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A MAP–TYPE GRONWALL INEQUALITY ON FUNCTIONAL DIFFERENTIAL EQUATIONS WITH STATE–DEPENDENCE

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Abstract. In this paper, in order to investigate a Gronwall inequality with state-dependence, another auxiliary map-type Gronwall inequality is discussed by modifying the technique of sequential monotonization on functions into the one on maps. Then we employ the state-dependent Gronwall inequality to give the estimate and boundedness of solutions for a functional differential equation with state-dependence. Finally, we exhibit a concrete example of bounded solutions as application.

1. Introduction

As useful tools of studying existence, uniqueness, continuous dependence, boundedness and stability of solutions, invariant manifolds and invariant foliations for differential equations, the development of integral inequalities is accompanied by the investigation of various sorts of differential equations, integral equations and difference equations. In order to obtain estimate and stability of solutions for linear differential equations, earliest integral inequalities were established by Gronwall ([4]) and Bellman ([2]) successively in early 1900's. So this type of integral inequalities was also called Gronwall inequalities or Gronwall-Bellman inequalities. Later, in 1956 Bihari ([3]) developed Gronwall inequality

$$u(t) \leqslant a + \int_0^t f(s)w(u(s))ds, \quad t \geqslant 0, \tag{1.1}$$

where constant $a > 0$, f is a nonnegative function and w is a nondecreasing positive function, to discuss a form of nonlinear differential equations. Then (1.1) was further improved to the case with a delay

$$u(t) \leqslant a + \int_{b(t_0)}^{b(t)} f(s)w(u(s))ds, \quad t \geqslant t_0,$$

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where $t_0 \leqslant b(t) \leqslant t$, by Lipovan ([6]) for the research of delay differential equations in 2000. To multi-delays differential equations, in 2005 a summed Gronwall inequality

$$u(t) \leqslant a(t) + \sum_{i=1}^{n} \int_{b_i(t_0)}^{b_i(t)} f_i(t,s) w_i(u(s)) ds, \ \ t_0 \leqslant t < T,$$

was provided by Agarwal, Deng and Zhang ([1]), where $\{w_i\}$ is a functions sequence satisfying the so-called *sequential monotonicity* motivated by Pinto ([9]). In 2016, Zhou, Shen and Zhang ([13]) indicated that the powered Gronwall inequality

$$u(t) \leqslant a(t) + \sum_{i=1}^{n} \{ \int_{b_i(t_0)}^{b_i(t)} f_i(t,s) \phi_i(u(s)) ds \}^{p_i}, \ \ t_0 \leqslant t < \infty,$$

where $p_i \geqslant 1$, can be applied to singular integral equations and stochastic differential equations, by modifying sequential monotonization raised by Wang ([11]) into powered sequential monotonization. In 2020, Zhou, Shen and Zhang ([14]) still employed the sequential monotonization to extend impulsive Gronwall inequalities to the following formula

$$u(t) \leqslant a(t) + \sum_{i=1}^{n} \int_{b_i(t_0)}^{b_i(t)} f_i(t,s) \phi_i(u(s)) ds + \sum_{\{t_j\} \bigcap (t_0,t)} h_j(t) \psi_j(u(t_j^-)) \tag{1.2}$$

for $0 \leqslant t_0 \leqslant t < \infty$, which can be used in nonautonomous impulsive differential equations. Besides, one can see the monograph [8] and references therein to know many other integral inequalities.

In this paper we discuss a map-type Gronwall inequality

$$x(t) \leqslant a(t) + \sum_{i=1}^{n} \int_{b_i(t_0)}^{b_i(t)} f_i(t,s) (\mathbf{w}_i x)(s) ds, \quad t \in [t_0, \infty), \tag{1.3}$$

where map $\mathbf{w}_i : C([t_0,\infty),[t_0,\infty)) \to C([t_0,\infty),\mathbb{R}_+ \backslash \{0\})$ is denoted by

$$(\mathbf{w}_i x)(t) := w_i(t, x(t)), \quad x \in C([t_0,\infty),[t_0,\infty)), \quad i = 1,\ldots,n \tag{1.4}$$

with $w_i \in C([t_0,\infty)^2, \mathbb{R}_+ \backslash \{0\})$. It is raised for $n = 1$, $b_i(t) := t$ and $f_i(t,s) := f_i(s)$ in reference [15] as a preliminary to study asymptotic behaviors of a functional differential equation with state-dependence. Relying on inequality (1.3), we can discuss the following Gronwall inequality with state-dependence

$$x(t) \leqslant \alpha(t) + \sum_{i=1}^{n} \int_{t_0}^{t} g_i(t,s) \omega_i(x(T_i(s,x(s)))) ds, \quad t \in [t_0, \infty), \tag{1.5}$$

where $T_i \in C([t_0,\infty)^2, [t_0,\infty))$ and $\omega_i \in C([t_0,\infty), \mathbb{R}_+ \backslash \{0\})$ for all $i = 1,\ldots,n$, which comes from a general functional differential equation with state-dependence

$$\dot{x}(t) = f(t, x(T_1(t,x(t))), \ldots, x(T_n(t,x(t)))). \tag{1.6}$$

Equation (1.6) is called the one with state-dependence, since the unknown state x in the right hand side of (1.6) does not depend on only time t but also itself. It is widely concerned in many articles, e.g. [5, 7, 10, 12]. More precisely, inequality (1.5) is significant to estimate the solutions of (1.6), and give the asymptotics of the solutions. The ideas of sequential monotonicity and powered sequential monotonicity of functions w_i s are further generalized to sequential monotonicity of maps $\mathbf{w}_i$ s by us. Moreover, different from the previous works on Gronwall inequalities, here we do not need the nonnegativity of all known and unknown functions.

