Journal of
Mathematical
Inequalities
Volume 17, Number 1 (2023), 127–139
doi:10.7153/jmi-2023-17-09

# THE BOUNDS OF EIGENVALUE FOR COMPLEX SINGULAR BOUNDARY VALUE PROBLEMS

FU SUN AND XIAOXUE HAN*

(*Communicated by R. Oinarov*)

*Abstract.* The present paper deals with “the perturbation of Legendre eigenvalue problem” with limit-circle type non-oscillation endpoints. The dissipative operators in limit-circle case are studied. Lower bounds on the real parts of all eigenvalues and the upper bounds on the imaginary parts of the non-real eigenvalues for this eigenvalue problem associated to a special separated boundary condition (see the below in (3.1b)) are obtained through a new method, partly inspired by the estimates obtained in Sun and Qi (Proc. Roy. Soc. Edinburgh A, 150:2607-2619, 2020).

## 1. Introduction

In this paper we consider the boundary value problem associated to the complex singular differential expression

$$-[(1-x^2)y']'+qy=\lambda wy \ \text{ in } \ L^2[0,1) \tag{1.1}$$

with suitable boundary value condition (see the below in (2.3)), where $q$ is complex function called the *potential function* and $w$ is real value function called the *weight function*, $0$ is the regular endpoint and $1$ is the limit-circle type non-oscillation endpoints (see Lemma 2.1). Since (1.1) and the classical Legendre equation [37, Example 8.3.1, p157]

$$-[(1-x^2)y']'=\lambda y,\ x\in(-1,1)$$

have the same first coefficient term $(1-x^2)$, so we call (1.1) as *“the perturbation of Legendre eigenvalue problem”* with boundary value condition. Let $L^2_w:=L^2_w[0,1)$ be the weighted Hilbert space of all Lebesgue measurable, complex-valued functions $f$ on $[0,1)$ satisfying $\int_0^1 w|f|^2<\infty$ with the norm $\|f\|_w:=\int_0^1 w|f|^2$ and the inner product $(f,g)_w:=\int_0^1 wf\overline{g}$.

Singular Sturm-Liouville boundary value problem attracted a lot of attentions in recent years since it is of great importance for quantum mechanics, diffusion processes and dynamic systems et al. It is generally known that several factors would induce the

*Mathematics subject classification* (2020): 34B05, 34B09, 34L15.

*Keywords and phrases*: The perturbation of Legendre eigenvalue problem, dissipative operators, limit-circle, a priori bounds.

* Corresponding author.

non-self-adjoint singular spectral problems, such as the non-symmetry of the differential expressions and the non-self-adjoint boundary conditions. The spectral properties have been widely investigated for the self-adjoint case, including regular, singular, definite and indefinite Sturm-Liouville problem. For the non-self-adjoint case, the authors in [3, 14] studied the dissipative singular Sturm-Liouville problems including the completeness of eigenvectors and associated vectors for the corresponding operators in Weyl's limit circle case using Livšic's theorem with the separated boundary conditions. And for other dissipative Sturm-Liouville problem, the authors have been studied fairly enough in many literatures, such as the Weyl's limit circle, transmission conditions and time scales case (see [4, 5, 31, 33, 34]).

It is well known that equation (1.1) is formally self-adjoint if and only if the coefficients are real-valued functions. If the potential function $q$ and $w$ are complex valued, the equation (1.1) is formally non-self-adjoint, and hence non-real eigenvalues may exist. For example, if we set $\mathrm{Im}\, q > 0$ in the equation (1.1) with Dirichlet boundary conditions, then it can be transferred to a strictly dissipative operator, and therefore, non-real eigenvalues exist.

In the case of complex Sturm-Liouville problems (i.e. $\mathrm{Im}\, q \neq 0$), the authors in [32] have given sufficient conditions to guarantee the eigenvalues of the problem (1.1) with different self-adjoint boundary conditions to be simple. Furthermore, the finiteness on eigenvalue of (1.1) with boundary value conditions under complex potential on half-line or whole line were investigated in [6, 21]. The classification results of non-self-adjoint, complex coefficients and non-symmetric Sturm-Liouville problems have been studied in [26, 29, 30, 35]. For other research topics of complex differential operators such as essential spectra and expansion of eigenfunctions et al. we mention [1, 2, 12, 17, 22, 27] and references cited therein.

A priori bounds of non-real eigenvalues of Sturm-Liouville problems was raised in [23] by Mingarelli and stressed by Kong, Möller, Wu and Zettl [20]. Determining a priori bounds and the exact number of non-real eigenvalues are an interesting and difficult problems in Sturm-Liouville theory. Recently, this open problems have been solved by Qi *et al.*, [7, 18, 25, 36] for the regular Sturm-Liouville problem and by Behrndt *et al.*, [8, 9, 10, 11, 28] for the singular case, respectively. The estimates on the bounds of eigenvalues for the complex Sturm-Liouville problems have been studied by the Rayleigh-Ritz method for $w \equiv 1$, $q > 0$ in [15] and for the general case in [16].

The present paper will focus on the perturbation of Legendre eigenvalue problem with limit-circle type non-oscillation endpoints associated to the boundary condition. The eigenvalues of dissipative operator with boundary value condition $\mathscr{B}y = 0$ (see the below) are studied and the bounds of eigenvalues for this complex value perturbation of Legendre eigenvalue problem with a special boundary condition are obtained.

This paper is organized as follows: Section 2 contains a basic discussion of the perturbation of Legendre eigenvalue problem with limit-circle type non-oscillation endpoints under the boundary value condition (2.3) and the dissipative operator problem (see Theorem 2.4 and 2.5). The estimate results (Theorem 3.4 and 3.5) on non-real eigenvalues are given in Section 3 with a special boundary condition (see the below in (3.1b)).

