in (3.4), we obtain

$$u(t) = \lambda\left(\frac{t-a}{b-a}\right)^{\mu}\mathfrak{I}^{\mu}_{a^+}u(b) - \lambda\mathfrak{I}^{\mu}_{a^+}u(t) + \left(\frac{t-a}{b-a}\right)^{\mu}\mathfrak{I}^{\eta+\mu}_{a^+}(qu)(b) - \mathfrak{I}^{\eta+\mu}_{a^+}(qu)(t),$$

i.e.,

$$\begin{aligned} u(t) &= \frac{\lambda}{\Gamma(\mu)}\int_a^b\left(\frac{t-a}{b-a}\right)^{\mu}(b-s)^{\mu-1}u(s)\,ds - \frac{\lambda}{\Gamma(\mu)}\int_a^t(t-s)^{\mu-1}u(s)\,ds \\ &\quad + \int_a^b\left(\frac{t-a}{b-a}\right)^{\mu}\frac{(b-s)^{\eta+\mu-1}}{\Gamma(\eta+\mu)}q(s)u(s)\,ds - \int_a^t\frac{(t-s)^{\eta+\mu-1}}{\Gamma(\eta+\mu)}q(s)u(s)\,ds \\ &= \frac{\lambda}{\Gamma(\mu)}\int_a^t\left[\left(\frac{t-a}{b-a}\right)^{\mu}(b-s)^{\mu-1} - (t-s)^{\mu-1}\right]u(s)\,ds \\ &\quad + \frac{\lambda}{\Gamma(\mu)}\int_t^b\left(\frac{t-a}{b-a}\right)^{\mu}(b-s)^{\mu-1}u(s)\,ds \end{aligned}$$

$$+\frac{1}{\Gamma(\eta+\mu)}\int_a^t\left(\left(\frac{t-a}{b-a}\right)^{\mu}(b-s)^{\eta+\mu-1}-(t-s)^{\eta+\mu-1}\right)q(s)u(s)\,ds$$
$$+\frac{1}{\Gamma(\eta+\mu)}\int_t^b\left(\frac{t-a}{b-a}\right)^{\mu}(b-s)^{\eta+\mu-1}q(s)u(s)\,ds$$
$$=\int_a^b\lambda G_\mu(t,s)u(s)\,ds+\int_a^b G_{\eta+\mu}(t,s)q(s)u(s)\,ds,$$

where $G_\delta(t,s)$ with $\delta\in\{\mu,\eta+\mu\}$ is given by (3.2). The proof is complete. □

LEMMA 5. ([10]) *Assume that* $0<\mu\leqslant 1$. *Then, The functions* $G_\delta,\delta\in\{\mu,\eta+\mu\}$ *defined in Lemma 4, satisfy the following properties:*

i)

$$\max_{t\in[a,b]}\int_a^b\left|G_\mu(t,s)\right|ds=\frac{(b-a)^\mu}{2^{2\mu-1}\Gamma(\mu+1)};\tag{3.5}$$

ii)

$$\max_{t,s\in[a,b]}\left|G_{\eta+\mu}(t,s)\right|=\frac{[(\eta+\mu-1)(b-a)]^{\eta+\mu-1}\mu^\mu}{(\eta+2\mu-1)^{\eta+2\mu-1}\Gamma(\eta+\mu)}.\tag{3.6}$$

Proof. i) It's clear that $\mu-1\leqslant 0$, so, we have

$$(b-s)^{\mu-1}\leqslant(t-s)^{\mu-1}$$
$$\Rightarrow\left(\frac{t-a}{b-a}\right)^{\mu}(b-s)^{\mu-1}\leqslant(t-s)^{\mu-1}$$
$$\Rightarrow\left(\frac{t-a}{b-a}\right)^{\mu}(b-s)^{\mu-1}-(t-s)^{\mu-1}\leqslant 0.$$