This paper is arranged as follows. In section 2 we solve the map-type Gronwall inequality (1.3) with sequential monotonicity of maps $\mathbf{w}_i$ s and without nonnegativity of functions known a and unknown x, and give the boundedness of its estimate. In section 3, as a corollary, we get the estimate of Gronwall inequality (1.5) with state-dependence and the boundedness of the estimate. Finally, as applications, we indicate the estimate and boundedness of solutions to a concrete (1.6).

2. Map-type Gronwall inequality

In this section, we discuss inequality (1.3). First, we provide some basic assumptions. As usual, denote set of positive integers $\{1,2,3,\dots\}$ and set of nonnegative real numbers by $\mathbb{N}$ and $\mathbb{R}_+$ respectively. Denote sets of nondecreasing functions, nondecreasing continuous functions and nondecreasing and continuously differentiable functions from $\mathbb{R}_+$ to itself by $\mathscr{I}(\mathbb{R}_+,\mathbb{R}_+)$, $C_{\mathscr{I}}(\mathbb{R}_+,\mathbb{R}_+)$ and $C^1_{\mathscr{I}}(\mathbb{R}_+,\mathbb{R}_+)$ respectively.

2.1. Estimate of solutions

As preliminary, we exhibit the definition of maps sequential nondecreasing, which is a generalization of functions sequential nondecreasing proposed in e.g. [1, 11, 14].

DEFINITION 2.1. An order relation $\propto$ is called maps sequential nondecreasing, if $\mathbf{w}_1 \propto \mathbf{w}_2$ for maps $\mathbf{w}_1$ and $\mathbf{w}_2$ defined as in (1.4), satisfying functions w_1 and w_2 are both nondecreasing in respect of their second variable, implies w_2/w_1 is also nondecreasing in respect of its second variable.

In what follows, we consider set $P = C([t_0,\infty),[t_0,\infty))$ and its partial order relation "$\leqslant$" is given by $x \leqslant y$ iff $x,y \in P$ and $x(t) \leqslant y(t)$ for all $t \in [t_0,\infty)$. Considering inequality (1.3), for $i = 1,\dots,n$ assume that

(A1) $a \in C^1_{\mathscr{I}}([t_0,\infty),[t_0,\infty))$, $f_i \in C([t_0,\infty)^2,\mathbb{R}_+)$ and $\mathbf{w}_i : C([t_0,\infty),[t_0,\infty)) \to C([t_0,\infty),\mathbb{R}_+\setminus\{0\})$;

(A2) $b_i \in C^1_{\mathscr{I}}([t_0,\infty),[t_0,\infty))$ satisfies $b_i(t) \leqslant t$ on $[t_0,\infty)$;

(A3) both functions w_i and f_i are nondecreasing with respect to their first variable;

(A4) maps sequence $\{\mathbf{w}_i\}_{i=1,\dots,n}$ defined as in (1.4) is sequential nondecreasing, i.e., $\mathbf{w}_i \propto \mathbf{w}_{i+1}$ for all $i = 1,\dots,n-1$.

If function a is not nondecreasing, then one can monotonize them as done in [1]; if functions f_i and w_i are not nondecreasing with respect to t, then one can monotonize them as done as

$$\tilde{f}_i(t,s) = \sup_{\tau\in[t_0,t]} f_i(\tau,s), \quad \tilde{w}_i(t,s) = \sup_{\tau\in[t_0,t]} w_i(\tau,s).$$

If maps sequence $\{\mathbf{w}_i\}_{i=1,\dots,n}$ is not sequential nondecreasing, we can enlarge the functions sequence $\{w_i\}$ into a new sequence $\{\upsilon_i\}$ such that $w_i(t,s) \leqslant \upsilon_i(t,s)$ for all $(t,s) \in [t_0,\infty)^2$ and $i = 1,\dots,n$, where υ_i s are all nondecreasing with respect to s, and $\upsilon_i \propto \upsilon_{i+1}$ for each $i = 1,\dots,n-1$. This replacement is called *maps sequential monotonization*, which can be done by setting

$$\upsilon_i(t,s) := \begin{cases} \max_{\tau\in[t_0,s]} w_1(t,\tau), & i=1, \\ \upsilon_{i-1}(t,s)\max_{\tau\in[t_0,s]} \dfrac{w_i(t,\tau)}{\upsilon_{i-1}(t,\tau)}, & i=2,\dots,n, \end{cases} \tag{2.7}$$

recursively as in [11, 13]. Let

$$(\mathscr{W}_i x)(t) = W_i(t,x(t)) := \int_{x_*}^{x(t)} \frac{ds}{w_i(t,s)}, \quad x\in P, \quad i=1,\dots,n, \tag{2.8}$$

where $x_* \in [t_0,\infty)$ is a given constant arbitrarily. The fact $w_i \in C([t_0,\infty)^2, \mathbb{R}_+\setminus\{0\})$ given in (A1) guarantees that (2.7) and (2.8) are both meaningful for all $(t,s)\in[t_0,\infty)^2$. If $w_i(t,s) = 0$ for some $(t,s)\in[t_0,\infty)^2$, only need to amplify w_i a little like [13, 14] as

$$\breve{w}_i(t,s) := w_i(t,s) + \varepsilon \tag{2.9}$$

with an arbitrarily chosen positive constant ε, which fulfills (A3)–(A4) as well. We give the main theorem as follows