## 2. Preliminary knowledge and results

In this section, we give some basic knowledge for the singular differential equation (1.1) under the conditions that

$$q\in L^2([0,1),\mathbb{C}\backslash\mathbb{R}),\ w\in L^2([0,1),\mathbb{R}),\ q=q_1+iq_2,\ i=\sqrt{-1},\ q_1,\ q_2\in\mathbb{R}. \tag{2.1}$$

Throughout this section the functions $q,w$ always satisfy (2.1). We first introduce some concepts (cf. [19, 24, 37]). The endpoint $1$ is *oscillatory* if every nontrivial real-valued solution has an infinite number of zeros in $(c,1)$ for any $c\in(0,1)$, and it is *non-oscillatory* otherwise. For fixed $\lambda\in\mathbb{R}$, a real solution $u$ of (1.1) is called a *principal solution* at $1$ if there exists $c\in[0,1)$ such that

$$u(x)\neq 0,\ x\in(c,1),\ \int_c^1\frac{1}{(1-x^2)u^2}=\infty.$$

A real solution $v$ of (1.1) is called a *non-principal solution* at $1$ if there exists $c\in[0,1)$ such that

$$v(x)\neq 0,\ x\in(c,1),\ \int_c^1\frac{1}{(1-x^2)v^2}<\infty.$$

If $u$ and $v$ are principal and non-principal solutions at $1$, respectively, then

$$\frac{u(x)}{v(x)}\to 0\ \text{ as }\ x\to 1.$$

We say that the endpoint $1$ is a *limit-circle type endpoint* if all solutions of (1.1) are in $L^2_w[c,1)$ for some $c\in[0,1)$. It is well known that the limit-circle type endpoint is independent of $\lambda\in\mathbb{R}$. The endpoint $1$ is *limit-circle type non-oscillation* if it is both limit-circle type and non-oscillation.

The main condition in this paper is

$$\gamma:=\sup_{x\in[0,1)}\left|\frac{1}{1-t^2}\int_t^x q(s)\mathrm{d}s\right|\in L^1[0,1),\ \int_c^x\frac{1}{1-t^2}\mathrm{d}t\in L^2_w[0,1) \tag{2.2}$$

for some (and hence for all) $c\in[0,1)$. It follows from Schwarz inequality that

$$\left|\int_t^x q(s)\mathrm{d}s\right|\leqslant\sqrt{1\pm t}\left(\int_0^1|q|^2\right)^{1/2}:=\sqrt{1\pm t}\|q\|_2,$$

where $\|\cdot\|_2$ denote the norm of the space $L^2[0,1)$ and hence

$$\left|\frac{1}{1-t^2}\int_t^x q(s)\mathrm{d}s\right|\leqslant\left(\sqrt{1\pm t}\right)^{-1}\|q\|_2\in L^1[0,1).$$

Clearly

$$\int_0^x\frac{1}{1-t^2}\mathrm{d}t=\frac{1}{2}\ln\frac{1+x}{1-x}\in L^2_w[0,1).$$

Therefore, the main condition (2.2) is well defined.

LEMMA 2.1. (*cf.* [28, Lemma 2.1]) *Assume that* (2.2) *holds. Then* (1.1) *is limit-circle type non-oscillation at endpoints* 1.

Let $v$ be the non-principal solutions of (1.1) at 1 for $\lambda = 0$, $u$ be the corresponding principal solutions, and satisfying $[u,v](x) \equiv 1$, where

$$[f,g] = fp\overline{g}' - \overline{g}pf',$$

$$f,g \in D_{\max} = \{f \in L^2_w(0,1) : f, pf' \in AC_{\mathrm{loc}}[0,1), \tau f/w \in L^2_w[0,1)\},$$

$AC_{\mathrm{loc}}[0,1)$ denotes the set of all complex-valued functions which are absolutely continuous on all compact subintervals of $[0,1)$. Then from Lemma 2.1 we can definite the boundary value condition at endpoints 1 (cf. [19] and [37, p.191])

$$\xi \cos\beta[y,v](1) - \sin\beta[y,u](1) = 0,\ 0 < \beta \leqslant \pi/2,\ \mathrm{Im}\,\xi > 0.$$

This together with the boundary value condition at regular endpoints 0

$$\cos\alpha y(0) - \sin\alpha py'(0) = 0,\ 0 \leqslant \alpha < \pi/2$$

implies that the boundary conditions in the form

$$\begin{cases} \cos\alpha y(0) - \sin\alpha py'(0) = 0, & 0 \leqslant \alpha < \pi/2, \\ \xi\cos\beta[y,v](1) - \sin\beta[y,u](1) = 0, & 0 < \beta \leqslant \pi/2,\ \mathrm{Im}\,\xi > 0. \end{cases} \tag{2.3}$$

Then the corresponding eigenvalue problem is

$$\begin{cases} \tau y := -[(1-x^2)y']' + qy = \lambda wy, \\ \mathscr{B}y = 0 : \begin{cases} \cos\alpha y(0) - \sin\alpha py'(0) = 0, \\ \xi\cos\beta[y,v](1) - \sin\beta[y,u](1) = 0. \end{cases} \end{cases} \tag{2.4}$$

In $L^2_w[0,1)$, define the operator $T$ with domain $D(T) = \{y \in L^2_w : y \in D_{\max},\ \mathscr{B}y = 0\}$ and the rule $Ty = \tau y/w$. Now we rewrite the problem (2.4) in the operator form $Ty = \lambda y$, $y \in D(T)$.

DEFINITION 2.2. (cf. [13, p. 175]) A linear operator $T$, acting in the Hilbert space with domain $D(T)$, is said to be dissipative if $\mathrm{Im}(Tf,f) \geqslant 0$, for all $f \in D(T)$.

