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A NEW HILBERT–TYPE INEQUALITY IN THE WHOLE PLANE

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Abstract. By means of the weight coefficients and the idea of introduced parameters, a new discrete Hilbert-type inequality in the whole plane is given, which is an extension of Hardy-Hilbert's inequality. The equivalent form is obtained. The equivalent statements of the best possible constant factor related to several parameters, the operator expressions and a few particular inequalities are considered.

1. Introduction

Suppose that $p > 1$, $\frac{1}{p}+\frac{1}{q}=1$, $a_m, b_n \geqslant 0$, $0 < \sum_{m=1}^{\infty} a_m^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$. We have the following Hardy-Hilbert's inequality with the best possible constant factor $\frac{\pi}{\sin(\pi/p)}$ (cf. [4], Theorem 315):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty} a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}}. \tag{1}$$

For $p = q = 2$, inequality (1) reduces to the well known Hilbert's inequality.

If $f(x), g(y) \geqslant 0$, $0 < \int_0^{\infty} f^p(x)dx < \infty$ and $0 < \int_0^{\infty} g^q(y)dy < \infty$, then we still have the integral analogue of (1) named in Hardy-Hilbert's integral inequality:

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin(\pi/p)}\left(\int_0^{\infty} f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^{\infty} g^q(y)dy\right)^{\frac{1}{q}}, \tag{2}$$

with the same best constant factor $\frac{\pi}{\sin(\pi/p)}$ (cf. [4], Theorem 316).

In 1998, by introducing an independent parameter $\lambda > 0$, Yang [31, 29] gave an extension of (2) (for $p = q = 2$) with the best possible constant factor. Inequalities (1) and (2) with their extensions play an important role in analysis and its applications (cf. [32, 30, 15, 20, 11, 5, 28, 25, 41, 26, 2, 1]).

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The following half-discrete Hilbert-type inequality was provided in 1934 (cf. [4], Theorem 351): If $K(x)$ $(x>0)$ is a decreasing function, $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $0<\phi(s)=\int_0^\infty K(t)t^{s-1}dt<\infty$, then for $a_n \geqslant 0$, $0<\sum_{n=1}^\infty a_n^p<\infty$, we have

$$\int_0^\infty x^{p-2}\left(\sum_{n=1}^\infty K(nx)a_n\right)^p dx < \phi^p\left(\frac{1}{q}\right)\sum_{n=1}^\infty a_n^p. \tag{3}$$

Some new extensions of (3) were provided by [23, 36, 21, 22, 14, 33].

In 2016, by the use of the technique of real analysis, Hong et al. [10] considered some equivalent statements of the extensions of (1) with the best possible constant factor related to several parameters. The other similar results about the extensions of (1)–(3) were given by [7, 9, 27, 8, 13, 6, 34, 39, 24, 38, 19, 12, 37, 18, 35].

In this paper, following the way of [10], by means of the weight coefficients and the idea of introduced parameters, a new discrete Hilbert-type in the whole plane is given as follows: for $r>1$, $\frac{1}{r}+\frac{1}{s}=1$,

$$\sum_{|n|=1}^\infty\sum_{|m|=1}^\infty \frac{a_m b_n}{|m|+|n|} < \frac{2\pi}{\sin(\pi/r)}\left(\sum_{|m|=1}^\infty |m|^{\frac{p}{r}-1}a_m^p\right)^{\frac{1}{p}}\left(\sum_{|n|=1}^\infty |n|^{\frac{q}{s}-1}b_n^q\right)^{\frac{1}{q}}, \tag{4}$$

which is an extension of (1). The general form as well as the equivalent form are obtained. The equivalent statements of the best possible constant factor related to several parameters, the operator expressions and a few particular inequalities are considered.

2. An example and some lemmas

EXAMPLE 1. (i) In view of the following expression (cf. [3]):

$$\cot x = \frac{1}{x}+\sum_{k=1}^\infty\left(\frac{1}{x-\pi k}+\frac{1}{x+\pi k}\right) \quad (x\in(0,\infty)),$$

for $b\in(0,1)$, by Lebesgue term by term theorem (cf. [17], we obtain

$$\begin{aligned}
A_b &:= \int_0^\infty \frac{u^{b-1}}{1-u}du = \int_0^1 \frac{u^{b-1}}{1-u}du + \int_1^\infty \frac{u^{b-1}}{1-u}du \\
&= \int_0^1 \frac{u^{b-1}}{1-u}du + \int_1^\infty \frac{v^{-b}}{1-v}dv = \int_0^1 \frac{u^{b-1}-u^{-b}}{1-u}du \\
&= \int_0^1 \sum_{k=0}^\infty (u^{k+b-1}-u^{k-b})du = \sum_{k=0}^\infty \int_0^1 (u^{k+b-1}-u^{k-b})du \\
&= \sum_{k=0}^\infty\left(\frac{1}{k+b}-\frac{1}{k+1-b}\right) \\
&= \pi\left[\frac{1}{\pi b}+\sum_{k=1}^\infty\left(\frac{1}{\pi b-\pi k}+\frac{1}{\pi b+\pi k}\right)\right] \\
&= \pi\cot\pi b \in \mathbf{R} = (-\infty,\infty).
\end{aligned}$$

Note. For $b \in (0, \frac{1}{2})$, $A_b > 0$; for $b \in (\frac{1}{2}, 1)$, $A_b < 0$. We also have $A_{1/2} = 0$.

(ii) For $\lambda, \eta > 0$, we set the homogeneous function of degree $-\lambda$ as follows:

$$k_\lambda^{(\eta)}(x,y) := \frac{x^\eta - y^\eta}{x^{\lambda+\eta} - y^{\lambda+\eta}} \quad (x,y > 0),$$

satisfying $k_\lambda^{(\eta)}(ux,uy) = u^{-\lambda} k_\lambda^{(\eta)}(x,y)$ $(u,x,y > 0)$ and

$$\begin{aligned} k_\lambda^{(\eta)}(v,v) &= \lim_{x \to v} \cdot \frac{x^\eta - v^\eta}{x^{\lambda+\eta} - v^{\lambda+\eta}} = \lim_{x \to v} \cdot \frac{\eta x^{\eta-1}}{(\lambda+\eta)x^{\lambda+\eta-1}} \\ &= \frac{\eta}{(\lambda+\eta)v^\lambda} \quad (v > 0). \end{aligned}$$