So,

$$\max_{t\in[a,b]}\int_a^b\left|G_\mu(t,s)\right|ds=\frac{1}{\Gamma(\mu)}\max_{t\in[a,b]}\left[\int_a^t-\left(\frac{t-a}{b-a}\right)^{\mu}(b-s)^{\mu-1}ds+\int_a^t(t-s)^{\mu-1}ds\right.$$
$$\left.+\int_t^b\left(\frac{t-a}{b-a}\right)^{\mu}(b-s)^{\mu-1}ds\right]$$
$$=\frac{1}{\Gamma(\mu)}\max_{t\in[a,b]}\left[\left(\frac{t-a}{b-a}\right)^{\mu}\left(\frac{(b-t)^\mu}{\mu}-\frac{(b-a)^\mu}{\mu}\right)+\frac{(t-a)^\mu}{\mu}\right.$$
$$\left.+\frac{1}{\mu(b-a)^\mu}[(t-a)(b-t)]^\mu\right]$$
$$=\frac{2}{\Gamma(\mu+1)(b-a)^\mu}\max_{t\in[a,b]}[(t-a)(b-t)]^\mu$$
$$=\frac{2}{\Gamma(\mu+1)(b-a)^\mu}\left(\frac{(b-a)^2}{4}\right)^{\mu}$$
$$=\frac{(b-a)^\mu}{2^{2\mu-1}\Gamma(\mu+1)}.$$

ii) For the second property, see the proof in [10]. □

We have the following Lyapunov-type inequality.

THEOREM 5. *Assume that* $0 < \mu \leqslant 1$. *If a nontrivial continuous solution to the fractional boundary value problem* (1.13)–(1.14) *exists. Then*

$$\int_a^b |q(s)|ds \geqslant \frac{(\eta+2\mu-1)^{\eta+2\mu-1}\,\Gamma(\eta+\mu+1)}{[(\eta+\mu-1)(b-a)]^{\eta+\mu-1}\,\mu^{\mu}}\left(1-\frac{|\lambda|\,(b-a)^{\mu}}{2^{2\mu-1}\Gamma(\mu+1)}\right). \qquad (3.7)$$

Proof. Using the above integral equation (3.1), we get

$$\begin{aligned} |u(t)| &\leqslant |\lambda|\int_a^b \left|G_{\mu}(t,s)\right| |u(s)|\,ds+\int_a^b \left|G_{\eta+\mu}(t,s)\right| |q(s)|\,|u(s)|\,ds \\ &\leqslant |\lambda|\,\|u\| \max_{t\in[a,b]}\int_a^b \left|G_{\mu}(t,s)\right| ds+\|u\| \max_{t,s\in[a,b]}\left|G_{\eta+\mu}(t,s)\right| \int_a^b |q(s)|\,ds, \end{aligned}$$

which yields

$$\|u\| \leqslant |\lambda|\,\|u\| \max_{t\in[a,b]}\int_a^b \left|G_{\mu}(t,s)\right| ds+\|u\| \max_{t,s\in[a,b]}\left|G_{\eta+\mu}(t,s)\right| \int_a^b |q(s)|\,ds.$$

Because $\|u\| \neq 0$ then

$$\frac{1-|\lambda| \max_{t\in[a,b]} \int_a^b \left|G_{\mu}(t,s)\right| ds}{\max_{t,s\in[a,b]}\left|G_{\eta+\mu}(t,s)\right|} \leqslant \int_a^b |q(s)|\,ds.$$

By using Lemma 5 to the last inequality, we get the inequality (3.7). The proof is complete □

REMARK 1. Notice that, if $\lambda=0$, we get the special case: Theorem 3.

3.1.2. Discussion of problem (1.13)–(1.15)

LEMMA 6. *Assume that* $0 < \mu \leqslant 1$. *And let* $u(t) \in C([a,b],\mathbb{R})$, *the problem* (1.13)–(1.15) *has equivalent to the fractional integral equation*

$$u(t)=\int_a^b \lambda \bar{G}(t,s)u(s)\,ds+\int_a^b (b-s)^{\eta+\mu-\gamma-1}\,G(t,s)q(s)u(s)\,ds, \qquad (3.8)$$

where

$$\bar{G}(t,s)=\frac{1}{\Gamma(\mu+1)}\begin{cases} \frac{(\mu-\gamma)(t-a)^{\mu}}{(b-a)^{\mu-\gamma}}\,(b-s)^{\mu-\gamma-1}-\mu\,(t-s)^{\mu-1}, & a\leqslant s<t\leqslant b, \\ \frac{(\mu-\gamma)(t-a)^{\mu}}{(b-a)^{\mu-\gamma}}\,(b-s)^{\mu-\gamma-1}, & a\leqslant t\leqslant s<b, \end{cases} \qquad (3.9)$$