THEOREM 2.1. *Suppose* (A1)–(A4) *hold and* $x\in P$ *satisfies* (1.3) *for all* $t\in[t_0,\infty)$. *If* $W_i(t,\infty)=\infty$ *for* $t\in[t_0,\infty)$ *and* $i=1,\dots,n$ *and* a_i *s are determined recursively by*

$$\begin{aligned} a_1(t) &:= a(t), \\ a_{i+1}(t) &:= \mathscr{W}_i^{-1}\{(\mathscr{W}_i a_i)(t) + \int_{b_i(t_0)}^{b_i(t)} f_i(t,s)ds\}, \quad i=1,\dots,n-1, \end{aligned}$$

then

$$x(t) \leqslant \mathscr{W}_n^{-1}\{(\mathscr{W}_n a_n)(t) + \int_{b_n(t_0)}^{b_n(t)} f_n(t,s)ds\}, \quad t\in[t_0,\infty). \tag{2.10}$$

Remark that, similarly to [1], distinct choices of x_* in W_i do not disturb the result above. In fact, for positive constant $y_* \neq x_*$, let

$$\breve{W}_i(t,x(t)) := \int_{y_*}^{x(t)} \frac{ds}{w_i(t,s)},$$

then $\breve{W}_i(t,x(t)) = W_i(t,x(t)) + \breve{W}_i(t,x_*)$. Let $y(t) := \breve{W}_i(t,x(t))$, then $y(t) - \breve{W}_i(t,x_*) = W_i(t,x(t))$. We further get

$$\mathscr{W}_i^{-1}(y(t) - \breve{W}_i(t,x_*)) = x(t) = (\breve{\mathscr{W}}_i^{-1}y)(t).$$

It yields that

$$\begin{aligned}
&\breve{\mathscr{W}}_i^{-1}\{(\breve{\mathscr{W}}_i x)(t) + \int_{b_i(t_0)}^{b_i(t)} f_i(t,s)ds\} \\
&= \mathscr{W}_i^{-1}\{(\breve{\mathscr{W}}_i x)(t) + \int_{b_i(t_0)}^{b_i(t)} f_i(t,s)ds - \breve{W}_i(t,x_*)\} \\
&= \mathscr{W}_i^{-1}\{(\mathscr{W}_i x)(t) + \int_{b_i(t_0)}^{b_i(t)} f_i(t,s)ds\}.
\end{aligned}$$

Hence, the result (2.10) is independent of the choice of $x_* \in [t_0,\infty)$ in $\mathscr{W}_i$.

Proof of Theorem 2.1. By (A3), it follows from (1.3) that

$$x(t) \leqslant a(t) + \sum_{i=1}^{n} \int_{b_i(t_0)}^{b_i(t)} f_i(T,s) w_i(T,x(s))ds, \quad \forall t \in [t_0,T], \tag{2.11}$$

where $T \in [t_0,\infty)$ is an arbitrarily chosen constant. We solve x from (2.11). Let

$$a_1(T,t) := a(t), \quad a_{i+1}(T,t) := W_i^{-1}\{T, W_i(T,a_i(T,t)) + \int_{b_i(t_0)}^{b_i(t)} f_i(T,s)ds\}$$

for $i = 1,\ldots,n-1$. We claim that solutions of (2.11) is

$$x(t) \leqslant W_n^{-1}\{T, W_n(T,a_n(T,t)) + \int_{b_n(t_0)}^{b_n(t)} f_n(T,s)ds\}, \quad \forall t \in [t_0,T]. \tag{2.12}$$

Clearly, recalling (A4), for $u_1,u_2 \in P$ satisfying order relation $u_1 \leqslant u_2$ compute

$$W_i(T,u_2(t)) - W_i(T,u_1(t)) = \int_{u_1(t)}^{u_2(t)} \frac{ds}{w_i(T,s)} \geqslant 0, \tag{2.13}$$

which also holds reversely. It follows from positive function w_i that W_i and W_i^{-1} are both strictly increasing in respect of their second variable, also implying invertibility of W_i to its second variable. For each $a_i(T,\cdot) \in [t_0,\infty)$, we have $W_i(T,t_0) \leqslant W_i(T,a_i(T,t)) + \int_{b_i(t_0)}^{b_i(t)} f_i(T,s)ds < \infty = W_i(T,\infty)$. By assumption of Theorem 2.1, the sum $W_i(T,a_i(T,t)) + \int_{b_i(t_0)}^{b_i(t)} f_i(T,s)ds$ lies in domain of $W_i^{-1}(T,\cdot)$ for any $T \in [t_0,\infty)$ and $t \in [t_0,T]$. We prove the assertion (2.12) as follows.

First, we verify (2.12) holds for $n=1$. For $n=1$, inequality (2.11) can be rewritten as

$$x(t) \leqslant a(t) + z_1(t), \tag{2.14}$$

where

$$z_1(t) := \int_{b_1(t_0)}^{b_1(t)} f_1(T,s)w_1(T,x(s))ds$$

is a nondecreasing function. From (A3) – (A4) we know

$$w_1(T,x(b_1(t))) \leqslant w_1(T,z_1(b_1(t)) + a(b_1(t))) \leqslant w_1(T,z_1(t) + a(t)). \tag{2.15}$$

Along with (2.15), for $t \in [t_0,T]$ compute

$$\begin{aligned}\frac{(z_1(t)+a(t))'}{w_1(T,z_1(t)+a(t))} &\leqslant \frac{f_1(T,b_1(t))w_1(T,x(b_1(t)))b_1'(t)}{w_1(T,z_1(t)+a(t))} + \frac{a'(t)}{w_1(T,z_1(t)+a(t))} \\ &\leqslant f_1(T,b_1(t))b_1'(t) + \frac{a'(t)}{w_1(T,a(t))}.\end{aligned}$$

Integrating both sides of inequality above from t_0 to t, by the fact $z_1(t_0) = 0$, we obtain

$$W_1(T,z_1(t)+a(t)) \leqslant W_1(T,a(t)) + \int_{b_1(t_0)}^{b_1(t)} f_1(T,s)ds, \quad \forall t \in [t_0,T]. \tag{2.16}$$

Recall that $W_1(T,a(t)) + \int_{b_1(t_0)}^{b_1(t)} f_1(T,s)ds$ lies in domain of $W_1^{-1}(T,\cdot)$. Combining (2.16) with (2.14), we obtain (2.12) is true for $n = 1$.