It follows that $k_\lambda^{(\eta)}(x,y)$ is a positive and continuous function with respect to $x,y > 0$. For $x \neq y$, we find

$$\frac{\partial}{\partial x} k_\lambda^{(\eta)}(x,y) = x^{-\eta}(x^{\lambda+\eta} - y^{\lambda+\eta})\varphi(x,y),$$

where, we set the following differentiable function:

$$\varphi(x,y) := \lambda x^{\lambda+\eta} - (\lambda+\eta)x^\lambda y^\eta + \eta y^{\lambda+\eta} \quad (x,y > 0).$$

We find that for $0 < x < y$,

$$\frac{\partial}{\partial x}\varphi(x,y) = \lambda(\lambda+\eta)x^{\lambda-1}(x^\eta - y^\eta) < 0;$$

for $x > y$. $\frac{\partial}{\partial x}\varphi(x,y)$. It follows that $\varphi(x,y)$ is strictly decreasing (resp. increasing) with respect to $x < y$ (resp. $x > y$). Since $\varphi(y,y) = \min_{x>0} \varphi(x,y) = 0$ $(y > 0)$, then $\varphi(x,y) > 0$ $(x \neq y)$, namely $\frac{\partial}{\partial x} k_\lambda^{(\eta)}(x,y) < 0$ $(x \neq y)$. Therefore, in view of $k_\lambda^{(\eta)}(x,y)$ is continuous at $x = y$, we conform that $k_\lambda^{(\eta)}(x,y)$ $(y > 0)$ is strictly decreasing with respect to $x > 0$. In the same way, we can show that $k_\lambda^{(\eta)}(x,y)$ $(x > 0)$ is also strictly decreasing with respect to $y > 0$.

(iii) For $\lambda_i \in (0,\lambda) \subset (0,\lambda+\eta)$ $(i = 1,2)$, since $k_\lambda^{(\eta)}(x,y) > 0$, by (i), we obtain

$$\begin{aligned} k_{\lambda,\eta}(\lambda_i) &:= \int_0^\infty k_\lambda^{(\eta)}(1,u) u^{\lambda_i - 1} du = \int_0^\infty \frac{1-u^\eta}{1-u^{\lambda+\eta}} u^{\lambda_i-1} du \\ &\overset{v=u^{\lambda+\eta}}{=} \frac{1}{\lambda+\eta}\left(\int_0^\infty \frac{v^{\frac{\lambda_i}{\lambda+\eta}-1} dv}{1-v} - \int_0^\infty \frac{v^{\frac{\lambda_i+\eta}{\lambda+\eta}-1} dv}{1-v} \right) \\ &= \frac{\pi}{\lambda+\eta}\left[\cot\left(\frac{\pi\lambda_i}{\lambda+\eta}\right) - \cot\left(\frac{\pi(\lambda_i+\eta)}{\lambda+\eta}\right)\right] \\ &= \frac{\pi}{\lambda+\eta}\left[\cot\left(\frac{\pi\lambda_i}{\lambda+\eta}\right) + \cot\left(\frac{\pi(\lambda-\lambda_i)}{\lambda+\eta}\right)\right] \in \mathbf{R}_+ = (0,\infty). \end{aligned} \quad (5)$$

In what follows, we suppose that $p>1$ $(q>1)$, $\frac{1}{p}+\frac{1}{q}=1$, $-1<\alpha,\beta<1$, $\lambda,\eta>0$, $\lambda_i\in(0,1]\cap(0,\lambda)$ $(i=1,2)$, $k_{\lambda,\eta}(\lambda_i)$ is indicated by (5). We still assume that $a_m,b_n\geqslant 0$ $(|m|,|n|\in\mathbf{N}=\{1,2,\cdots\})$, such that for $c:=\lambda-\lambda_1-\lambda_2$, $0<\sum_{|m|=1}^{\infty}(|m|+\alpha m)^{p[1-\lambda_1)-c-1}a_m^p<\infty$ and

$$0<\sum_{|n|=1}^{\infty}(|n|+\beta n)^{q[1-\lambda_2)-c-1}b_n^q<\infty,$$

where, $\sum_{|j|=1}^{\infty}\cdots=\sum_{j=-1}^{-\infty}\cdots+\sum_{j=1}^{\infty}\cdots(j=m,n)$.

LEMMA 1. *For $\gamma>0$, we have the following inequalities:*

$$\frac{1}{\gamma}\left[(1-\alpha)^{-\gamma-1}+(1+\alpha)^{-\gamma-1}\right]<\sum_{|m|=1}^{\infty}(|m|+\alpha m)^{-\gamma-1}<\frac{1}{\gamma}\left[(1-\alpha)^{-\gamma-1}+(1+\alpha)^{-\gamma-1}\right](\gamma+1). \quad (6)$$

Proof. By the decreasing property of series, we find

$$\begin{aligned}\sum_{|m|=1}^{\infty}(|m|+\alpha m)^{-\gamma-1}&=\sum_{m=-1}^{-\infty}[(1-\alpha)(-m)]^{-\gamma-1}+\sum_{m=1}^{\infty}[(1+\alpha)m]^{-\gamma-1}\\&=\sum_{m=1}^{\infty}[(1-\alpha)m]^{-\gamma-1}+\sum_{m=1}^{\infty}[(1+\alpha)m]^{-\gamma-1}\\&=[(1-\alpha)^{-\gamma-1}+(1+\alpha)^{-\gamma-1}]\left(1+\sum_{m=2}^{\infty}m^{-\gamma-1}\right)\\&<[(1-\alpha)^{-\gamma-1}+(1+\alpha)^{-\gamma-1}]\left(1+\int_1^{\infty}x^{-\gamma-1}dx\right)\\&=\frac{1}{\gamma}[(1-\alpha)^{-\gamma-1}+(1+\alpha)^{-\gamma-1}](\gamma+1),\end{aligned}$$

$$\begin{aligned}\sum_{|m|=1}^{\infty}(|m|+\alpha m)^{-\gamma-1}&=\sum_{m=-1}^{-\infty}[(1-\alpha)(-m)]^{-\gamma-1}+\sum_{m=1}^{\infty}[(1+\alpha)m]^{-\gamma-1}\\&=[(1-\alpha)^{-\gamma-1}+(1+\alpha)^{-\gamma-1}]\sum_{m=1}^{\infty}m^{-\gamma-1}\\&>[(1-\alpha)^{-\gamma-1}+(1+\alpha)^{-\gamma-1}]\int_1^{\infty}x^{-\gamma-1}dx\\&=\frac{1}{\gamma}[(1-\alpha)^{-\gamma-1}+(1+\alpha)^{-\gamma-1}].\end{aligned}$$

Hence, we have (6).

