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SHARPED JORDAN'S TYPE INEQUALITIES WITH EXPONENTIAL APPROXIMATIONS

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Abstract. In this paper, we establish two exponential double inequalities generalized Jordan's inequality.

1. Introduction

Jordan's inequality [12] in pp. 33 is known as that

$$\frac{2}{\pi} \leqslant \frac{\sin x}{x} < 1$$

for $0 < x \leqslant \frac{\pi}{2}$ and many mathematicians showed various extension, refinement and application of Jordan's inequality [1], [4]–[11], [13]–[25], Especially, Zhu [24] showed the following inequality: for $0 < x \leqslant \frac{\pi}{2}$, we have

$$\frac{\sin x}{x} \leqslant \frac{2}{\pi} + \frac{\pi - 2}{\pi^3}(\pi^2 - 4x^2),$$

where the constant $\frac{\pi-2}{\pi^3} \cong 0.0368181$ is the best possible. Based on the above Zhu's generalized Jordan's inequality, the author [13] proved Theorems 1, 2 and 3.

THEOREM 1. *For $0 < x < \frac{\pi}{2}$, we have*

$$\left(\frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2)\right)^{\theta} < \frac{\sin x}{x} < \left(\frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2)\right)^{\vartheta},$$

where the constants $\theta = 1$ and $\vartheta = 0$ are the best possible.

THEOREM 2. *For $0 < x < \frac{\pi}{2}$, we have*

$$\left(\frac{2}{\pi} + \frac{\pi - 2}{\pi^3}(\pi^2 - 4x^2)\right)^{\theta} < \frac{\sin x}{x} < \left(\frac{2}{\pi} + \frac{\pi - 2}{\pi^3}(\pi^2 - 4x^2)\right)^{\vartheta},$$

where the constants $\theta = \frac{\pi^3}{24(\pi-2)} \cong 1.13169$ and $\vartheta = 1$ are the best possible.

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THEOREM 3. *For $0 < x < \frac{\pi}{2}$, we have*

$$\frac{\sin x}{x} < \left(\frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2)\right)^{\vartheta(x)},$$

where

$$\vartheta(x) = \frac{4x^2}{\pi^2}.$$

In [13], the author stated the following problem which is Problem 3.1 of [13].

PROBLEM 4. *For $0 < x < \frac{\pi}{2}$, we have*

$$\frac{\sin x}{x} > \left(\frac{2}{\pi} + \frac{\pi - 2}{\pi^3}(\pi^2 - 4x^2)\right)^{\theta(x)},$$

where

$$\theta(x) = \frac{\pi^3}{24(\pi-2)} - \frac{(48 - 24\pi + \pi^3)x^3}{3(\pi-2)\pi^3}.$$

Problem 4 has been solved by Malesevic [11] et al. In this paper, we use Theorem 5 to give a different proof from their proof.

THEOREM 5. *For $0 < x < \frac{\pi}{2}$, we have*

$$e^{-\frac{x^2}{6} + \theta(x)} < \frac{\sin x}{x} < e^{-\frac{x^2}{6} + \frac{2x}{\pi}\theta(x)},$$

where

$$\theta(x) = \frac{2x^4}{3\pi^2} + \frac{16x^4}{\pi^4}\ln\frac{2}{\pi}.$$

Bagul and Chesneau [2], [3] proved the results like Theorem 5 involving exponentials functions and $\frac{\sin x}{x}$: for $0 < x < \frac{\pi}{2}$, we have

$$e^{-kx^2} < \frac{\sin x}{x} < e^{-\frac{x^2}{6}},$$

where

$$k = \frac{-\ln\frac{2}{\pi}}{(\frac{\pi}{2})^2}.$$

The double inequality of Theorem 5 has equal left, middle and right sides for $x = \frac{\pi}{2}$. However, their above result does not hold that. The right-hand side of the double inequality in Theorem 6 is a new relational expression, which is different from Jordan type inequalities known.