LEMMA 2.3. *For arbitrary* $f,g,u,v \in D_{\max}$, *where* $D_{\max}$ *is defined above, we have*

$$\begin{vmatrix} [f,\overline{u}](x) & [f,v](x) \\ [\overline{g},\overline{u}](x) & [\overline{g},v](x) \end{vmatrix} = [f,g](x)[u,v](x),\ x \in [0,1).$$

*Proof.* By direct calculation we can get the above equality. □

THEOREM 2.4. *The operator $T$ is dissipative, i.e.,* $\operatorname{Im}(Tf,f) \geqslant 0$, $\forall f \in D(T)$.

*Proof.* For each $f \in D(T)$, integrating by parts implies

$$2i\operatorname{Im}(Tf,f) = (Tf,f) - (f,Tf) = [f,f](1) - [f,f](0). \tag{2.5}$$

From the boundary condition at $0$, we have

$$[f,f](0) = 0. \tag{2.6}$$

It follows from the boundary condition at endpoint $1$ and Lemma 2.3 that

$$[f,f](1) = \frac{2i\operatorname{Im}\xi\cos\beta}{\sin\beta}\,|[f,v](1)|^2. \tag{2.7}$$

Now, substituting (2.6) and (2.7) into (2.5) implies that

$$\operatorname{Im}(Tf,f) = \frac{\operatorname{Im}\xi\cos\beta}{\sin\beta}\,|[f,v](1)|^2. \tag{2.8}$$

Since $\beta \in (0,\frac{\pi}{2}]$ and $\operatorname{Im}\xi > 0$, $\operatorname{Im}(Tf,f) \geqslant 0$. □

THEOREM 2.5. *The operator $T$ has no real eigenvalue, i.e., all its eigenvalues lie in the upper half-plane.*

*Proof.* If the statement in Theorem 2.5 was false, then there exists a real eigenvalue $\lambda_0$ of $T$, and let $\varphi := \varphi(x,\lambda_0)$ be the corresponding eigenfunction. It follows from $\operatorname{Im}(T\varphi,\varphi) = \operatorname{Im}(\lambda_0\|\varphi\|^2) = 0$ and (2.8) that $[\varphi,v](1) = 0$. Using the boundary condition at $1$, i.e., $i\cos\beta[\varphi,v](1) + \sin\beta[\varphi,u](1) = 0$, one sees that $[\varphi,u](1) = 0$. Let $\varphi$ and $\psi$ be the solutions of

$$\begin{cases} -[(1-x^2)\varphi']' + q\varphi = \lambda w\varphi, \\ \varphi(0,\lambda) = \sin\alpha, \ \ \varphi'(0,\lambda) = \cos\alpha, \end{cases} \quad \begin{cases} -[(1-x^2)\psi']' + q\psi = \lambda w\psi, \\ \psi(0,\lambda) = -\cos\alpha, \ \ \psi'(0,\lambda) = \sin\alpha, \end{cases}$$

respectively. Then from Lemma 2.3 that

$$1 = [\varphi,\psi] = [\varphi,u][\psi,v] - [\varphi,v][\psi,u] = 0,$$

and hence we get a contradiction. This completes the proof. □

## 3. Estimates on the lower and upper bounds of non-real eigenvalues

In this section, we give the bounds on the non-real eigenvalues of dissipative operator $T$ for the special case $\beta = \pi/2$ in (2.3), i.e., $[y,u](1) = 0$, and the corresponding eigenvalue problem is

$$\begin{cases} \tau y := -[(1-x^2)y']' + qy = \lambda wy, & \text{(3.1a)} \\ \mathscr{B}_0 y = 0 : \cos\alpha y(0) - \sin\alpha py'(0) = 0, \ [y,u](1) = 0, & \text{(3.1b)} \end{cases}$$

where $q$ satisfied (2.1). In the following we will give the estimates on the bounds of non-real eigenvalues for the singular eigenvalue problem (3.1a) and (3.1b). Firstly, we prepare some lemmas in the following.

LEMMA 3.1. (*cf.* [28, Lemma 2.3]) *Assume that* (2.2) *holds, $u$ is a principal solution of* (3.1a) *at* 1 *for* $\lambda = 0$. *Let $y$ be an eigenfunction of* (3.1a) *and* (3.1b) *corresponding to the eigenvalue $\lambda$, then $y$ is bounded and*

$$[y,u](1) = 0 \Leftrightarrow ((1-x^2)y')(x)y(x) \to 0 \text{ as } x \to 1.$$

The following lemma is the estimates of $\|\sqrt{1-x^2}\phi'\|_2$, where $\phi$ is an eigenfunction of (3.1a) and (3.1b) corresponding to a non-real eigenvalue $\lambda$. That is $\mathscr{B}_0\phi = 0$ and

$$-[(1-x^2)\phi']' + q\phi = \lambda w\phi. \tag{3.2}$$

Since the problem (3.1a) and (3.1b) is a linear system and $\phi$ is continuous, we can choose $\phi$ satisfies $\int_0^1 |\phi(x)|^2 \mathrm{d}x = 1$ in the following discussion. To simplify the statements, let

$$\begin{aligned} &\Gamma^2_{\alpha,q_1^-} = 2\left(|\cot\alpha| + \|q_1^-\|_2 + 2\|q_1^-\|_2^2\right), \ q_k^- = -\min\{q_k, 0\}, \\ &q_k^+ = \max\{q_k, 0\}, \ q_1 = \operatorname{Re} q, \ q_2 = \operatorname{Im} q. \end{aligned} \tag{3.3}$$

Since $w(x) > 0$ a.e. on $[0,1)$, we can choose $\varepsilon_1 > 0$ such that

$$\Omega(\varepsilon_1) = \{x \in [0,1): \ w(x) < \varepsilon_1\}, \ m(\varepsilon_1) = \operatorname{mes}\Omega(\varepsilon_1). \tag{3.4}$$

And $w^2(x) > 0$ a.e. on $[0,1)$, we can choose $\varepsilon_2 > 0$ such that

$$\Omega(\varepsilon_2) = \{x \in [0,1): \ w^2(x) < \varepsilon_2\}, \ m(\varepsilon_2) = \operatorname{mes}\Omega(\varepsilon_2). \tag{3.5}$$

Then $m(\varepsilon_1) \to 0$ and $m(\varepsilon_2) \to 0$ as $\varepsilon \to 0$.