The lemma is proved. □

DEFINITION 1. We set

$$k(m,n) := k_{\lambda}^{(\eta)}(|m|+\alpha m, |n|+\beta n) = \frac{(|m|+\alpha m)^{\eta}-(|n|+\beta n)^{\eta}}{(|m|+\alpha m)^{\lambda+\eta}-(|n|+\beta n)^{\lambda+\eta}} \quad (|m|,|n| \in \mathbf{N}),$$

and define the following weight coefficients:

$$\varpi(\lambda_2,m) := (|m|+\alpha m)^{\lambda-\lambda_2} \sum_{|n|=1}^{\infty} k(m,n)(|n+\beta n)^{\lambda_2-1} \quad (|m| \in \mathbf{N}). \tag{7}$$

$$\omega(\lambda_1,n) := (|n|+\beta n)^{\lambda-\lambda_1} \sum_{|m|=1}^{\infty} k(m,n)(|m|+\alpha m)^{\lambda_1-1} \quad (|n| \in \mathbf{N}). \tag{8}$$

LEMMA 2. *The following inequalities are valid:*

$$0 < \frac{2}{1-\beta^2} k_{\lambda,\eta}(\lambda_2)(1-\theta(\lambda_2,m)) < \varpi(\lambda_2,m) < \frac{2}{1-\beta^2} k_{\lambda,\eta}(\lambda_2) \quad (|m| \in \mathbf{N}), \tag{9}$$

where, we indicate that

$$\theta(\lambda_2,m) := \frac{1}{k_{\lambda,\eta}(\lambda_2)} \int_0^{\frac{1+|\beta|}{|m|+\alpha m}} \frac{(1-u^{\eta})u^{\lambda_2-1}}{1-u^{\lambda+\eta}} du = O\left(\frac{1}{(|m|+\alpha m)^{\lambda_2}}\right) > 0.$$

Proof. For fixed $|m| \in \mathbf{N}$, we set

$$k^{(1)}(m,y) := k_{\lambda}^{(\eta)}(|m|+\alpha m, (1-\beta)(-y)), \quad y<0,$$
$$k^{(2)}(m,y) := k_{\lambda}^{(\eta)}(|m|+\alpha m, (1+\beta)y), \quad y>0.$$

where from, for $y>0$, $k^{(1)}(m,-y) = k_{\lambda}^{(\eta)}(|m|+\alpha m, (1-\beta)y)$. We find

$$\begin{aligned}\varpi(\lambda_2,m) &= (|m|+\alpha m)^{\lambda-\lambda_2} \left\{ \sum_{n=-1}^{-\infty} k^{(1)}(m,n)[(1-\beta)(-n)]^{\lambda_2-1} + \sum_{n=1}^{\infty} k^{(2)}(m,n)[(1+\beta)n]^{\lambda_2-1} \right\} \\ &= (|m|+\alpha m)^{\lambda-\lambda_2} \left\{ \sum_{n=1}^{\infty} k^{(1)}(m,-n)[(1-\beta)n]^{\lambda_2-1} + \sum_{n=1}^{\infty} k^{(2)}(m,n)[(1+\beta)n]^{\lambda_2-1} \right\}\end{aligned}$$

$$= (|m| + \alpha m)^{\lambda-\lambda_2} \Bigg[(1-\beta)^{\lambda_2-1} \sum_{n=1}^{\infty} k^{(1)}(m,-n) n^{\lambda_2-1} + (1+\beta^{\lambda_2-1} \sum_{n=1}^{\infty} k^{(2)}(m,n) n^{\lambda_2-1} \Bigg].$$

It is evident that for fixed $|m| \in \mathbf{N}$, $\lambda_2 \leqslant 1$, by Example 1 (ii), both $k^{(1)}(m,-y)y^{\lambda_2-1}$ and $k^{(2)}(m,y)y^{\lambda_2-1}$ are strictly decreasing with respect to $y \in (0,\infty)$. By the decreasing property of series, we have

$$\varpi(\lambda_2,m) < (|m| + \alpha m)^{\lambda-\lambda_2} \Bigg[(1-\beta)^{\lambda_2-1} \int_0^{\infty} k^{(1)}(m,-y) y^{\lambda_2-1} dy + (1+\beta^{\lambda_2-1} \int_0^{\infty} k^{(2)}(m,y) y^{\lambda_2-1} \Bigg],$$

$$\varpi(\lambda_2,m) > (|m| + \alpha m)^{\lambda-\lambda_2} \Bigg[(1-\beta)^{\lambda_2-1} \int_1^{\infty} k^{(1)}(m,-y) y^{\lambda_2-1} dy + (1+\beta^{\lambda_2-1} \int_0^{\infty} k^{(2)}(m,y) y^{\lambda_2-1} \Bigg].$$

Setting $u = \frac{(1-\beta)y}{|m|+\alpha m}$ (resp. $u = \frac{(1+\beta)y}{|m|+\alpha m}$) in the above first (resp. second) integrals, we obtain

$$\varpi(\lambda_2,m) < [(1-\beta)^{-1} + (1+\beta)^{-1}] \int_0^{\infty} k_{\lambda}^{(\eta)}(1,u) u^{\lambda_2-1} du = \frac{2k_{\lambda,\eta}(\lambda_2)}{1-\beta^2},$$

$$\begin{aligned} \varpi(\lambda_2,m) &> \frac{1}{1-\beta} \int_{\frac{1-\beta}{|m|+\alpha m}}^{\infty} k_{\lambda}^{(\eta)}(1,u) u^{\lambda_2-1} du \\ &\quad + \frac{1}{1+\beta} \int_{\frac{1+\beta}{|m|+\alpha m}}^{\infty} k_{\lambda}^{(\eta)}(1,u) u^{\lambda_2-1} du \\ &> \frac{2}{1-\beta^2} \int_{\frac{1+|\beta|}{|m|+\alpha m}}^{\infty} k_{\lambda}^{(\eta)}(1,u) u^{\lambda_2-1} du \\ &= \frac{2k_{\lambda,\eta}(\lambda_2)}{1-\beta^2} (1-\theta(\lambda_2,m)) > 0, \end{aligned}$$

where, $\theta(\lambda_2,m) = \frac{1}{k_{\lambda,\eta}(\lambda_2)} \int_0^{\frac{1+|\beta|}{|m|+\alpha m}} \frac{(1-u^{\eta})u^{\lambda_2-1}}{1-u^{\lambda+\eta}} du$, satisfying

$$\begin{aligned} 0 &< \int_0^{\frac{1+|\beta|}{|m|+\alpha m}} k_{\lambda}^{(\eta)}(1,u) u^{\lambda_2-1} du \\ &\leqslant \int_0^{\frac{1+|\beta|}{|m|+\alpha m}} k_{\lambda}^{(\eta)}(1,0) u^{\lambda_2-1} du \\ &= \int_0^{\frac{1+|\beta|}{|m|+\alpha m}} u^{\lambda_2-1} du = \frac{1}{\lambda_2} \left(\frac{1+|\beta|}{|m|+\alpha m} \right)^{\lambda_2}. \end{aligned}$$

Hence, we have (9).