In order to prove (2.12) by induction, suppose that (2.12) holds for $n = m$. Then inequality (2.11) for $n = m+1$ can be rewritten as

$$x(t) \leqslant a(t) + z_2(t), \tag{2.17}$$

where

$$z_2(t) := \sum_{i=1}^{m+1} \int_{b_i(t_0)}^{b_i(t)} f_i(T,s)w_i(T,x(s))ds$$

is a nondecreasing function. Let

$$(\mathbf{u}_{i+1}x)(t) = u_{i+1}(T,x(t)) := \frac{w_{i+1}(T,x(t))}{w_1(T,x(t))}, \quad i = 1,\dots,m.$$

From (A3) – (A4) we see that z_2 also satisfies (2.15) like z_1. Then

$$\begin{aligned}\frac{(z_2(t)+a(t))'}{w_1(T,z_2(t)+a(t))} &\leqslant \sum_{i=1}^{m+1} \frac{f_i(T,b_i(t))w_i(T,x(b_i(t)))b_i'(t)}{w_1(T,z_2(t)+a(t))} + \frac{a'(t)}{w_1(T,z_2(t)+a(t))} \\ &\leqslant \sum_{i=1}^{m+1} \frac{f_i(T,b_i(t))w_i(T,z_2(b_i(t))+a(b_i(t)))b_i'(t)}{w_1(T,z_2(b_i(t))+a(b_i(t)))} + \frac{a'(t)}{w_1(T,a(t))} \\ &\leqslant f_1(T,b_1(t))b_1'(t) + \frac{a'(t)}{w_1(T,a(t))} \\ &\quad + \sum_{i=1}^{m} f_{i+1}(T,b_{i+1}(t))u_{i+1}(T,z_2(b_{i+1}(t))+a(b_{i+1}(t)))b_{i+1}'(t).\end{aligned}$$

Integrating both sides of the inequality above from t_0 to t, by the fact $z_2(t_0)=0$ we have

$$W_1(T,z_2(t)+a(t)) \leqslant W_1(T,a(t)) + \int_{b_1(t_0)}^{b_1(t)} f_1(T,s)ds + \sum_{i=1}^{m} \int_{b_{i+1}(t_0)}^{b_{i+1}(t)} f_{i+1}(T,s)u_{i+1}(T,z_2(s)+a(s))ds \quad (2.18)$$

for any $t \in [t_0,T]$.

Let $\xi(t) := W_1(T,z_2(t)+a(t))$ and $\gamma_1(T,t) := W_1(T,a(t)) + \int_{b_1(t_0)}^{b_1(t)} f_1(T,s)ds$. Then (2.18) can be rewritten as

$$\xi(t) \leqslant \gamma_1(T,t) + \sum_{i=1}^{m} \int_{b_{i+1}(t_0)}^{b_{i+1}(t)} f_{i+1}(T,s)u_{i+1}(T,W_1^{-1}(T,\xi(s)))ds. \quad (2.19)$$

Recalling W_1^{-1} is nondecreasing in respect of its second variable and

$$u_{i+1}(T,W_1^{-1}(T,x(t))) = \frac{w_{i+1}(T,W_1^{-1}(T,x(t)))}{w_1(T,W_1^{-1}(T,x(t)))}, \quad i=1,\dots,m,$$
$$\frac{u_{i+1}(T,W_1^{-1}(T,x(t)))}{u_i(T,W_1^{-1}(T,x(t)))} = \frac{w_{i+1}(T,W_1^{-1}(T,x(t)))}{w_i(T,W_1^{-1}(T,x(t)))}, \quad i=2,\dots,m,$$

one can verify that $\{\mathbf{u}_{i+1} \circ \mathscr{W}_1^{-1}\}_{i=1,\dots,m}$ is sequential nondecreasing. It yields that (2.19) is of same form as (2.11) for $n=m$ and satisfies inductive assumption. Hence,

$$\xi(t) \leqslant U_m^{-1}\{T,U_m(T,\gamma_m(T,t)) + \int_{b_{m+1}(t_0)}^{b_{m+1}(t)} f_{m+1}(T,s)ds\}, \quad t \in [t_0,\infty), \quad (2.20)$$

where

$$\gamma_{i+1}(T,t) := U_i^{-1}\{T,U_i(T,\gamma_i(T,t)) + \int_{b_{i+1}(t_0)}^{b_{i+1}(t)} f_{i+1}(T,s)ds\},$$
$$\begin{aligned} U_i(T,x(t)) &:= \int_{W_1(T,x_{i+1}(t_0))}^{x(t)} \frac{ds}{u_{i+1}(T,W_1^{-1}(T,s))} = \int_{W_1(T,x_{i+1}(t_0))}^{x(t)} \frac{w_1(T,W_1^{-1}(T,s))}{w_{i+1}(T,W_1^{-1}(T,s))}ds \\ &= \int_{W_1(T,x_{i+1}(t_0))}^{x(t)} \frac{dW_1^{-1}(T,s)}{w_{i+1}(T,W_1^{-1}(T,s))} = \int_{x_{i+1}(t_0)}^{W_1^{-1}(T,x(t))} \frac{ds}{w_{i+1}(T,s)} \\ &= W_{i+1}(T,W_1^{-1}(T,x(t))), \quad i=1,\dots,m. \end{aligned}$$