LEMMA 3.2. *Let $\lambda$ and $\phi$ be defined as above with* $\operatorname{Re}\lambda \leqslant 0$, *then*

$$\int_0^1 (1-x^2)|\phi'|^2 \leqslant \Gamma^2_{\alpha,q_1^-}, \quad \int_0^1 q_1^{\pm}|\phi|^2 \leqslant \|q_1^{\pm}\|_2\left(1 + 2\Gamma_{\alpha,q_1^-}\right).$$

*Proof.* Let $\lambda$ be a non-real eigenvalue of (3.1a) and (3.1b) and $\phi$ the corresponding eigenfunction with $\|\phi\|_2 = 1$. It follows from Lemma 3.1 that $\phi$ is bounded and satisfies

$$(1-x^2)\phi'(x)\phi(x) \to 0 \ \text{ as } x \to 1. \tag{3.6}$$

Multiplying both sides of (3.2) by $\overline{\phi}$ and integrating by parts over the interval $[0,1]$, then from $\mathscr{B}_0\phi = 0$ we have

$$\cot\alpha|\phi(0)|^2 + \int_0^1 (1-x^2)|\phi'|^2 + \int_0^1 q|\phi|^2 = \lambda\int_0^1 w|\phi|^2. \tag{3.7}$$

Here (3.6) is used. Separating the real parts of (3.7) yields

$$\operatorname{Re}\lambda\int_0^1 w|\phi|^2 = \cot\alpha|\phi(0)|^2 + \int_0^1 (1-x^2)|\phi'|^2 + \int_0^1 q_1|\phi|^2. \tag{3.8}$$

From $\operatorname{Re}\lambda \leqslant 0$ and $w(x) > 0$ a.e. on $[0,1)$ one sees that

$$\cot\alpha|\phi(0)|^2 + \int_0^1 (1-x^2)|\phi'|^2 + \int_0^1 q_1|\phi|^2 \leqslant 0. \tag{3.9}$$

Set $q_1^- = -\min\{q_1, 0\}$ and

$$Q(x) = \int_0^x q_1^-(t)\mathrm{d}t - x\int_0^1 q_1^-(t)\mathrm{d}t,\ x \in [0,1]. \tag{3.10}$$

Then one can verify that

$$Q(0) = 0 = Q(1),\ Q'(x) = q_1^-(x) - \int_0^1 q_1^-(t)\mathrm{d}t \text{ a.e. } x \in [0,1], \tag{3.11}$$

and the condition $q_1^- \in L^2[0,1)$ implies that $\max|Q(x)| \leqslant \sqrt{1-x^2}\|q_1^-\|_2$. Then

$$\begin{aligned}
\int_0^1 q_1^-|\phi|^2 &= \int_0^1 \left(Q' + \int_0^1 q_1^-(t)\mathrm{d}t\right)|\phi|^2 \\
&\leqslant \|q_1^-\|_2 - 2\operatorname{Re}\left(\int_0^1 Q\phi'\overline{\phi}\right) \\
&\leqslant \|q_1^-\|_2 + 2\|q_1^-\|_2\left(\int_0^1 (1-x^2)|\phi'|^2\right)^{1/2}\left(\int_0^1 |\phi|^2\right)^{1/2} \\
&\leqslant \|q_1^-\|_2 + \frac{1}{2}\int_0^1 (1-x^2)|\phi'|^2 + 2\|q_1^-\|_2^2.
\end{aligned} \tag{3.12}$$

This together with (3.9) and $|\phi(x)| \leqslant 1,\ x \in [0,1]$ implies that

$$\int_0^1 (1-x^2)|\phi'|^2 \leqslant \Gamma^2_{\alpha,q_1^-}, \tag{3.13}$$

where $\Gamma^2_{\alpha,q_1^-}$ is defined in (3.3). It follows from the penultimate inequality of (3.12) and (3.13) that

$$\int_0^1 q_1^-|\phi|^2 \leqslant \|q_1^-\|_2 + 2\Gamma_{\alpha,q_1^-}\|q_1^-\|_2.$$

Setting $q_1^+ = \max\{q_1, 0\}$, similar to (3.10), (3.11) and (3.12), we can prove that

$$\int_0^1 q_1^+|\phi|^2 \leqslant \|q_1^+\|_2 + 2\|q_1^+\|_2\left(\int_0^1 (1-x^2)|\phi'|^2\right)^{1/2},$$

which, together with (3.13), gives that

$$\int_0^1 q_1^+|\phi|^2 \leqslant \|q_1^+\|_2 + 2\Gamma_{\alpha,q_1^-}\|q_1^+\|_2.$$

This completes the proof of Lemma 3.2. □

LEMMA 3.3. *Let $\lambda$ and $\phi$ be defined as above and (2.2) holds. Then $\int_{\Omega(\varepsilon)} |\phi(x)|^2 dx < 1/2$ for any $\varepsilon > 0$.*

*Proof.* Since $\int_0^1 |\phi(x)| \mathrm{d}x = 1$, there must exist $c \in [0,1)$ such that $|\phi(c)| \leqslant 1$, and hence it follows from Lemma 3.2 that

$$|\phi(x)| \leqslant |\phi(c)| + \left|\int_c^x \phi'\right| \leqslant 1 + \left|\int_c^x \frac{1}{1-t^2}\right|^{\frac{1}{2}} \left|\int_c^x (1-t^2)|\phi'|^2\right|^{\frac{1}{2}}$$
$$\leqslant 1 + \Gamma_{\alpha,q_1^-} \left|\int_c^x \frac{1}{1-t^2}\right|^{\frac{1}{2}},$$

which implies that

$$|\phi(x)|^2 \leqslant 2 + 2\Gamma^2_{\alpha,q_1^-} \left|\int_c^x \frac{1}{1-t^2}\right|.$$