The lemma is proved. □

Note. In the same way, by the symmetric property, we still have

$$\begin{aligned} 0 &< \frac{2}{1-\alpha^2}k_{\lambda,\eta}(\lambda_1)(1-\vartheta(\lambda_1,n)) < \varpi(\lambda_1,n) \\ &< \frac{2}{1-\alpha^2}k_{\lambda,\eta}(\lambda_1) \quad (|n| \in \mathbf{N}), \end{aligned} \tag{10}$$

where, we indicate that

$$\vartheta(\lambda_1,n) := \frac{1}{k_{\lambda,\eta}(\lambda_1)} \int_0^{\frac{1+|\alpha|}{|n|+\beta n}} \frac{(1-u^\eta)u^{\lambda_2-1}}{1-u^{\lambda+\eta}} du = O\left(\frac{1}{(|n|+\beta n)^{\lambda_1}}\right) > 0. \quad \square \tag{11}$$

LEMMA 3. *The following inequality follows:*

$$\begin{aligned} H &:= \sum_{|n|=1}^{\infty} \sum_{|m|=1}^{\infty} k(m,n) a_m b_n \\ &< \frac{2(k_{\lambda,\eta}(\lambda_2))^{\frac{1}{p}} (k_{\lambda,\eta}(\lambda_1))^{\frac{1}{q}}}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}} \left[\sum_{|m|=1}^{\infty} (|m|+\alpha m)^{p[1-\lambda_1)-c-1} a_m^p\right]^{\frac{1}{p}} \\ &\quad \times \left[\sum_{|n|=1}^{\infty} (|n|+\beta n)^{q[1-\lambda_2)-c-1} b_n^q\right]^{\frac{1}{q}}. \end{aligned} \tag{12}$$

Proof. By Hölder's inequality with weight (cf. [2]), we find

$$\begin{aligned} H &= \sum_{|n|=1}^{\infty} \sum_{|m|=1}^{\infty} k(m,n) \left[\frac{(|n|+\beta n)^{(\lambda_2-1)/p}}{(|m|+\alpha m)^{(\lambda_1-1)/q}} a_m\right] \left[\frac{(|m|+\alpha m)^{(\lambda_1-1)/q}}{(|n|+\beta n)^{(\lambda_2-1)/p}} b_n\right] \\ &\leqslant \left[\sum_{|m|=1}^{\infty} \sum_{|n|=1}^{\infty} k(m,n) \frac{(|n|+\beta n)^{(\lambda_2-1)}}{(|m|+\alpha m)^{(\lambda_1-1)(p-1)}} a_m^p\right]^{\frac{1}{p}} \\ &\quad \times \left[\sum_{|n|=1}^{\infty} \sum_{|m|=1}^{\infty} k(m,n) \frac{(|m|+\alpha m)^{(\lambda_1-1)}}{(|n|+\beta n)^{(\lambda_2-1)(q-1)}} b_n^q\right]^{\frac{1}{q}} \\ &= \left[\sum_{|m|=1}^{\infty} \varpi(\lambda_2,m)(|m|+\alpha m)^{p[1-\lambda_1)-c-1} a_m^p\right]^{\frac{1}{p}} \\ &\quad \times \left[\sum_{|n|=1}^{\infty} \omega(\lambda_1,n)(|n|+\beta n)^{q[1-\lambda_2)-c-1} b_n^q\right]^{\frac{1}{q}}. \end{aligned}$$

Then by (9) and (10), we have (12).

The lemma is proved. □

REMARK 1. (i) By (12), for $\lambda_1+\lambda_2=\lambda$, we find

$$k_{\lambda,\eta}(\lambda_1)=k_{\lambda,\eta}(\lambda_2)=\frac{\pi}{\lambda+\eta}\left[\cot\left(\frac{\pi\lambda_1}{\lambda+\eta}\right)+\cot\left(\frac{\pi\lambda_2}{\lambda+\eta}\right)\right],$$
$$0<\sum_{|m|=1}^{\infty}(|m|+\alpha m)^{p[1-\lambda_1)-1}a_m^p<\infty,$$
$$0<\sum_{|n|=1}^{\infty}(|n|+\beta n)^{q[1-\lambda_2)-1}b_n^q<\infty,$$

and the following inequality:

$$\sum_{|n|=1}^{\infty}\sum_{|m|=1}^{\infty}k(m,n)a_mb_n<\frac{2k_{\lambda,\eta}(\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}\left[\sum_{|m|=1}^{\infty}(|m|+\alpha m)^{p[1-\lambda_1)-1}a_m^p\right]^{\frac{1}{p}}$$
$$\times\left[\sum_{|n|=1}^{\infty}(|n|+\beta n)^{q[1-\lambda_2)-1}b_n^q\right]^{\frac{1}{q}}. \tag{13}$$

In particular, for $\alpha=\beta=0$, $a_{-m}=a_m$, $b_{-n}=b_n$ $(m,n\in\mathbf{N})$ in (13), we have (cf. [40])

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{m^{\eta}-n^{\eta}}{m^{\lambda+\eta}-n^{\lambda+\eta}}a_mb_n$$
$$<k_{\lambda,\eta}(\lambda_1)\left[\sum_{m=1}^{\infty}m^{p[1-\lambda_1)-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}n^{q[1-\lambda_2)-1}b_n^q\right]^{\frac{1}{q}}. \tag{14}$$

(ii) For $\lambda=1$, $\lambda_1=\frac{1}{q}$, $\lambda_2=\frac{1}{p}$ in (14), we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{m^{\eta}-n^{\eta}}{m^{1+\eta}-n^{1+\eta}}a_mb_n$$
$$<\frac{\pi}{1+\eta}\left[\cot\frac{\pi}{q(1+\eta)}+\cot\frac{\pi}{p(1+\eta)}\right]\left(\sum_{m=1}^{\infty}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^q\right)^{\frac{1}{q}}; \tag{15}$$

for $\lambda=1$, $\lambda_1=\frac{1}{p}$, $\lambda_2=\frac{1}{q}$ in (14), we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{m^{\eta}-n^{\eta}}{m^{1+\eta}-n^{1+\eta}}a_mb_n<\frac{\pi}{1+\eta}\left[\cot\frac{\pi}{q(1+\eta)}+\cot\frac{\pi}{p(1+\eta)}\right]$$
$$\times\left(\sum_{m=1}^{\infty}m^{p-2}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}n^{q-2}b_n^q\right)^{\frac{1}{q}}; \tag{16}$$

for $p=q=2$, both (15) and (16) reduce to

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{m^{\eta}-n^{\eta}}{m^{1+\eta}-n^{1+\eta}}a_m b_n < \frac{2\pi}{1+\eta}\cot\frac{\pi}{2(1+\eta)}\left(\sum_{m=1}^{\infty}a_m^2\sum_{n=1}^{\infty}b_n^2\right)^{\frac{1}{2}}. \tag{17}$$