Here one can verify that $U_i(T,\infty)=\infty$. In fact, by the assumption $W_i(T,\infty)=\infty$ we get $W_i^{-1}(T,\infty)=\infty$. From the formula of U_i above, we have

$$U_i(T,\infty) = W_{i+1}(T,W_1^{-1}(T,\infty)) = W_{i+1}(T,\infty) = \infty.$$

Then, like the fact $W_i(T,a_i(T,t))+\int_{b_i(t_0)}^{b_i(t)} f_i(T,s)ds$ lies in domain of $W_i^{-1}(T,\cdot)$, we know that $U_i(T,\gamma_i(T,t))+\int_{b_{i+1}(t_0)}^{b_{i+1}(t)} f_{i+1}(T,s)ds$ is in the domain of $U_i^{-1}(T,\cdot)$. Therefore, we obtain from (2.17) and (2.20) that

$$\begin{aligned} x(t) &\leqslant z_2(t)+a(t)=W_1^{-1}(T,\xi(t)) \\ &\leqslant W_m^{-1}\{T,W_m(T,W_1^{-1}(T,\gamma_m(T,t)))+\int_{b_{m+1}(t_0)}^{b_{m+1}(t)} f_{m+1}(T,s)ds\} \end{aligned} \tag{2.21}$$

for all $t\in[t_0,T]$. To simplify (2.21), we claim that $W_1^{-1}(T,\gamma_i(T,t))=a_{i+1}(T,t)$ for $i=1,\ldots,m$. It is easy to testify that

$$W_1^{-1}(T,\gamma_1(T,t))=W_1^{-1}\{T,W_1(T,a_1(T,t))+\int_{b_1(t_0)}^{b_1(t)} f_1(T,s)ds\}=a_2(T,t),$$

i.e., the assertion is true for $i=1$. Suppose that the assertion holds for $i=k$. Then from the inductive assumption,

$$\begin{aligned} W_1^{-1}(T,\gamma_{k+1}(T,t)) &= W_1^{-1}\{T,U_k^{-1}\{T,U_k(T,\gamma_k(T,t))+\int_{b_{k+1}(t_0)}^{b_{k+1}(t)} f_{k+1}(T,s)ds\}\} \\ &= W_{k+1}^{-1}\{T,W_{k+1}(T,W_1^{-1}(T,\gamma_k(T,t)))+\int_{b_{k+1}(t_0)}^{b_{k+1}(t)} f_{k+1}(T,s)ds\} \\ &= W_{k+1}^{-1}\{T,W_{k+1}(T,a_{k+1}(T,t))+\int_{b_{k+1}(t_0)}^{b_{k+1}(t)} f_{k+1}(T,s)ds\} \\ &= a_{k+2}(T,t), \end{aligned}$$

which proves the assertion by induction. It follows from the assertion and inequality (2.21) that

$$x(t)\leqslant W_{m+1}^{-1}\{T,W_{m+1}(T,a_{m+1}(T,t))+\int_{b_{m+1}(t_0)}^{b_{m+1}(t)} f_{m+1}(T,s)ds\}$$

for all $t\in[t_0,T]$. Thus, by induction the assertion (2.12) is true.

Further, letting $t=T$ in inequality (2.12), we have

$$\begin{aligned} x(T) &\leqslant \mathscr{W}_n^{-1}\{(\mathscr{W}_n a_n)(T,T)+\int_{b_n(t_0)}^{b_n(T)} f_n(T,s)ds\}, \\ a_i(T,T) &= \mathscr{W}_i^{-1}\{(\mathscr{W}_i a_i)(T,T)+\int_{b_i(t_0)}^{b_i(T)} f_i(T,s)ds\}, \\ a_0(T,T) &= a(T), \quad \forall T\in[t_0,\infty), \end{aligned}$$

implying (2.10) by the arbitrariness of T. Thus, Theorem 2.1 is proved. $\square$

2.2. Boundedness of estimate

In what follows we discuss boundedness of estimate (2.10) in Theorem 2.1. Suppose that

(B1) $a \in C^1_{\mathscr{I}}([t_0,\infty),[t_0,\infty))$ is bounded;

(B2) all $f_i \in C([t_0,\infty)^2,\mathbb{R}_+)$ and $b_i \in C^1_{\mathscr{I}}([t_0,\infty),[t_0,\infty))$ satisfy that $\int_{b_i(t_0)}^{b_i(\infty)} f_i(\infty,\ s)ds < \infty$.

THEOREM 2.2. *Suppose that* (A1)–(A4) *and* (B1)–(B2) *hold. If* $W_i(t,\infty) = \infty$ *for* $i = 1,\dots,n$ *and* $t \in [t_0,\infty)$*, then* $x(t)$ *in* (2.10) *is upper bounded for* $t \in [t_0,\infty)$.