As a result

$$\int_{\Omega(\varepsilon)} |\phi(x)|^2 \mathrm{d}x \leqslant 2m(\varepsilon) + 2\Gamma^2_{\alpha,q_1^-} \int_{\Omega(\varepsilon)} \left|\int_c^x \frac{1}{1-t^2}\right| \mathrm{d}x.$$

From the main condition (2.2) and $m(\varepsilon) \to 0$ as $\varepsilon \to 0$ one sees that the last term of the above inequality tends to $0$ as $\varepsilon \to 0$ by the continuity of the integral $\int_{\Omega(\varepsilon)} \left|\int_c^x \frac{1}{1-t^2}\right| \mathrm{d}x$. The proof is finished. □

THEOREM 3.4. *Assume that (2.1), (2.2) and (3.4) hold. If $\lambda$ is a eigenvalue of (3.1a) and (3.1b) with $\operatorname{Re}\lambda \leqslant 0$, then*

$$\begin{aligned} \operatorname{Re}\lambda &\geqslant -\frac{2}{\varepsilon_1}\left(|\cot\alpha| + \|q_1^+\|_2 + \Gamma_{\alpha,q_1^-}(\Gamma_{\alpha,q_1^-} + 2\|q_1^+\|_2)\right), \\ |\operatorname{Im}\lambda| &\leqslant \frac{2}{\varepsilon_1}\|q_2^+\|_2\left(1 + 4\Gamma_{\alpha,q_1^-}\right). \end{aligned} \tag{3.14}$$

*If $\lambda$ is a eigenvalue of (3.1a) and (3.1b) with $\operatorname{Re}\lambda > 0$, then*

$$|\operatorname{Im}\lambda| \leqslant \frac{2}{\varepsilon_1}\|q_2^+\|_2\left(1 + 2N_{\Gamma,\operatorname{Re}\widetilde{\lambda}}\right), \tag{3.15}$$

*where $N^2_{\Gamma,\operatorname{Re}\widetilde{\lambda}} = 2(\Gamma^2_{\alpha,q_1^-} + \operatorname{Re}\widetilde{\lambda}\|w\|_2 + 4|\operatorname{Re}\widetilde{\lambda}|^2\|w\|_2^2)$.*

*Proof.* It follows from $\int_0^1 |\phi|^2 = 1$, (3.4) and Lemma 3.3 that

$$\int_0^1 w|\phi|^2 \quad \geqslant \varepsilon_1 \left(\int_0^1 |\phi|^2 - \int_{\Omega(\varepsilon_1)} |\phi|^2\right) \geqslant \varepsilon_1 (1 - 1/2) \geqslant \varepsilon_1/2. \tag{3.16}$$

Since $\int_0^1 |\phi|^2 = 1$, there must satisfied $|\phi(x)|^2 \leqslant 1$ for $x \in [0,1)$. Using (3.8) and Lemma 3.2, we can get

$$\begin{aligned} |\operatorname{Re}\lambda| \int_0^1 w|\phi|^2 &\leqslant |\cot\alpha| + \Gamma^2_{\alpha,q_1^-} + \int_0^1 q_1^+|\phi|^2 \\ &\leqslant |\cot\alpha| + \|q_1^+\|_2 + \Gamma_{\alpha,q_1^-}\left(\Gamma_{\alpha,q_1^-} + 2\|q_1^+\|_2\right). \end{aligned} \tag{3.17}$$

Separating the imaginary parts of (3.7) yields that

$$\operatorname{Im}\lambda \int_0^1 w|\phi|^2 = \int_0^1 q_2|\phi|^2. \tag{3.18}$$

Let $Q_2(x) = \int_0^x q_2(t)\mathrm{d}t - x\int_0^1 q_2(t)\mathrm{d}t$. Then similar to (3.12) we have

$$\int_0^1 q_2|\phi|^2 \leqslant \|q_2^+\|_2 + 2\|q_2^+\|_2 \left(\int_0^1 (1-x^2)|\phi'|^2\right)^{1/2}, \tag{3.19}$$

and hence from Lemma 3.2 we have

$$\int_0^1 q_2|\phi|^2 \leqslant \|q_2^+\|_2 + 2\|q_2^+\|_2 \Gamma_{\alpha,q_1^-}, \tag{3.20}$$

which, together with (3.18) yields that

$$|\operatorname{Im}\lambda| \int_0^1 w|\phi|^2 \leqslant \|q_2^+\|_2 + 2\|q_2^+\|_2 \Gamma_{\alpha,q_1^-}. \tag{3.21}$$

So the inequalities in (3.14) follows (3.16), (3.17) and (3.21) immediately.

Now, if $\widetilde{\lambda}$ is an eigenvalue of (3.1a) and (3.1b) with $\operatorname{Re}\widetilde{\lambda} > 0$, then we consider the eigenvalue problem

$$-[(1-x^2)\phi']' + (q - \operatorname{Re}\widetilde{\lambda} w)\phi = \lambda w\phi, \ \ \mathscr{B}_0\phi = 0. \tag{3.22}$$

It can be easily verify that $\widetilde{\lambda} - \operatorname{Re}\widetilde{\lambda}$ is also an eigenvalue of (3.22).