and

$$G(t,s)=\begin{cases}\frac{\Gamma(\mu+1-\gamma)(t-a)^{\mu}}{\Gamma(\mu+1)\Gamma(\eta+\mu-\gamma)(b-a)^{\mu-\gamma}}-\frac{(t-s)^{\eta+\mu-1}}{\Gamma(\eta+\mu)(b-s)^{\eta+\mu-\gamma-1}}, & a\leqslant s\leqslant t\leqslant b,\\ \frac{\Gamma(\mu+1-\gamma)(t-a)^{\mu}}{\Gamma(\mu+1)\Gamma(\eta+\mu-\gamma)(b-a)^{\mu-\gamma}}, & a\leqslant t\leqslant s\leqslant b.\end{cases} \tag{3.10}$$

Proof. The general solution of the equation ${}^{\mathfrak{LC}}\mathfrak{D}^{\eta}_{a^+}\left({}^{\mathfrak{LC}}\mathfrak{D}^{\mu}_{a^+}+\lambda\right)u(t)=-q(t)u(t)$ is given by

$$u(t)=-\lambda\mathfrak{I}^{\mu}_{a^+}u(t)-\mathfrak{I}^{\eta+\mu}_{a^+}(qu)(t)+d_0\frac{(t-a)^{\mu}}{\Gamma(\mu+1)}+d_1,$$

where $d_0,d_1\in\mathbb{R}$. From the boundary condition $u(a)=0$, we get $d_1=0$, so

$$u(t)=-\lambda\mathfrak{I}^{\mu}_{a^+}u(t)-\mathfrak{I}^{\eta+\mu}_{a^+}(qu)(t)+d_0\frac{(t-a)^{\mu}}{\Gamma(\mu+1)}. \tag{3.11}$$

Applying the operator on ${}^{\mathfrak{LC}}\mathfrak{D}^{\gamma}_{a^+}$, on (3.11), we get

$$\begin{aligned}{}^{\mathfrak{LC}}\mathfrak{D}^{\gamma}_{a^+}u(t)&=-\lambda\,{}^{\mathfrak{LC}}\mathfrak{D}^{\gamma}_{a^+}\mathfrak{I}^{\mu}_{a^+}u(t)-{}^{\mathfrak{LC}}\mathfrak{D}^{\gamma}_{a^+}\mathfrak{I}^{\eta+\mu}_{a^+}(qu)(t)+d_0\frac{{}^{\mathfrak{LC}}\mathfrak{D}^{\gamma}_{a^+}(t-a)^{\mu}}{\Gamma(\mu+1)}\\&=-\lambda\left({}^{\mathfrak{LC}}\mathfrak{D}^{\gamma}_{a^+}\mathfrak{I}^{\gamma}_{a^+}\right)\mathfrak{I}^{\mu-\gamma}_{a^+}u(t)-\left({}^{\mathfrak{LC}}\mathfrak{D}^{\gamma}_{a^+}\mathfrak{I}^{\gamma}_{a^+}\right)\mathfrak{I}^{\eta+\mu-\gamma}_{a^+}(qu)(t)\\&\quad+d_0\frac{{}^{\mathfrak{LC}}\mathfrak{D}^{\gamma}_{a^+}(t-a)^{\mu}}{\Gamma(\mu+1)},\end{aligned}$$

using the property (2.4) and Lemma 2, we obtain

$${}^{\mathfrak{LC}}\mathfrak{D}^{\gamma}_{a^+}u(t)=-\lambda\mathfrak{I}^{\mu-\gamma}_{a^+}u(t)-\mathfrak{I}^{\eta+\mu-\gamma}_{a^+}(qu)(t)+d_0\frac{(t-a)^{\mu-\gamma}}{\Gamma(\mu+1-\gamma)}.$$

From the boundary condition ${}^{\mathfrak{LC}}\mathfrak{D}^{\gamma}_{a^+}u(b)=0$, we get

$$d_0=\frac{\Gamma(\mu+1-\gamma)}{(b-a)^{\mu-\gamma}}\left[\lambda\mathfrak{I}^{\mu-\gamma}_{a^+}u(b)+\mathfrak{I}^{\eta+\mu-\gamma}_{a^+}(qu)(b)\right].$$