(iii) For $\lambda=N\in\mathbf{N}$, $\eta=1$, $\lambda_1+\lambda_2=\lambda=N$ in (14), we have $k_{N,1}(\lambda_1)=\frac{\pi}{N+1}\left(\cot\frac{\lambda_1\pi}{N+1}+\cot\frac{\lambda_2\pi}{N+1}\right)$ and

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\sum_{j=0}^{N}m^{N-j}n^{j}} < k_{N,1}(\lambda_1)\left[\sum_{m=1}^{\infty}m^{p[1-\lambda_1)-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}n^{q[1-\lambda_2)-1}b_n^q\right]^{\frac{1}{q}}. \tag{18}$$

For $N=1$, $\lambda_1=\frac{1}{q}$, $\lambda_2=\frac{1}{p}$ in (18), we have (1). Hence, (18) is an extensions of (1).

(iv) For $\alpha=\beta=0$, $\lambda=\eta=1$, $\lambda_1=\frac{1}{r}$, $\lambda_2=\frac{1}{s}$ $(r>1,\ \frac{1}{r}+\frac{1}{s}=1)$, (12) reduces (5); for $r=q$, $s=p$, $a_{-m}=a_m$, $b_{-n}=b_n$ $(m,n\in\mathbf{N})$, (5) reduces to (1). Hence, (12) is an extension of (1).

LEMMA 4. *The constant factor* $\frac{2k_{\lambda,\eta}(\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}$ *in* (13) *is the best possible.*

Proof. For any $0<\varepsilon<p\lambda_1$, we set

$$\widetilde{a}_m := (|m|+\alpha m)^{\lambda_1-\frac{\varepsilon}{p}-1},\ \ \widetilde{b}_n := (|n|+\beta n)^{\lambda_2-\frac{\varepsilon}{q}-1}\ \ (|m|,|n|\in\mathbf{N}).$$

If there exists a constant M $(\leqslant \frac{2k_{\lambda,\eta}(\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}})$, such that (13) is valid when we replace $\frac{2k_{\lambda,\eta}(\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}$ by M, then in particular, we have

$$\widetilde{H} := \sum_{|n|=1}^{\infty}\sum_{|m|=1}^{\infty}k(m,n)\widetilde{a}_m\widetilde{b}_n < M\left[\sum_{|m|=1}^{\infty}(|m|+\alpha m)^{p[1-\lambda_1)-1}\widetilde{a}_m^p\right]^{\frac{1}{p}}\left[\sum_{|n|=1}^{\infty}(|n|+\beta n)^{q[1-\lambda_2)-1}\widetilde{b}_n^q\right]^{\frac{1}{q}}.$$

By Lemma 1, we obtain

$$\widetilde{H} < M\left[\sum_{|m|=1}^{\infty}(|m|+\alpha m)^{-\varepsilon-1}\right]^{\frac{1}{p}}\left[\sum_{|n|=1}^{\infty}(|n|+\beta n)^{-\varepsilon-1}\right]^{\frac{1}{q}} < \frac{M}{\varepsilon}(\varepsilon+1)[(1-\alpha)^{-\varepsilon-1}+(1+\alpha)^{-\varepsilon-1}]^{\frac{1}{p}}[(1-\beta)^{-\varepsilon-1}+(1+\beta)^{-\varepsilon-1}]^{\frac{1}{q}}.$$

By (8) (for $(\lambda_1-\frac{\varepsilon}{p})+(\lambda_2+\frac{\varepsilon}{p})=\lambda$), (10) and Lemma 1, we find

$$\begin{aligned}\widetilde{H}&=\sum_{|n|=1}^{\infty}\sum_{|m|=1}^{\infty}k(m,n)(|m|+\alpha m)^{\lambda_1-\frac{\varepsilon}{p}-1}(|n|+\beta n)^{\lambda_2-\frac{\varepsilon}{q}-1}\\&=\sum_{|n|=1}^{\infty}\varpi(\lambda_1-\frac{\varepsilon}{p},n)(|n|+\beta n)^{-\varepsilon-1}\\&>\frac{2k_{\lambda,\eta}(\lambda_1-\frac{\varepsilon}{p})}{1-\alpha^2}\sum_{|n|=1}^{\infty}\left(1-O(\frac{1}{(|n|+\beta n)^{\lambda_1-\frac{\varepsilon}{p}}})\right)(|n|+\beta n)^{-\varepsilon-1}\\&=\frac{2k_{\lambda,\eta}(\lambda_1-\frac{\varepsilon}{p})}{1-\alpha^2}\left[\sum_{|n|=1}^{\infty}(|n|+\beta n)^{-\varepsilon-1}-\sum_{|n|=1}^{\infty}O(\frac{1}{(|n|+\beta n)^{\lambda_1+\frac{\varepsilon}{q}+1}})\right]\\&=\frac{2k_{\lambda,\eta}(\lambda_1-\frac{\varepsilon}{p})}{\varepsilon(1-\alpha^2)}\left[(1-\beta)^{-\varepsilon-1}+(1-\beta)^{-\varepsilon-1}-\varepsilon O(1)\right].\end{aligned}$$

In view of the above results, we have

$$\begin{aligned}&\frac{2k_{\lambda,\eta}(\lambda_1-\frac{\varepsilon}{p})}{1-\alpha^2}\left[(1-\beta)^{-\varepsilon-1}+(1-\beta)^{-\varepsilon-1}-\varepsilon O(1)\right]<\varepsilon\widetilde{H}\\&<M(\varepsilon+1)[(1-\alpha)^{-\varepsilon-1}+(1+\alpha)^{-\varepsilon-1}]^{\frac{1}{p}}[(1-\beta)^{-\varepsilon-1}+(1+\beta)^{-\varepsilon-1}]^{\frac{1}{q}}.\end{aligned}$$

For $\varepsilon\to0^+$, by Fatou lemma (cf. [17]), we find that

$$\frac{4k_{\lambda,\eta}(\lambda_1)}{(1-\alpha^2)(1-\beta^2)}\leqslant\frac{2M}{(1-\alpha^2)^{1/p}(1-\beta^2)^{1/q}},$$

namely, $\frac{2k_{\lambda,\eta}(\lambda_1)}{(1-\alpha^2)^{1/q}(1-\beta^2)^{1/p}}\leqslant M$. Hence, $M=\frac{2k_{\lambda,\eta}(\lambda_1)}{(1-\alpha^2)^{1/q}(1-\beta^2)^{1/p}}$ is the best possible constant factor of (13).