Proof. We claim that a_i in Theorem 2.1 is bounded for $i = 1,\dots,n$. It is true for $i = 1$ from (B1). Assuming that a_i is bounded for $i = k$, we verify it is true for $i = k+1$. Note that

$$a_{k+1}(t) = W_k^{-1}\{t, W_k(t,a_k(t)) + \int_{b_k(t_0)}^{b_k(t)} f_k(t,s)ds\}, \quad t \in [t_0,\infty). \tag{2.22}$$

Recall below (2.13) that W_i and W_i^{-1} are both nondecreasing in respect of their second variable. Since $W_i(t,\infty) = \infty$, by inductive assumption we get

$$W_k(t,a_k(t)) < W_k(t,\infty) = \infty, \quad t \in [t_0,\infty).$$

Along with (B2), we obtain

$$W_k(t,a_k(t)) + \int_{b_k(t_0)}^{b_k(t)} f_k(t,s)ds < \infty, \quad t \in [t_0,\infty).$$

From the monotonicity of W_i in respect of its second variable, the fact $W_k(t,\infty) = \infty$ follows that $W_k^{-1}(t,\infty) = \infty$. By the monotonicity of W_i^{-1} with respect to its second variable, we get

$$W_k^{-1}\{t, W_k(t,a_k(t)) + \int_{b_k(t_0)}^{b_k(t)} f_k(t,s)ds\} < W_k^{-1}(t,\infty) = \infty, \quad t \in [t_0,\infty),$$

which follows from (2.22) that a_{k+1} is bounded for $t \in [t_0,\infty)$. By induction the assertion is true. It follows from (2.10) that $x(t)$ is bounded for any $t \in [t_0,\infty)$, and Theorem 2.2 is proved. □

3. Gronwall inequality with state-dependence

In this section, we generalize our result to the Gronwall inequality with state-dependence, coming from a functional differential equations with state-dependence. Considering inequality (1.5), for $i = 1,\dots,n$ suppose that

(C1) $\alpha \in C^1_{\mathscr{I}}([t_0,\infty),[t_0,\infty))$, $g_i \in C([t_0,\infty)^2,\mathbb{R}_+)$, $T_i \in C([t_0,\infty)^2,[t_0,\ \infty))$ and $\omega_i \in C_{\mathscr{I}}([t_0,\infty),\mathbb{R}_+\setminus\{0\})$;

(C2) both g_i and T_i are nondecreasing with respect to its first variable;

(C3) α is bounded on $[t_0,\infty)$;

(C4) each g_i satisfies that $\int_{t_0}^{\infty} g_i(\infty,s)ds < \infty$.

The following corollary give the estimate of x in inequality (1.5).

COROLLARY 3.1. *Suppose* (C1) – (C2) *hold and* $x \in P$ *satisfies* (1.5) *for any* $t \in [t_0,\infty)$. *If there exists a* $\varphi \in C_{\mathscr{I}}([t_0,\infty),[a(t_0),\infty))$ *satisfying* $V_i(t,\infty) = \infty$ *for* $i = 1,\dots,n$ *and* $t \in [t_0,\infty)$, *where*

$$(\mathscr{V}_i x)(t) \;=\; V_i(t,x(t)) := \int_{x_*}^{x(t)} \frac{ds}{\varpi_i(t,s)}, \quad x \in P, \quad i = 1,\dots,n, \tag{3.23}$$

$$\varpi_i(t,s) := \begin{cases} \max_{\tau\in[t_0,s]} \omega_i\circ\varphi\circ T_i(t,\tau), & i=1, \\ \varpi_{i-1}(t,s)\max_{\tau\in[t_0,s]} \dfrac{\omega_i\circ\varphi\circ T_i(t,\tau)}{\varpi_{i-1}(t,\tau)}, & i=2,\dots,n, \end{cases} \tag{3.24}$$

and $x_* \in [t_0,\infty)$ *is an arbitrarily given constant, then all* x, *satisfying* $x(t) \leqslant \varphi(t)$ *for* $t \in [t_0,\infty)$, *can be estimated by*

$$x(t) \leqslant \min\{\varphi(t), \mathscr{V}_n^{-1}\{(\mathscr{V}_n\alpha_n)(t) + \int_{t_0}^{t} g_n(t,s)ds\}\}, \quad t \in [t_0,\infty), \tag{3.25}$$

where functions α_i *s are determined recursively by*

$$\begin{aligned} \alpha_1(t) &:= \alpha(t), \\ \alpha_{i+1}(t) &:= \mathscr{V}_i^{-1}\{(\mathscr{V}_i\alpha_i)(t) + \int_{t_0}^{t} g_i(t,s)ds\}, \quad i = 1,\dots,n-1. \end{aligned}$$

Proof. Considering $x(t) \leqslant \varphi(t)$, it follows from inequality (1.5) and (3.24) that

$$\begin{aligned} x(t) &\leqslant \alpha(t) + \sum_{i=1}^{n} \int_{t_0}^{t} g_i(t,s)\omega_i(x(T_i(s,x(s))))ds \\ &\leqslant \alpha(t) + \sum_{i=1}^{n} \int_{t_0}^{t} g_i(t,s)\omega_i\circ\varphi\circ T_i(s,x(s))ds \\ &\leqslant \alpha(t) + \sum_{i=1}^{n} \int_{t_0}^{t} g_i(t,s)\varpi_i(s,x(s))ds, \quad t \in [t_0,\infty). \end{aligned} \tag{3.26}$$

The function $x(t)$ fulfills integral inequality (1.3), where

$$a(t) := \alpha(t), \quad f_i(t,s) := g_i(t,s), \quad w_i(s,x(s)) := \varpi_i(s,x(s)).$$

It is easy to testify that (A1) holds by (C1). From (C2) and the monotonicity of ω_i and φ_i, it follows that (A3) is true. Through the transformation (3.24), one can verify that (A4) is satisfied. Thus, employing Theorem 2.1 in (3.26), we get the estimate

$$x(t) \leqslant \mathscr{V}_n^{-1}\{(\mathscr{V}_n\alpha_n)(t) + \int_{t_0}^{t} g_n(t,s)ds\}, \quad t \in [t_0,\infty).$$

Along with the fact $x(t) \leqslant \varphi(t)$, (3.25) is gotten and Corollary 3.1 is proved. □

From Theorem 2.2 we can conclude result on boundedness of the estimate as follows.