Substituting the value of d_0 in (3.11), by means of the Proof of Lemma 3.1 [11]), we obtain

$$\begin{aligned}u(t)&=\lambda\frac{(t-a)^{\mu}}{\Gamma(\mu+1)}\frac{\Gamma(\mu+1-\gamma)}{(b-a)^{\mu-\gamma}}\mathfrak{I}^{\mu-\gamma}_{a^+}u(b)-\lambda\mathfrak{I}^{\mu}_{a^+}u(t)\\&\quad+\frac{(t-a)^{\mu}}{\Gamma(\mu+1)}\frac{\Gamma(\mu+1-\gamma)}{(b-a)^{\mu-\gamma}}\mathfrak{I}^{\eta+\mu-\gamma}_{a^+}(qu)(b)-\mathfrak{I}^{\eta+\mu}_{a^+}(qu)(t)\\&=\frac{\lambda}{\Gamma(\mu+1)}\left[\int_a^b\frac{(\mu-\gamma)(t-a)^{\mu}}{(b-a)^{\mu-\gamma}}(b-s)^{\mu-\gamma-1}u(s)\,ds-\int_a^t\mu(t-s)^{\mu-1}u(s)\,ds\right]\\&\quad+\int_a^b(b-s)^{\eta+\mu-\gamma-1}G(t,s)q(s)u(s)\,ds\\&=\int_a^b\lambda\bar{G}(t,s)u(s)\,ds+\int_a^b(b-s)^{\eta+\mu-\gamma-1}G(t,s)q(s)u(s)\,ds,\end{aligned}$$

where $\bar{G}(t,s)$ and $G(t,s)$ are given by (3.9) and (3.10) respectively. The proof is complete. □

LEMMA 7. *Assume that* $0<\mu\leqslant 1$. *Then, The functions* $\bar{G}$ *and* G *defined in Lemma* 6*, satisfy the following properties:*

i) For any $t\in[a,b]$,

$$\int_a^b \left|\bar{G}(t,s)\right| ds \leqslant \frac{2(b-a)^{\mu}}{\Gamma(\mu+1)}; \tag{3.12}$$

ii)

$$\max_{t,s\in[a,b]} |G(t,s)| = C, \tag{3.13}$$

where C *is given by (*1.12*).*

Proof. i) For the first property, we have

$$\begin{aligned}
\int_a^b \left|\bar{G}(t,s)\right| ds &\leqslant \frac{1}{\Gamma(\mu+1)}\left\{\int_a^t \left|\frac{(\mu-\gamma)(t-a)^{\mu}}{(b-a)^{\mu-\gamma}}(b-s)^{\mu-\gamma-1}-\mu(t-s)^{\mu-1}\right| ds\right.\\
&\qquad \left. + \int_t^b \frac{(\mu-\gamma)(t-a)^{\mu}}{(b-a)^{\mu-\gamma}}(b-s)^{\mu-\gamma-1}ds\right\}\\
&\leqslant \frac{1}{\Gamma(\mu+1)}\left\{\int_a^b \frac{(\mu-\gamma)(t-a)^{\mu}}{(b-a)^{\mu-\gamma}}(b-s)^{\mu-\gamma-1}ds+\int_a^t \mu(t-s)^{\mu-1}ds\right\}\\
&= \frac{2(t-a)^{\mu}}{\Gamma(\mu+1)}\\
&\leqslant \frac{2(b-a)^{\mu}}{\Gamma(\mu+1)}.
\end{aligned}$$

ii) For the second property, see the proof in [11]. □

We have the following Lyapunov-type inequality.

THEOREM 6. *Assume that* $0<\mu\leqslant 1$. *If a nontrivial continuous solution to the fractional boundary value problem* (1.13)–(1.15) *exists. Then*

$$\int_a^b (b-s)^{\eta+\mu-\gamma-1}|q(s)|ds \geqslant \frac{1}{C}\left(1-\frac{2|\lambda|(b-a)^{\mu}}{\Gamma(\mu+1)}\right), \tag{3.14}$$

where C *is given by* (1.12).