The lemma is proved. □

REMARK 2. For $\widehat{\lambda}_1=\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}$, $\widehat{\lambda}_2=\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}$, we find

$$\begin{aligned}\widehat{\lambda}_1+\widehat{\lambda}_2&=\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}+\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}=\lambda,\\0&<\widehat{\lambda}_1,\widehat{\lambda}_2<\lambda.\end{aligned}$$

and by Hölder's inequality (cf. [16]), we obtain

$$\begin{aligned}0<k_{\lambda,\eta}(\widehat{\lambda}_2)&=k_{\lambda,\eta}(\widehat{\lambda}_1)=k_{\lambda,\eta}\left(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}\right)\\&=\int_0^{\infty}k_\lambda^{(\eta)}(u,1)u^{\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}-1}du\\&=\int_0^{\infty}k_\lambda^{(\eta)}(u,1)(u^{\frac{\lambda-\lambda_2-1}{p}})(u^{\frac{\lambda_1-1}{q}})du\end{aligned}$$

$$\leqslant \left(\int_0^\infty k_\lambda^{(\eta)}(u,1)u^{\lambda-\lambda_2-1}du\right)^{\frac{1}{p}} \left(\int_0^\infty k_\lambda^{(\eta)}(u,1)u^{\lambda_1-1}du\right)^{\frac{1}{q}}$$
$$= \left(\int_0^\infty k_\lambda^{(\eta)}(1,v)v^{\lambda_2-1}dv\right)^{\frac{1}{p}} \left(\int_0^\infty k_\lambda^{(\eta)}(u,1)u^{\lambda_1-1}du\right)^{\frac{1}{q}}$$
$$= \left(k_{\lambda,\eta}(\lambda_2)\right)^{\frac{1}{p}} \left(k_{\lambda,\eta}(\lambda_1)\right)^{\frac{1}{q}} < \infty. \tag{19}$$

For $c \leqslant \min\{p(1-\lambda_1), q(1-\lambda_2)\}$, we have $\widehat{\lambda}_i \leqslant 1 \ (i=1,2)$, and then we can reduce (12) as follows:

$$H < \frac{2(k_{\lambda,\eta}(\lambda_2))^{\frac{1}{p}}(k_{\lambda,\eta}(\lambda_1))^{\frac{1}{q}}}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}} \left[\sum_{|m|=1}^{\infty}(|m|+\alpha m)^{p[1-\widehat{\lambda}_1)-1}a_m^p\right]^{\frac{1}{p}}$$
$$\times \left[\sum_{|n|=1}^{\infty}(|n|+\beta n)^{q[1-\widehat{\lambda}_2)-1}b_n^q\right]^{\frac{1}{q}}. \tag{20}$$

LEMMA 5. *If* $c \leqslant \min\{p(1-\lambda_1), q(1-\lambda_2)\}$, *the constant factor*

$$\frac{2(k_{\lambda,\eta}(\lambda_2))^{\frac{1}{p}}(k_{\lambda,\eta}(\lambda_1))^{\frac{1}{q}}}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}$$

in (12) *(or* (20) *is the best possible, then we have* $\lambda_1+\lambda_2=\lambda$.

Proof. If the constant factor $\frac{2(k_{\lambda,\eta}(\lambda_2))^{\frac{1}{p}}(k_{\lambda,\eta}(\lambda_1))^{\frac{1}{q}}}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}$ in (12) (or (20)) is the best possible, then in (20) and (13) (for $\lambda_i = \widehat{\lambda}_i \ (i=1,2)$), we have the following inequality:

$$\frac{2(k_{\lambda,\eta}(\lambda_2))^{\frac{1}{p}}(k_{\lambda,\eta}(\lambda_1))^{\frac{1}{q}}}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}} \leqslant \frac{2k_{\lambda,\eta}(\widehat{\lambda}_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}} (\in \mathbf{R}_+)$$

namely, $(k_{\lambda,\eta}(\lambda_2))^{\frac{1}{p}}(k_{\lambda,\eta}(\lambda_1))^{\frac{1}{q}} \leqslant k_{\lambda,\eta}(\widehat{\lambda}_1)$, which follows that (19) keeps the form of equality.

We observe that (19) keeps the form of equality if and only if there exist constants A and B, such that they are not both zero and (cf. [16])

$$Au^{\lambda-\lambda_2-1} = Bu^{\lambda_1-1} \text{ a.e. in } \mathbf{R}_+.$$

Assuming that $A \neq 0$, it follows that $u^{\lambda-\lambda_2-\lambda_1} = \frac{B}{A}$ a.e. in $\mathbf{R}_+$, and $\lambda-\lambda_2-\lambda_1 = 0$, namely, $\lambda_1+\lambda_2 = \lambda$.

The lemma is proved. □

3. Main results

THEOREM 1. *Inequality* (12) *is equivalent to the following inequality:*

$$L := \left[\sum_{|n|=1}^{\infty}(|n|+\beta n)^{p(\lambda_2+c)-c-1}\left(\sum_{|m|=1}^{\infty}k(m,n)a_m\right)^p\right]^{\frac{1}{p}}$$
$$< \frac{2(k_{\lambda,\eta}(\lambda_2))^{\frac{1}{p}}(k_{\lambda,\eta}(\lambda_1))^{\frac{1}{q}}}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}\left[\sum_{|m|=1}^{\infty}(|m|+\alpha m)^{p(1-\lambda_1)-c-1}a_m^p\right]^{\frac{1}{p}}. \tag{21}$$

Proof. Suppose that (21) is valid. By Hölder's inequality (cf. [16]), we find

$$H = \sum_{|n|=1}^{\infty}\left[(|n|+\beta n)^{\lambda_2+\frac{c}{q}-\frac{1}{p}}\sum_{|m|=1}^{\infty}k(m,n)a_m\right]\left[(|n|+\beta n)^{\frac{1}{p}-\lambda_2-\frac{c}{q}}b_n\right]$$
$$\leqslant L\left[\sum_{|n|=1}^{\infty}(|n|+\beta n)^{q[1-\lambda_2)-c-1}b_n^q\right]^{\frac{1}{q}}. \tag{22}$$

Then by (21), we obtain (12).