COROLLARY 3.2. *Suppose that* (C1)–(C4) *hold. If* $V_i(t,\infty) = \infty$ *for* $i = 1,\dots,n$ *and* $t \in [t_0,\infty)$*, then* $x(t)$ *in* (3.25) *is upper bounded for* $t \in [t_0,\infty)$.

4. Applications

In this section, we employ our integral inequality to estimate solutions of a concrete functional differential equation (1.6), and obtain the boundedness of its solutions. Consider Cauchy problem as follows

$$\begin{cases} \dot{x}(t) = f(t,x(T_1(t,x(t))),\dots,x(T_n(t,x(t)))), & t \in [t_0,\infty), \\ x(t_0) = x_0, \end{cases} \tag{4.27}$$

where constant $x_0 \geqslant t_0$, $f \in C([t_0,\infty)^{n+1},\mathbb{R}_+)$ and $T_i \in C([t_0,\infty)^2,[t_0,\infty))$ is nondecreasing in respect of its first variable for $1 \leqslant i \leqslant n$. Suppose that

(D1) $0 \leqslant f(t,x_1,\dots,x_n) \leqslant \beta(t) + \sum_{i=1}^{n} \gamma_i(t)\omega_i(x_i)$ for all $(t,x_1,\dots,x_n) \in [t_0,\infty)^{n+1}$;

(D2) $f(t,x_1,\dots,x_n) \leqslant \varphi'(t)$ for all $(t,x_1,\dots,x_n) \in [t_0,\infty)^{n+1}$;

(D3) β satisfies $\int_{t_0}^{\infty} \beta(s)ds < \infty$;

(D4) each γ_i satisfies that $\int_{t_0}^{\infty} \gamma_i(s)ds < \infty$,

where $\beta \in C([t_0,\infty),[t_0,\infty))$, $\gamma_i \in C([t_0,\infty),\mathbb{R}_+)$ and $\omega_i \in C_{\mathscr{I}}([t_0,\infty),\mathbb{R}_+\backslash\{0\})$ for all $1 \leqslant i \leqslant n$. The following corollary give the estimate and boundedness of the solution x of Cauchy problem (4.27).

COROLLARY 4.1. *Suppose* (D1)–(D2) *hold and* $\varphi \in C_{\mathscr{I}}([t_0,\infty),[t_0,\infty))$ *satisfies* $V_i(t,\infty) = \infty$ *for* $i = 1,\dots,n$ *and* $t \in [t_0,\infty)$*, where* V_i *is defined as in* (3.23)*. Then for* $x_0 \in [t_0,\varphi(t_0))$*, all solutions* x *of* (4.27) *can be estimated by*

$$x(t) \leqslant \mathscr{V}_n^{-1}\{(\mathscr{V}_n\alpha_n)(t) + \int_{t_0}^{t} \gamma_n(s)ds\}, \quad t \in [t_0,\infty), \tag{4.28}$$

where functions α_i *s are determined recursively by*

$$\alpha_1(t) := x_0 + \int_{t_0}^{t} \beta(s)ds,$$
$$\alpha_{i+1}(t) := \mathscr{V}_i^{-1}\{(\mathscr{V}_i\alpha_i)(t) + \int_{t_0}^{t} \gamma_i(s)ds\}, \quad i = 1,\ldots,n-1.$$

Additionally, if (D3)–(D4) *hold, then all the solutions* x *are bounded.*

Proof. The equivalent integral equation of Cauchy problem (4.27) is

$$x(t) = x_0 + \int_{t_0}^{t} f(s,x(T_1(s,x(s))),\ldots,x(T_n(s,x(s))))ds, \tag{4.29}$$

which follows from (D1) that

$$x(t) \leqslant x_0 + \int_{t_0}^{t} \beta(s)ds + \sum_{i=1}^{n} \int_{t_0}^{t} \gamma_i(s)\omega_i(x(T_i(s,x(s))))ds. \tag{4.30}$$

The function $x(t)$ fulfills integral inequality (1.5), where

$$\alpha(t) := x_0 + \int_{t_0}^{t} \beta(s)ds, \quad g_i(t,s) := \gamma_i(s).$$

One can verify (C1) holds by (D1). (C2) also holds naturally. Apply Corollary 3.1 to (4.30), then all x, satisfying $x(t) \leqslant \varphi(t)$ for $t \in [t_0,\infty)$, can be estimated by (4.28). On the other hand, combining (4.27) with (D2), we get

$$x(t) \leqslant x_0 + \varphi(t) - \varphi(t_0) \leqslant \varphi(t), \quad t \in [t_0,\infty).$$

Therefore, all solutions x of (4.27) can be estimated by (4.28).