Proof. Using the integral equation (3.8), we get

$$\begin{aligned}
|u(t)| &\leqslant |\lambda|\int_a^b \left|\bar{G}(t,s)\right| |u(s)|ds + \int_a^b (b-s)^{\eta+\mu-\gamma-1}|G(t,s)||q(s)||u(s)|ds\\
&\leqslant |\lambda|\,\|u\|\max_{t\in[a,b]}\int_a^b \left|\bar{G}(t,s)\right| ds + \|u\|\max_{t,s\in[a,b]}|G(t,s)|\int_a^b (b-s)^{\eta+\mu-\gamma-1}|q(s)|ds,
\end{aligned}$$

In the same way as proving theorem 5 we obtain

$$\frac{1-|\lambda| \max\limits_{t\in[a,b]} \int_a^t \left|\bar{G}(t,s)\right| ds}{\max\limits_{t,s\in[a,b]} |G(t,s)|} \leqslant \int_a^b (b-s)^{\eta+\mu-\gamma-1} |q(s)|\, ds,$$

where $\max\limits_{t,s\in[a,b]} |G(t,s)| = C$ with C is given by (1.12) (See Proposition 3.2 in [11]). By Lemma 7 and the last inequality, we get the inequality (3.14). The proof is complete. □

REMARK 2. Note that, if $\lambda = 0$, we get the special case: Theorem 4.

3.2. Case $1 < \mu \leqslant 2$

3.2.1. Discussion of problem (1.13)–(1.14)

LEMMA 8. *Assume that* $1 < \mu \leqslant 2$*. And let* $u(t) \in C^2([a;b],\mathbb{R})$*, the problem* (1.13)–(1.14) *has equivalent to the fractional integral equation*

$$u(t) = \int_a^b \lambda \widehat{G}_\mu(t,s)u(s)\, ds + \int_a^b \widehat{G}_{\eta+\mu}(t,s)q(s)u(s)\, ds, \tag{3.15}$$

where

$$\widehat{G}_\delta(t,s) = \frac{1}{\Gamma(\delta)} \begin{cases} \frac{t-a}{b-a}(b-s)^{\delta-1} - (t-s)^{\delta-1}, & a \leqslant s \leqslant t \leqslant b, \\ \frac{t-a}{b-a}(b-s)^{\delta-1}, & a \leqslant t \leqslant s \leqslant b, \end{cases} \tag{3.16}$$

with $\delta \in \{\mu, \eta+\mu\}$.

Proof. Using Lemma 3, we can write the equation ${}^{\mathfrak{LC}}\mathfrak{D}^{\eta}_{a^+}\left({}^{\mathfrak{LC}}\mathfrak{D}^{\mu}_{a^+} + \lambda\right)u(t) = -q(t)u(t)$ as follows

$${}^{\mathfrak{LC}}\mathfrak{D}^{\eta+\mu}_{a^+}u(t) + \lambda\, {}^{\mathfrak{LC}}\mathfrak{D}^{\eta}_{a^+}u(t) = -q(t)u(t), \tag{3.17}$$

Applying the operator $\mathfrak{I}^{\eta+\mu}_{a^+}$ on the equation (3.17), and using the properties (2.6) and (2.7), we get

$$u(t) - u(a) - u'(a)(t-a) + \lambda \mathfrak{I}^{\mu}_{a^+}(u(t) - u(a)) = -\mathfrak{I}^{\eta+\mu}_{a^+}(qu)(t). \tag{3.18}$$

Substituting the value of the boundary condition $u(a) = 0$ in (3.18) we get

$$u(t) = u'(a)(t-a) - \lambda \mathfrak{I}^{\mu}_{a^+}u(t) - \mathfrak{I}^{\eta+\mu}_{a^+}(qu)(t), \tag{3.19}$$

and using the boundary condition $u(b) = 0$ we obtain

$$u'(a) = \frac{1}{b-a}\left[\lambda \mathfrak{I}^{\mu}_{a^+}u(b) + \mathfrak{I}^{\eta+\mu}_{a^+}(qu)(b)\right].$$

Substituting the value of $u'(a)$ in (3.19), we get

$$u(t) = \frac{t-a}{b-a}\left[\lambda \mathfrak{I}^{\mu}_{a^+}u(b) + \mathfrak{I}^{\eta+\mu}_{a^+}(qu)(b)\right] - \lambda \mathfrak{I}^{\mu}_{a^+}u(t) - \mathfrak{I}^{\eta+\mu}_{a^+}(qu)(t).$$