Note that $\operatorname{Re}\left(\widetilde{\lambda} - \operatorname{Re}\widetilde{\lambda}\right) = 0$, and hence

$$0 = \operatorname{Re}\left(\widetilde{\lambda} - \operatorname{Re}\widetilde{\lambda}\right)\int_0^1 w|\phi|^2 = \cot\alpha|\phi(0)|^2 + \int_0^1 (1-x^2)|\phi'|^2 + \int_0^1 (q_1 - \operatorname{Re}\widetilde{\lambda} w)|\phi|^2. \tag{3.23}$$

Set $\widetilde{Q}(x) = \int_0^x (q_1^- + \operatorname{Re}\widetilde{\lambda} w) - x\int_0^1 (q_1^- + \operatorname{Re}\widetilde{\lambda} w)$, similar to (3.11) and (3.12) we have

$$\begin{aligned}\int_0^1 (q_1^- + \operatorname{Re}\widetilde{\lambda} w)|\phi|^2 \leqslant 4\|q_1^-\|_2^2 + 4|\operatorname{Re}\widetilde{\lambda}|^2\|w\|_2^2 + \|q_1^-\|_2 \\ + \operatorname{Re}\widetilde{\lambda}\|w\|_2 + \frac{1}{2}\int_0^1 (1-x^2)|\phi'|^2,\end{aligned}$$

which together with (3.23) implies that

$$\int_0^1 (1-x^2)|\phi'|^2 \leqslant N_{\Gamma,\operatorname{Re}\widetilde{\lambda}}^2, \tag{3.24}$$

where $N_{\Gamma,\operatorname{Re}\widetilde{\lambda}}^2 := 2\left(\Gamma_{\alpha,q_1^-}^2 + \operatorname{Re}\widetilde{\lambda}\|w\|_2 + 4|\operatorname{Re}\widetilde{\lambda}|^2\|w\|_2^2\right)$. It follows from (3.19) and (3.24) that

$$\begin{aligned}|\operatorname{Im}\widetilde{\lambda}|\int_0^1 w|\phi|^2 &= |\operatorname{Im}(\widetilde{\lambda} - \operatorname{Re}\widetilde{\lambda})|\int_0^1 w|\phi|^2 = \int_0^1 q_2|\phi|^2 \\ &\leqslant \|q_2^+\|_2\left(1 + 2\left(\int_0^1 (1-x^2)|\phi'|^2\right)^{1/2}\right) \\ &\leqslant \|q_2^+\|_2\left(1 + 2N_{\Gamma,\operatorname{Re}\widetilde{\lambda}}\right).\end{aligned}$$

So the inequality in (3.15) holds. The proof of Theorem 3.4 is complete. $\square$

Suppose that $w \in AC(a,b)$ and $w' \in L^2_p(a,b)$, we have the following result.

THEOREM 3.5. *Assume that* (2.1), (2.2) *and* (3.5) *hold. Let* $w \in AC[0,1)$ *and* $w' \in L^2_p[0,1)$, *if* $\lambda$ *is a eigenvalue of* (3.1a) *and* (3.1b) *with* $\operatorname{Re}\lambda \leqslant 0$, *then*

$$\begin{aligned} &\operatorname{Re}\lambda \geqslant -\frac{2}{\varepsilon_2}\left(\|w\|_c\left[|\cot\alpha| + \|q_1^+\|_2 + \Gamma_{\alpha,q_1^-}(\Gamma_{\alpha,q_1^-} + 2\|q_1^+\|_2)\right] + \widehat{w}\Gamma_{\alpha,q_1^-}\right),\\ &|\operatorname{Im}\lambda| \leqslant \frac{2}{\varepsilon_2}\left(\|w\|_c\|q_2^+\|_2(1+2\Gamma_{\alpha,q_1^-}) + \widehat{w}\Gamma_{\alpha,q_1^-}\right). \end{aligned} \tag{3.25}$$

*If* $\lambda$ *is a eigenvalue of* (3.1a) *and* (3.1b) *with* $\operatorname{Re}\lambda > 0$, *then*

$$|\operatorname{Im}\lambda| \leqslant \frac{2}{\varepsilon_2}\left(\|w\|_c\|q_2^+\|_2(1+2N_{\Gamma,\operatorname{Re}\widetilde{\lambda}}) + \widehat{w}N_{\Gamma,\operatorname{Re}\widetilde{\lambda}}\right). \tag{3.26}$$

*where* $N^2_{\Gamma,\operatorname{Re}\widetilde{\lambda}} = 2(\Gamma^2_{\alpha,q_1^-} + \operatorname{Re}\widetilde{\lambda}\|w\|_2 + 4|\operatorname{Re}\widetilde{\lambda}|^2\|w\|_2^2)$.

*Proof.* Let $\lambda$ be an eigenvalue of (3.1a) and (3.1b) and $\phi$ the corresponding eigenfunction defined as above, that is $\mathscr{B}_0 y = 0$ and

$$-[(1-x^2)\phi']' + q\phi = \lambda w\phi. \tag{3.27}$$

Multiplying both sides of (3.27) by $w\overline{\phi}$ and integrating by parts over the interval $[0,1)$, then from $(1-x^2)\phi'(x)\phi(x) \to 0$ as $x \to 1$ in Lemma 3.1 one sees that

$$w(0)\cot\alpha|\phi(0)|^2 + \int_0^1 w(1-x^2)|\phi'|^2 + \int_0^1 wq|\phi|^2 + \int_0^1 w'(1-x^2)\phi'\overline{\phi} = \lambda\int_0^1 w^2|\phi|^2.$$

Separating the real and imaginary parts yields that

$$\begin{aligned} \operatorname{Re}\lambda\int_0^1 w^2|\phi|^2 =& w(0)\cot\alpha|\phi(0)|^2 + \int_0^1 w(1-x^2)|\phi'|^2 + \int_0^1 wq_1|\phi|^2\\ &+ \operatorname{Re}\left(\int_0^1 w'(1-x^2)\phi'\overline{\phi}\right), \end{aligned} \tag{3.28}$$

$$\operatorname{Im}\lambda\int_0^1 w^2|\phi|^2 = \operatorname{Im}\left(\int_0^1 w'(1-x^2)\phi'\overline{\phi}\right) + \int_0^1 wq_2|\phi|^2. \tag{3.29}$$