It is easy to testify that (C3)–(C4) hold by (D3)–(D4). Applying Corollary 3.2 to (4.28), all solutions x of (4.27) is upper bounded for $t \in [t_0,\infty)$. Recalling the fact $f(t,x_1,\ldots,x_n) \geqslant 0$, it follows from (4.29) that $x(t) \geqslant x_0$ for all $t \in [t_0,\infty)$. Thus, all these solutions x are bounded and Corollary 4.1 is proved. □

Next, we exhibit a concrete (4.27), that is Cauchy problem

$$\begin{cases} \dot{x}(t) = \dfrac{1}{4(t+1)^2}\Big\{2 + \dfrac{2\arctan\{x(tx(t))\}}{\pi} + \dfrac{4\arctan^2\{x(tx(t))\}}{\pi^2}\Big\}, & t \in \mathbb{R}_+, \\ x(0) = x_0, \end{cases} \tag{4.31}$$

where constant $x_0 \geqslant 0$. Set

$$(\mathscr{V}_i x)(t) := \int_0^{x(t)} \frac{ds}{\arctan^i\left(2 - \dfrac{1}{ts+1}\right)}, \quad x \in C(\mathbb{R}_+,\mathbb{R}_+), \quad i = 1,2.$$

CONCLUSION 1. *For $x_0 \in [0,1)$, all solutions x of (4.31) can be estimated by*

$$x(t) \leqslant \mathscr{V}_2^{-1}\Big\{\mathscr{V}_2 \circ \mathscr{V}_1^{-1}\Big\{\mathscr{V}_1\Big(x_0 + \frac{1}{2} - \frac{1}{2(t+1)}\Big) + \frac{1}{2\pi} - \frac{1}{2\pi(t+1)}\Big\} + \frac{1}{\pi^2} - \frac{1}{\pi^2(t+1)}\Big\},$$

for all $t \in \mathbb{R}_+$, and these solutions are all bounded.

Proof. One can easily verify that (D1) – (D4) hold, where

$$\beta(t) = \frac{1}{2(t+1)^2}, \quad \gamma_1(t) = \frac{1}{2\pi(t+1)^2}, \quad \gamma_2(t) = \frac{1}{\pi^2(t+1)^2},$$

$$\omega_i(x) = \arctan^i x, \quad n = 2, \quad \varphi(t) = 2 - \frac{1}{t+1}, \quad T_i(t, x(t)) = tx(t),$$

$$\int_0^\infty \beta(s)ds = \frac{1}{2}, \quad \int_0^\infty \gamma_1(s)ds = \frac{1}{2\pi}, \quad \int_0^\infty \gamma_2(s)ds = \frac{1}{\pi^2},$$

$$\varpi_i(t,s) = \arctan^i\Big(2 - \frac{1}{ts+1}\Big), \quad i = 1, 2.$$

One can verify that $\varpi_1 \propto \varpi_2$. In fact, the formula

$$\frac{\varpi_2(t,s)}{\varpi_1(t,s)} = \arctan\Big(2 - \frac{1}{ts+1}\Big)$$

is nondecreasing in respect of the variable s for all $(t,s) \in \mathbb{R}_+^2$. One can also verify that $V_i(t,\infty) = \infty$ for all $t \in \mathbb{R}_+$ and $i = 1, 2$. In fact,

$$(\mathscr{V}_i x)(t) := \int_0^{x(t)} \frac{ds}{\arctan^i\Big(2 - \frac{1}{ts+1}\Big)} \geqslant \int_0^{x(t)} \frac{ds}{\arctan^i 2}, \quad x \in C(\mathbb{R}_+, \mathbb{R}_+), \quad i = 1, 2,$$

which indicates that

$$V_i(t,\infty) \geqslant \int_0^\infty \frac{ds}{\arctan^i 2} = \infty, \quad \forall t \in \mathbb{R}_+, \quad i = 1, 2.$$

Applying Corollary 4.1, the proof of Conclusion 1 is completed. □

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REFERENCES

[1] R. P. Agarwal, S. Deng, W. Zhang, *Generalization of a retarded Gronwall-like inequality and its applications*, Appl. Math. Comput., **165** (2005), 599–612.

[2] R. Bellman, *The stability of solutions of linear differential equations*, Duke Math. J., **10** (1943), 643–647.

[3] I. A. Bihari, *A generalization of a lemma of Bellman and its application to uniqueness problem of differential equation*, Acta Math. Acad. Sci. Hung., **7** (1956), 81–94.

[4] T. H. Gronwall, *Note on the derivatives with respect to a parameter of the solutions of a system of differential equations*, Ann. Math., **20** (1919), 292–296.

[5] J. Härterich, B. Sandstede, A. Scheel, *Exponential dichotomies for linear non-autonomous functional differential equations of mixed type*, Indiana Univ. Math. J., **51** (2020), 1081–1109.

[6] O. Lipovan, *A retarded Gronwall-like inequality and its applications*, J. Math. Anal. Appl., **252** (2000), 389–401.

[7] J. Mallet-Paret, *The Fredholm alternative for functional differential equations of mixed type*, J. Dyn. Diff. Eq., **11** (1999), 1–47.

[8] B. G. Pachpatte, *Inequalities for Differential and Integral Equations*, Academic Press, New York, 1998.

[9] M. Pinto, *Integral inequalities of Bihari-type and applications*, Funkcial. Ekvac., **33** (1990), 387–403.

[10] A. Rustichini, *Hopf bifurcation for functional differential equations of mixed type*, J. Dyn. Diff. Eq., **1** (1989), 145–177.

[11] W. Wang, *A generalized retarded Gronwall-like inequality in two variables and applications to BVP*, Appl. Math. Comput., **191** (2007), 144–154.

[12] Y. Zeng, P. Zhang, T.-T. Lu, W. Zhang, *Existence of solutions for a mixed type differential equation with state-dependence*, J. Math. Anal. Appl., **453** (2017), 629–644.

[13] J. Zhou, J. Shen, W. Zhang, *A powered Gronwall-type inequality and applications to stochastic differential equations*, Disc. Contin. Dyn. Sys., **36** (2016), 7207–7234.

[14] J. Zhou, J. Shen, W. Zhang, *Growth in an impulsive integral inequality*, Appl. Math. Comput., **377** (2020), 125152.

[15] J. Zhou, *Solutions of mixed-type functional differential equations with state-dependence*, J. Differ. Equ., **312** (2022), 148–175.

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