Using the same method as in the above proofs, we obtain the required result. □

LEMMA 9. *Assume that* $1 < \mu \leqslant 2$. *The functions* $\widehat{G}_{\delta}, \delta \in \{\mu, \eta+\mu\}$ *defined in Lemma 8, satisfy the following properties:*

i) For any $t \in [a,b]$ *we have*

$$\int_a^b |\widehat{G}_{\mu}(t,s)|ds \leqslant \frac{2(b-a)^{\mu}}{\Gamma(\mu+1)}; \tag{3.20}$$

ii)

$$\max_{t,s\in[a,b]} |\widehat{G}_{\eta+\mu}(t,s)| = \frac{[(\eta+\mu-1)(b-a)]^{\eta+\mu-1}}{\Gamma(\eta+\mu)(\eta+\mu)^{\eta+\mu}}. \tag{3.21}$$

Proof. i) By the Green's function (3.16), when $\delta = \mu$ we have

$$\begin{aligned}
\int_a^b |\widehat{G}_{\mu}(t,s)|ds &= \frac{1}{\Gamma(\mu)}\int_a^t \left|\frac{(t-a)}{(b-a)}(b-s)^{\mu-1} - (t-s)^{\mu-1}\right| ds \\
&\quad + \frac{1}{\Gamma(\mu)}\int_t^b \left|\frac{(t-a)}{(b-a)}(b-s)^{\mu-1}\right| ds \\
&\leqslant \frac{1}{\Gamma(\mu)}\int_a^t \frac{(t-a)}{(b-a)}(b-s)^{\mu-1}ds + \frac{1}{\Gamma(\mu)}\int_a^t (t-s)^{\mu-1}ds \\
&\quad + \frac{1}{\Gamma(\mu)}\int_t^b \frac{(t-a)}{(b-a)}(b-s)^{\mu-1}ds \\
&= \frac{1}{\Gamma(\mu)}\left(\int_a^b \frac{(t-a)}{(b-a)}(b-s)^{\mu-1}ds + \int_a^t (t-s)^{\mu-1}ds\right) \\
&= \frac{1}{\Gamma(\mu+1)}\left((t-a)(b-a)^{\mu-1} + (t-a)^{\mu}\right) \\
&\leqslant \frac{2(b-a)^{\mu}}{\Gamma(\mu+1)}.
\end{aligned}$$

ii) For the second property, see the paper [12]. □

We have the following Lyapunov-type inequality.

THEOREM 7. *Assume that* $1 < \mu \leqslant 2$. *If a nontrivial continuous* $2nd$ *derivative on* $[a,b]$ *solution to the fractional boundary value problem* (1.13)–(1.14) *exists. Then*

$$\int_a^b |q(s)|ds \geqslant \frac{\Gamma(\eta+\mu)(\eta+\mu)^{\eta+\mu}}{[(\eta+\mu-1)(b-a)]^{\eta+\mu-1}}\left(1 - \frac{2|\lambda|(b-a)^{\mu}}{\Gamma(\mu+1)}\right). \tag{3.22}$$

Proof. By Lemmas 8 and 9 and taking the similar procedures in the Proof of Theorem 5, we can obtain (3.22). □

REMARK 3. If $\lambda = 0$ and $\eta + \mu = 2$, we get the classical case: Theorem 1.

3.2.2. Discussion of problem (1.13)–(1.15)

LEMMA 10. *Assume that* $1 < \mu \leqslant 2$. *And let* $u(t) \in C^2([a,b],\mathbb{R})$, *the problem* (1.13)–(1.15) *has equivalent to the fractional integral equation*

$$u(t) = \int_a^b \lambda \widetilde{G}_1(t,s)u(s)\,ds - \int_a^b (b-s)^{\eta+\mu-\gamma-1}\widetilde{G}_2(t,s)q(s)u(s)\,ds, \tag{3.23}$$

where

$$\widetilde{G}_1(t,s) = \begin{cases} \frac{\Gamma(2-\gamma)(t-a)}{\Gamma(\mu-\gamma)(b-a)^{1-\gamma}}(b-s)^{\mu-\gamma-1} - \frac{1}{\Gamma(\mu)}(t-s)^{\mu-1}, & a \leqslant s \leqslant t \leqslant b, \\ \frac{\Gamma(2-\gamma)(t-a)}{\Gamma(\mu-\gamma)(b-a)^{1-\gamma}}(b-s)^{\mu-\gamma-1}, & a \leqslant t \leqslant s \leqslant b, \end{cases} \tag{3.24}$$

and

$$\widetilde{G}_2(t,s) = \begin{cases} \frac{\Gamma(2-\gamma)(t-a)}{\Gamma(\eta+\mu-\gamma)(b-a)^{1-\gamma}} - \frac{(t-s)^{\eta+\mu-1}}{\Gamma(\eta+\mu)(b-s)^{\eta+\mu-\gamma-1}}, & a \leqslant s \leqslant t \leqslant b, \\ \frac{\Gamma(2-\gamma)(t-a)}{\Gamma(\eta+\mu-\gamma)(b-a)^{1-\gamma}}, & a \leqslant t \leqslant s \leqslant b. \end{cases} \tag{3.25}$$