It follows from $|\phi(x)| \leqslant 1$ and Lemma 3.2 yields that

$$\begin{aligned} &\left|w(0)\cot\alpha|\phi(0)|^2 + \int_0^1 w(1-x^2)|\phi'|^2 + \int_0^1 wq_1|\phi|^2\right|\\ &\leqslant \|w\|_c\left(|\cot\alpha| + \Gamma^2_{\alpha,q_1^-} + \|q_1^+\|_2 + 2\|q_1^+\|_2\Gamma_{\alpha,q_1^-}\right), \end{aligned} \tag{3.30}$$

and from (3.20) one sees that

$$\int_0^1 wq_2|\phi|^2 \leqslant \|w\|_c\|q_2^+\|_2\left(1+2\Gamma_{\alpha,q_1^-}\right), \tag{3.31}$$

where $\|\cdot\|_c$ denote the maximum norm of $C(0,1)$. From $w' \in L_p^2(0,1)$, $\widehat{w} = (\int_0^1 (1-x^2)|w'|^2)^{1/2}$, Lemma 3.2 and Cauchy-Schwarz inequality, one sees that

$$\left|\int_0^1 w'(1-x^2)\phi'\overline{\phi}\right| \leqslant \left(\int_0^1 (1-x^2)|\phi'|^2\right)^{1/2}\left(\int_0^1 (1-x^2)|w'|^2\right)^{1/2} \leqslant \widehat{w}\Gamma_{\alpha,q_1^-}. \tag{3.32}$$

In view of Lemma 3.3, one sees that

$$\int_0^1 w^2|\phi|^2 \geqslant \varepsilon_2 \int_{[0,1)\setminus\Omega(\varepsilon_2)} |\phi|^2 = \varepsilon_2\left(\int_0^1 |\phi|^2 - \int_{\Omega(\varepsilon_2)} |\phi|^2\right) \geqslant \frac{\varepsilon_2}{2}. \tag{3.33}$$

It follows from (3.28), (3.30), (3.32) and (3.33) that

$$|\operatorname{Re}\lambda| \leqslant \frac{2}{\varepsilon_2}\left(\|w\|_c\left[|\cot\alpha| + \|q_1^+\|_2 + \Gamma_{\alpha,q_1^-}(\Gamma_{\alpha,q_1^-} + 2\|q_1^+\|_2)\right] + \widehat{w}\Gamma_{\alpha,q_1^-}\right). \tag{3.34}$$

From (3.29), (3.31), (3.32) and (3.33), one can verify that

$$|\operatorname{Im}\lambda| \leqslant \frac{2}{\varepsilon_2}\left(\|w\|_c\|q_2^+\|_2(1+2\Gamma_{\alpha,q_1^-}) + \widehat{w}\Gamma_{\alpha,q_1^-}\right). \tag{3.35}$$

So the inequalities in (3.25) hold follows from (3.34) and (3.35) immediately.

Furthermore, if $\widehat{\lambda}$ is an eigenvalue of (3.1a) and (3.1b) with $\operatorname{Re}\widehat{\lambda} > 0$, then (3.22) holds. Using (3.19), (3.24), (3.29), (3.32), (3.33), we obtain (3.26) with a similar argument in the proof of Theorem 3.4. This completes the proof. □

*Acknowledgements.* The authors deeply thank the editors and reviewers. The authors would like to thank Professor Jiangang Qi and Zhaowen Zheng for their generous help. This research was partially supported by the National Natural Science Foundation of China (Grant/Award Number: 12101356, 12101357, 12071254) and Natural Science Foundation of Shandong Province (Grant/Award Number: ZR2021QA065, ZR2021QA080).

## REFERENCES

[1] S. ALBEVERIO, R. O. HRYNIV AND YA. MYKYTYUK, *On spectra of non-self-adjoint Sturm-Liouville operators*, Sel. Math., New Ser., **13** (2008), 571–599.

[2] E. K. ARPAT, *An eigenfunction expansion of the non-selfadjoint Sturm-Liouville operator with a singular potential*, J. Math. Chem., **51** (2013), 2196–2213.

[3] E. BAIRAMOV AND A. M. KRALL, *Dissipative operators generated by the Sturm-Liouville differential expression in the Weyl limit circle case*, J. Math. Anal. Appl., **254** (2001), 178–190.

[4] E. BAIRAMOV AND E. UGURLU, *Kreins theorems for a dissipative boundary value transmission problem*, Complex Analysis and Operator Theory, **7** (2013), 831–842.

[5] E. BAIRAMOV AND E. UGURLU, *The determinants of dissipative Sturm-Liouville operators with transmission conditions*, Math. Comput. Model., **53** (2011), 805–813.

[6] G. BAŞCANBAZ-TUNCA, *Spectral expansion of a non-self-adjoint differential operator on the whole axis*, J. Math. Anal. Appl., **252** (2000), 278–297.

[7] J. BEHRNDT, S. CHEN, F. PHILIPP AND J. QI, *Estimates on the non-real eigenvalues of regular indefinite Sturm-Liouvilleproblems*, Proc. Roy. Soc. Edinburgh A, **144** (2014), 1113–1126.

[8] J. BEHRNDT, Q. KATATBEH AND C. TRUNK, *Non-real eigenvalues of singular indefinite Sturm-Liouville operators*, Proc. Amer. Math. Soc., **137** (2009), 3797–3806.

[9] J. BEHRNDT, F. PHILIPP AND C. TRUNK, *Bounds on the non-real spectrum of differential operators with indefinite weights*, Math. Ann., **357** (2013), 185–213.

[10] J. BEHRNDT, P. SCHMITZ AND C. TRUNK, *Spectral bounds for singular indefinite Sturm-Liouville operators with $L^1$-potentials*, Proc. Amer. Math. Soc., **146** (2018), 3935–3942.