On the other hand, assuming that (12) is valid, we set

$$b_n := (|n|+\beta n)^{p(\lambda_2+c)-c-1}\left(\sum_{|m|=1}^{\infty}k(m,n)a_m\right)^{p-1}, \quad |n|\in \mathbf{N}.$$

Then we have

$$L^p = \sum_{|n|=1}^{\infty}(|n|+\beta n)^{q[1-\lambda_2)-c-1}b_n^q = H. \tag{23}$$

If $L=0$, then (21) is naturally valid; if $L=\infty$, then it is impossible that makes (21) valid, namely, $L<\infty$. Suppose that $0<L<\infty$. By (12), it follows that

$$\sum_{|n|=1}^{\infty}(|n|+\beta n)^{q[1-\lambda_2)-c-1}b_n^q$$
$$= L^p = H < \frac{2(k_{\lambda,\eta}(\lambda_2))^{\frac{1}{p}}(k_{\lambda,\eta}(\lambda_1))^{\frac{1}{q}}}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}$$
$$\times\left[\sum_{|m|=1}^{\infty}(|m|+\alpha m)^{p[1-\lambda_1)-c-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{|n|=1}^{\infty}(|n|+\beta n)^{q[1-\lambda_2)-c-1}b_n^q\right]^{\frac{1}{q}},$$

$$L = \left[\sum_{|n|=1}^{\infty}(|n|+\beta n)^{q[1-\lambda_2)-c-1}b_n^q\right]^{\frac{1}{p}}$$
$$< \frac{2(k_{\lambda,\eta}(\lambda_2))^{\frac{1}{p}}(k_{\lambda,\eta}(\lambda_1))^{\frac{1}{q}}}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}\left[\sum_{|m|=1}^{\infty}(|m|+\alpha m)^{p[1-\lambda_1)-c-1}a_m^p\right]^{\frac{1}{p}},$$

namely, (21) follows, which is equivalent to (12).

The theorem is proved. □

THEOREM 2. *The following statements (i), (ii), (iii), (iv) and (v) are equivalent:*

(i) Both $k_{\lambda,\eta}^{\frac{1}{p}}(\lambda_2)k_{\lambda,\eta}^{\frac{1}{q}}(\lambda_1)$ *and* $k_{\lambda,\eta}(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})$ *are independent of* p,q*;*

(ii)

$$k_{\lambda,\eta}^{\frac{1}{p}}(\lambda_2)k_{\lambda,\eta}^{\frac{1}{q}}(\lambda_1) \leqslant k_{\lambda,\eta}\left(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}\right);$$

(iii) $\lambda_1+\lambda_2=\lambda$*;*

(iv) for $c \leqslant \min\{p(1-\lambda_1),q(1-\lambda_2)\}$*, the constant factor* $\frac{2(k_{\lambda,\eta}(\lambda_2))^{\frac{1}{p}}(k_{\lambda,\eta}(\lambda_1))^{\frac{1}{q}}}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}$ *in (12) is the best possible constant factor;*

(v) for $c \leqslant \min\{p(1-\lambda_1),q(1-\lambda_2)\}$*, the constant factor* $\frac{2(k_{\lambda,\eta}(\lambda_2))^{\frac{1}{p}}(k_{\lambda,\eta}(\lambda_1))^{\frac{1}{q}}}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}$ *in* (21) *is the best possible constant factor.*

If the statement (iv) follows, namely, $\lambda_1+\lambda_2=\lambda$ *(or* $c=0$*), then we have* (13) *and the following equivalent inequality with the best possible constant factor* $\frac{2k_{\lambda,\eta}(\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}$*:*

$$\left[\sum_{|n|=1}^{\infty}(|n|+\beta n)^{p\lambda_2-1}\left(\sum_{|m|=1}^{\infty}k(m,n)a_m\right)^p\right]^{\frac{1}{p}} < \frac{2k_{\lambda,\eta}(\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}\left[\sum_{|m|=1}^{\infty}(|m|+\alpha m)^{p(1-\lambda_1)-1}a_m^p\right]^{\frac{1}{p}}. \tag{24}$$

In particular, for $\alpha=\beta=0$*,* $a_{-m}=a_m$*,* $b_{-n}=b_n$ $(m,n\in\mathbf{N})$ *in* (24)*, we have the following inequality equivalent to* (14)*:*

$$\left[\sum_{n=1}^{\infty}n^{p\lambda_2-1}\left(\sum_{m=1}^{\infty}\frac{m^{\eta}-n^{\eta}}{m^{\lambda+\eta}-n^{\lambda+\eta}}a_m\right)^p\right]^{\frac{1}{p}} < k_{\lambda,\eta}(\lambda_1)\left[\sum_{m=1}^{\infty}m^{p(1-\lambda_1)-1}a_m^p\right]^{\frac{1}{p}}. \tag{25}$$

Proof. $(i)\Rightarrow(ii)$. Since both $k_{\lambda,\eta}^{\frac{1}{p}}(\lambda_2)k_{\lambda,\eta}^{\frac{1}{q}}(\lambda_1)$ and $k_{\lambda,\eta}(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})$ are independent of p,q, we find

$$k_{\lambda,\eta}^{\frac{1}{p}}(\lambda_2)k_{\lambda,\eta}^{\frac{1}{q}}(\lambda_1) = \lim_{p\to\infty}\lim_{q\to 1^+}k_{\lambda,\eta}^{\frac{1}{p}}(\lambda_2)k_{\lambda,\eta}^{\frac{1}{q}}(\lambda_1) = k_{\lambda,\eta}(\lambda_1),$$

and by Fatou lemma, we have the following inequality:

$$k_{\lambda,\eta}\left(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}\right) = \lim_{p\to\infty}\lim_{q\to 1^+} k_{\lambda,\eta}\left(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}\right)$$
$$\geqslant k_{\lambda,\eta}(\lambda_1) = k_{\lambda,\eta}^{\frac{1}{p}}(\lambda_2)k_{\lambda,\eta}^{\frac{1}{q}}(\lambda_1).$$

$(ii)\Rightarrow(iii)$. If $k_{\lambda,\eta}^{\frac{1}{p}}(\lambda_2)k_{\lambda,\eta}^{\frac{1}{q}}(\lambda_1)\leqslant k_{\lambda,\eta}(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})$, then (19) keeps the form of equality. Based on the proof of Lemma 5, it follows that $\lambda_1+\lambda_2=\lambda$.