Proof. Because $u(a) = 0$, we can use the equation (3.19). Then, by applying the operator ${}^{\mathfrak{LC}}\mathfrak{D}^{\gamma}_{a^+}$ on (3.19), we get

$$\begin{aligned} {}^{\mathfrak{LC}}\mathfrak{D}^{\gamma}_{a^+}u(t) &= u'(a)\,{}^{\mathfrak{LC}}\mathfrak{D}^{\gamma}_{a^+}(t-a) - \lambda\mathfrak{I}^{\mu-\gamma}_{a^+}u(t) - \mathfrak{I}^{\eta+\mu-\gamma}_{a^+}(qu)(t) \\ &= u'(a)\frac{1}{\Gamma(2-\gamma)}(t-a)^{1-\gamma} - \lambda\mathfrak{I}^{\mu-\gamma}_{a^+}u(t) - \mathfrak{I}^{\eta+\mu-\gamma}_{a^+}(qu)(t), \end{aligned}$$

using the boundary condition ${}^{\mathfrak{LC}}\mathfrak{D}^{\gamma}_{a^+}u(b) = 0$, we obtain

$$u'(a) = \frac{\Gamma(2-\gamma)}{(b-a)^{1-\gamma}}\left[\lambda\mathfrak{I}^{\mu-\gamma}_{a^+}u(b) + \mathfrak{I}^{\eta+\mu-\gamma}_{a^+}(qu)(b)\right].$$

Substituting the value of $u'(a)$ in (3.19), we obtain

$$\begin{aligned} u(t) = &\frac{\lambda\Gamma(2-\gamma)(t-a)}{(b-a)^{1-\gamma}}\mathfrak{I}^{\mu-\gamma}_{a^+}u(b) - \lambda\mathfrak{I}^{\mu}_{a^+}u(t) \\ &+ \frac{\Gamma(2-\gamma)(t-a)}{(b-a)^{1-\gamma}}\mathfrak{I}^{\eta+\mu-\gamma}_{a^+}(qu)(b) - \mathfrak{I}^{\eta+\mu}_{a^+}(qu)(t), \end{aligned}$$

i.e.,

$$\begin{aligned} u(t) = &\frac{\lambda}{\Gamma(\mu-\gamma)}\frac{\Gamma(2-\gamma)(t-a)}{(b-a)^{1-\gamma}}\int_a^b (b-s)^{\mu-\gamma-1}u(s)\,ds - \frac{\lambda}{\Gamma(\mu)}\int_a^t (t-s)^{\mu-1}u(s)\,ds \\ &+ \frac{1}{\Gamma(\eta+\mu-\gamma)}\frac{\Gamma(2-\gamma)(t-a)}{(b-a)^{1-\gamma}}\int_a^b (b-s)^{\eta+\mu-\gamma-1}q(s)u(s)\,ds \\ &- \frac{1}{\Gamma(\eta+\mu)}\int_a^t (t-s)^{\eta+\mu-1}q(s)u(s)\,ds \end{aligned}$$