[11] J. BEHRNDT, P. SCHMITZ AND C. TRUNK, *Spectral bounds for indefinite singular Sturm-Liouville operators with uniformly locally integrable potentials*, J. Differential Equations, **267** (2019), 468–493.

[12] I. M. GLAZMAN, *Direct Methods of Qualitative Spectral Analysis of Singular Differential Operators*, Israel Program for Scientific Translations, 1965.

[13] I. C. GOHBERG AND M. G. KREIN, *Introduction to the theory of linear non-self-adjoint operators*, Amer. Math. Soc, Providence, Rhode Island, 1969.

[14] G. S. GUSEINOV AND H. TUNCAY, *The determinants of perturbation connected with a dissipative Sturm-Liouville operators*, J. Math. Anal. Appl., **194** 1995, 39–49.

[15] C. O. HORGAN AND S. NEMAT-NASSER, *Bounds on eigenvalues of Sturm-Liouville problems with discontinuous coefficients*, J. Appl. Math. Phys., **30** (1979), 77–86.

[16] W. JIAN AND H. SUN, *On Bounds of Eigenvalues of Complex Sturm-Liouville Boundary Value Problems*, Abstract and Applied Analysis, Vol. 2014 (2014), 1–4.

[17] Y. KAMIMURA, *A criterion for the complete continuity of a $2n$-th order differential operator with complex coefficients*, Proc. Royal Soc. Edinburgh A, **116** (1990), 161–176.

[18] M. KIKONKO AND A. B. MINGARELLI, *Bounds on real and imaginary parts of non-real eigenvalues of a non-definite Sturm-Liouville problem*, J. Differential Equations, **261** (2016), 6221–6232.

[19] L. KONG, Q. KONG, H. WU AND A. ZETTL, *Regular approximations of sigular Sturm-Liouville problems with limit-circle endpoints*, Result. Math., **45** (2004), 274–292.

[20] Q. KONG, M. MÖLLER, H. WU AND A. ZETTL, *Indefinite Sturm-Liouville problems*, Proc. Roy. Soc. Edinburgh A, **133** (2003), 639–652.

[21] A. M. KRALL, E. BAIRAMOV AND Ö. ÇAKAR, *Spectrum and spectral singularities of a quadratic pencil of a Schrödinger operator with a general boundary condition*, J. Differential Equations, **151** (1999), 252–267.

[22] V. A. MARCHENKO, *Expansion in eigenfunctions of non-selfadjoint singular second order differential oeprators*, Amer. Math. Soc. Transl., **25** (1963), 77–130.

[23] A. B. MINGARELLI, *A survey of the regular weighted Sturm-Liouville problem – The non-definite case*, arXiv:1106.6013v1 [math.CA], 2011.

[24] H. D. NIESSEN AND A. ZETTL, *Singular Sturm-Liouville problem: The Friedrichs extension and comparison of eigenvalues*, Proc. London Math. Soc., **64** (1992), 545–578.

[25] J. QI AND S. CHEN, *A priori bounds and existence of non-real eigenvalues of indefinite Sturm-Liouville problems*, J. Spectr. Theory, **4** (2014), 53–63.

[26] J. QI, Z. ZHENG AND H. SUN, *Classification of Sturm-Liouville differential equations with complex coefficients and operator realizations*, Proc. Roy. Soc. A, **467** (2011), 1835–1850.

[27] D. RACE, *On the location of essential spactra and regularity fields of complex Sturm-Liouville operators*, Proc. Roy. Soc. Edinburgh A, **85** (1980), 1–14.

[28] F. SUN AND J. QI, *A priori bounds and existence of non-real eigenvalues for singular indefinite Sturm-Liouville problems with limit-circle type endpoints*, Proc. Roy. Soc. Edinburgh A, **150** (2020), 2607–2619.

[29] H. SUN AND J. QI, *On classification of second-order differential equations with complex coefficients*, J. Math. Anal. Appl., **372** (2010), 585–597.

[30] H. SUN, J. QI AND H. JING, *Classification of non-self-adjoint singular Sturm-Liouville difference equations*, Appl. Math. Comput., **217** (2011), 8020–8030.

[31] H. TUNA, *Completeness of the rootvectors of a dissipative Sturm-Liouville operators on time scales*, Appl. Math. Comput., **228** (2014), 108–115.

[32] O. A. VELIEV, *On the simplicity of the eigenvalues of the non-self-adjoint Mathieu-Hill operators*, Applied and Computational Mathematics, **13** (2013), 122–134.

[33] Z. WANG AND H. WU, *Dissipative non-self-adjoint Sturm-Liouville operators and completeness of their eigenfunctions*, J. Math. Anal. Appl., **394** (2012), 1–12.

[34] Z. WANG AND H. WU, *The completeness of eigenfunctions of perturbation connected with Sturm-Liouville operators*, J. Syst. Sci. Complex, **19** (2006), 1–12.

[35] B. XIE, X. GUO AND J. QI, *Classification of non-symmetric Sturm-Liouville systems and operator realizations*, Acta Applicandae Mathematicae, **125** (2013), 27–47.
[36] B. XIE AND J. QI, *Non-real eigenvalues of indefinite Sturm-Liouville problems*, J. Differential Equations, **8** (2013), 2291–2301.
[37] A. ZETTL, *Sturm-Liouville Theory*, Math. Surveys Monogr., vol. 121, Providence, RI: Amer. Math. Soc., 2005.

(Received March 17, 2022)

*Fu Sun*
*School of Statistics and Data Science*
*Qufu Normal University*
*Qufu, 273165, China*
*e-mail:* sfmath@163.com

*Xiaoxue Han*
*School of Statistics and Data Science*
*Qufu Normal University*
*Qufu, 273165, China*
*e-mail:* hanxiaoxue1229@163.com