$(iii)\Rightarrow(i)$. If $\lambda_1+\lambda_2=\lambda$, then we have

$$k_{\lambda,\eta}^{\frac{1}{p}}(\lambda_2)k_{\lambda,\eta}^{\frac{1}{q}}(\lambda_1) = k_{\lambda,\eta}\left(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}\right) = k_{\lambda,\eta}(\lambda_1).$$

Both $k_{\lambda,\eta}^{\frac{1}{p}}(\lambda_2)k_{\lambda,\eta}^{\frac{1}{q}}(\lambda_1)$ and $k_{\lambda,\eta}(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})$ are independent of p,q.

Hence, it follows that $(i)\Leftrightarrow(ii)\Leftrightarrow(iii)$.

$(iii)\Leftrightarrow(iv)$. By Lemma 4, 5, we obtain the conclusions

$(iv)\Leftrightarrow(v)$. If the constant factor in (12) is the best possible, then so is constant factor in (21). Otherwise, by (22), we would reach a contradiction that the constant factor in (12) is not the best possible. On the other-hand, If the constant factor in (21) is the best possible, then so is constant factor in (12). Otherwise, by (23), we would reach a contradiction that the constant factor in (21) is not the best possible.

Therefore, the statements (i), (ii), (iii), (iv) and (v) are equivalent.

The theorem is proved. □

4. Operator expressions

We set functions

$$\varphi(m) := (|m|+\alpha m)^{p(1-\lambda_1)-c-1}, \quad \psi(n) := (|n|+\beta n)^{q(1-\lambda_2)-c-1},$$

where from,

$$\psi^{1-p}(n) = (|n|+\beta n)^{p(\lambda_2+c)-c-1} \ \ (|m|,|n|\in\mathbf{N}).$$

Define the following real normed spaces:

$$l_{p,\varphi} := \left\{ a=\{a_m\}_{|m|=1}^{\infty}; ||a||_{p,\varphi} := \left(\sum_{|m|=1}^{\infty}\varphi(m)|a_m|^p\right)^{\frac{1}{p}} < \infty \right\},$$

$$l_{q,\psi} := \left\{ b=\{b_n\}_{|n|=1}^{\infty}; ||b||_{q,\psi} := \left(\sum_{|n|=1}^{\infty}\psi(n)|b_n|^q\right)^{\frac{1}{q}} < \infty \right\},$$

$$l_{p,\psi^{1-p}} := \left\{ c=\{c_n\}_{|n|=1}^{\infty}; ||c||_{p,\psi^{1-p}} := \left(\sum_{|n|=1}^{\infty}\psi^{1-p}(n)|c_n|^p\right)^{\frac{1}{p}} < \infty \right\}.$$

Assuming that $a \in l_{p,\varphi}$, setting

$$c = \{c_n\}_{|n|=1}^{\infty}, \ \ c_n := \sum_{|m|=1}^{\infty} k(m,n)a_m, \ \ |n| \in \mathbf{N},$$

we can rewrite (21) as follows:

$$||c||_{p,\psi^{1-p}} < \frac{2(k_{\lambda,\eta}(\lambda_2))^{\frac{1}{p}}(k_{\lambda,\eta}(\lambda_1))^{\frac{1}{q}}}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}||a||_{p,\varphi} < \infty,$$

namely, $c \in l_{p,\psi^{1-p}}$.

DEFINITION 2. Define a Hilbert-type operator $T : l_{p,\varphi} \to l_{p,\psi^{1-p}}$ as follows: For any $a \in l_{p,\varphi}$, there exists a unique representation $Ta = c \in l_{p,\psi^{1-p}}$, satisfying for any $|n| \in \mathbf{N}$, $Ta(n) = c_n$. Define the formal inner product of Ta and $b \in l_{q,\psi}$, and the norm of T as follows:

$$(Ta,b) := \sum_{|n|=1}^{\infty}\left(\sum_{|m|=1}^{\infty} k(m,n)a_m\right) b_n = H,$$

$$||T|| := \sup_{a(\neq 0)\in l_{p,\varphi}} \frac{||Ta||_{p,\psi^{1-p}}}{||a||_{p,\varphi}}.$$

By Theorem 1 and Theorem 2, we have

THEOREM 3. *If $a \in l_{p,\varphi}$, $b \in l_{q,\psi}$, $||a||_{p,\varphi}$, $||b||_{q,\psi} > 0$, then we have the following equivalent inequalities:*

$$(Ta,b) < \frac{2(k_{\lambda,\eta}(\lambda_2))^{\frac{1}{p}}(k_{\lambda,\eta}(\lambda_1))^{\frac{1}{q}}}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}||a||_{p,\varphi}||b||_{q,\psi}, \tag{26}$$

$$||Ta||_{p,\psi^{1-p}} < \frac{2(k_{\lambda,\eta}(\lambda_2))^{\frac{1}{p}}(k_{\lambda,\eta}(\lambda_1))^{\frac{1}{q}}}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}||a||_{p,\varphi}. \tag{27}$$

Moreover, if $\lambda_1 + \lambda_2 = \lambda$, then the constant factor $\frac{2(k_{\lambda,\eta}(\lambda_2))^{\frac{1}{p}}(k_{\lambda,\eta}(\lambda_1))^{\frac{1}{q}}}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}$ in (26) and (27) is the best possible, namely,

$$||T|| = \frac{2k_{\lambda,\eta}(\lambda_1)}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}} = \frac{2}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}} \times \frac{\pi}{\lambda+\eta}\left[\cot\left(\frac{\pi\lambda_1}{\lambda+\eta}\right) + \cot\left(\frac{\pi\lambda_2}{\lambda+\eta}\right)\right]. \tag{28}$$

On the other hand, if the constant factor $\frac{2(k_{\lambda,\eta}(\lambda_2))^{\frac{1}{p}}(k_{\lambda,\eta}(\lambda_1))^{\frac{1}{q}}}{(1-\beta^2)^{1/p}(1-\alpha^2)^{1/q}}$ in (26) or (27) is the best possible, then for $c \leqslant \min\{p(1-\lambda_1), q(1-\lambda_2)\}$, we have $\lambda_1 + \lambda_2 = \lambda$.

5. Conclusions

In this paper, by means of the weight coefficients and the idea of introduced parameters, a new discrete Hilbert-type inequality in the whole plane is obtained in Lemma 2, which is an extension of (1). The equivalent form is given in Theorem 1. The equivalent statements of the best possible constant factor related to several parameters are considered in Theorem 2. The operator expressions, some particular inequalities are provided in Theorem 3 and Remark 1. The lemmas and theorems provide an extensive account of this type of inequalities.

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