$$
\begin{aligned}
&= \lambda \int_a^t \left(\frac{\Gamma(2-\gamma)(t-a)}{\Gamma(\mu-\gamma)(b-a)^{1-\gamma}} (b-s)^{\mu-\gamma-1} - \frac{(t-s)^{\mu-1}}{\Gamma(\mu)} \right) u(s)\, ds \\
&\quad + \lambda \int_t^b \frac{\Gamma(2-\gamma)(t-a)}{\Gamma(\mu-\gamma)(b-a)^{1-\gamma}} (b-s)^{\mu-\gamma-1} u(s)\, ds \\
&\quad + \int_a^t (b-s)^{\eta+\mu-\gamma-1} \left(\frac{\Gamma(2-\gamma)(t-a)}{\Gamma(\eta+\mu-\gamma)(b-a)^{1-\gamma}} - \frac{(t-s)^{\eta+\mu-1}}{\Gamma(\eta+\mu)(b-s)^{\eta+\mu-\gamma-1}} \right) \\
&\quad \times q(s)u(s)\, ds + \int_t^b (b-s)^{\eta+\mu-\gamma-1} \frac{\Gamma(2-\gamma)(t-a)}{\Gamma(\eta+\mu-\gamma)(b-a)^{1-\gamma}} q(s)u(s)\, ds \\
&= \lambda \int_a^b \widetilde{G}_1(t,s)u(s)\, ds + \int_a^b (b-s)^{\eta+\mu-\gamma-1} \widetilde{G}_2(t,s)q(s)u(s)\, ds. \quad \square
\end{aligned}
\tag{3.26}
$$

LEMMA 11. *Assume that $1<\mu\leqslant 2$. The functions $\widetilde{G}_1$ and $\widetilde{G}_2$ defined in Lemma 10, satisfy the following properties:*

i) For any $t\in[a,b]$ we have

$$
\int_a^b |\widetilde{G}_1(t,s)|ds \leqslant \left(\frac{\Gamma(2-\gamma)}{\Gamma(\mu-\gamma+1)} + \frac{1}{\Gamma(\mu+1)} \right)(b-a)^{\mu}; \tag{3.27}
$$

ii)

$$
\max_{t,s\in[a,b]} |\widetilde{G}_2(t,s)| = \widetilde{C}, \tag{3.28}
$$

where $\widetilde{C}=C$ (here C is given by (1.12 with $1<\mu\leqslant 2$)).

Proof. i) For the first property, we have

$$
\begin{aligned}
\int_a^b |\widetilde{G}_1(t,s)|ds &= \int_a^t \left| \frac{\Gamma(2-\gamma)(t-a)}{\Gamma(\mu-\gamma)(b-a)^{1-\gamma}} (b-s)^{\mu-\gamma-1} - \frac{1}{\Gamma(\mu)} (t-s)^{\mu-1} \right| ds \\
&\quad + \int_t^b \left| \frac{\Gamma(2-\gamma)(t-a)}{\Gamma(\mu-\gamma)(b-a)^{1-\gamma}} (b-s)^{\mu-\gamma-1} \right| ds \\
&\leqslant \int_a^b \frac{\Gamma(2-\gamma)(t-a)}{\Gamma(\mu-\gamma)(b-a)^{1-\gamma}} (b-s)^{\mu-\gamma-1} ds + \int_a^t \frac{1}{\Gamma(\mu)} (t-s)^{\mu-1} ds \\
&= \frac{\Gamma(2-\gamma)(b-a)^{\mu-1}(t-a)}{\Gamma(\mu-\gamma+1)} + \frac{(t-a)^{\mu}}{\Gamma(\mu+1)} \\
&\leqslant \frac{\Gamma(2-\gamma)(b-a)^{\mu-1}(b-a)}{\Gamma(\mu-\gamma+1)} + \frac{(b-a)^{\mu}}{\Gamma(\mu+1)} \\
&= \left(\frac{\Gamma(2-\gamma)}{\Gamma(\mu-\gamma+1)} + \frac{1}{\Gamma(\mu+1)} \right)(b-a)^{\mu}.
\end{aligned}
\tag{3.29}
$$

ii) For the second property, it can be derived by the fact that the proof of Proposition 3.2 in [11] also remains valid for $\mu\in(1,2]$. $\square$

We have the following Lyapunov-type inequality.

THEOREM 8. *Assume that* $1 < \mu \leqslant 2$. *If a nontrivial continuous* 2*nd derivative on* $[a,b]$ *solution to the fractional boundary value problem* (1.13)–(1.14) *exists. Then*

$$\int_a^b (b-s)^{\eta+\mu-\gamma-1}|q(s)|ds \geqslant \frac{1}{\widetilde{C}}\left(1-|\lambda|(b-a)^{\mu}\left(\frac{\Gamma(2-\gamma)}{\Gamma(\mu-\gamma+1)}+\frac{1}{\Gamma(\mu+1)}\right)\right), \tag{3.30}$$

where $\widetilde{C}$ *is given by* (3.28).

Proof. By Lemmas 10 and 11 and taking the similar procedures in the Proof of Theorem 5, we can obtain (3.30). □

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