THEOREM 6. *For $0 < x < \frac{\pi}{2}$, we have*

$$\left(\frac{2}{\pi}+\frac{\pi-2}{\pi^3}(\pi^2-4x^2)\right)^{\theta(x)} < \frac{\sin x}{x} < \left(\frac{2}{\pi}+\frac{\pi-2}{\pi^3}(\pi^2-4x^2)\right)^{\vartheta(x)},$$

where

$$\theta(x)=\frac{\pi^3}{24(\pi-2)}-\frac{(48-24\pi+\pi^3)x^3}{3(\pi-2)\pi^3}$$

and

$$\vartheta(x)=\frac{\pi^3}{24(\pi-2)}-\frac{(48-24\pi+\pi^3)x}{12(\pi-2)\pi^3}.$$

2. Preliminaries

In this section, we show Lemma 1 needed to prove Theorems 5 and 6.

LEMMA 1. *For $0 < x < \frac{\pi}{2}$, we have*

$$\begin{aligned}&\left(\frac{32(\pi-2)(3\pi-2)}{\pi^8}+\frac{64\ln\frac{2}{\pi}}{\pi^6}\right)x^6\\ &\quad<\ln\left(\frac{2}{\pi}+\frac{(\pi-2)(\pi^2-4x^2)}{\pi^3}\right)+\frac{4(\pi-2)x^2(\pi^3-4x^2+2\pi x^2)}{\pi^6}\\ &\quad<\left(\frac{64(\pi-2)(3\pi-2)}{\pi^9}+\frac{128\ln\frac{2}{\pi}}{\pi^7}\right)x^7.\end{aligned}$$

Proof. First, we will show that

$$\begin{aligned}f_1(x)=&\left(\frac{32(\pi-2)(3\pi-2)}{\pi^8}+\frac{64\ln\frac{2}{\pi}}{\pi^6}\right)x^6-\ln\left(\frac{2}{\pi}+\frac{(\pi-2)(\pi^2-4x^2)}{\pi^3}\right)\\ &-\frac{4(\pi-2)x^2(\pi^3-4x^2+2\pi x^2)}{\pi^6}<0.\end{aligned}$$

The derivative of $f_1(x)$ is

$$\begin{aligned}f_1'(x)&=x^5\left(\frac{64(\pi-2)\left(8\pi^2-14\pi^3+11\pi^4-48x^2+96\pi x^2-36\pi^2x^2\right)}{\pi^8\left(\pi^3+8x^2-4\pi x^2\right)}+\frac{384\ln\frac{2}{\pi}}{\pi^6}\right)\\ &=x^5f_2(x). \qquad (2.1)\end{aligned}$$

Since the derivative of $f_2(x)$ is

$$f_2'(x)=\frac{1024(\pi-2)^4x}{\pi^6\left(\pi^3+8x^2-4\pi x^2\right)^2}>0,$$

$f_2(x)$ is strictly increasing for $0<x<\frac{\pi}{2}$. From

$$\begin{aligned}\lim_{x\to 0+0} f_2(x) &= \frac{64(\pi-2)\left(8\pi^2-14\pi^3+11\pi^4\right)}{\pi^{11}}+\frac{384\ln\frac{2}{\pi}}{\pi^6}\\ &= \frac{64}{\pi^9}\left(-16+36\pi-36\pi^2+11\pi^3+6\pi^3\ln\frac{2}{\pi}\right)\cong -\frac{64}{\pi^9}\times 1.15077<0\end{aligned}$$

and

$$\begin{aligned}\lim_{x\to\frac{\pi}{2}-0} f_2(x) &= \frac{32(\pi-2)\left(-4\pi^2+10\pi^3+2\pi^4\right)}{\pi^{10}}+\frac{384\ln\frac{2}{\pi}}{\pi^6}\\ &= \frac{64}{\pi^8}\left(4-12\pi+3\pi^2+\pi^3+6\pi^2\ln\frac{2}{\pi}\right)\cong \frac{64}{\pi^8}\times 0.174322>0,\end{aligned}$$

there exists uniquely a number x_0 such that $f_2(x)<0$ for $0<x<x_0$ and $f_2(x)>0$ for $x_0<x<\frac{\pi}{2}$. Therefore, $f_1(x)$ is strictly decreasing for $0<x<x_0$ and strictly increasing for $x_0<x<\frac{\pi}{2}$. See (2.1) for the relationship between $f_1'(x)$ and $f_2(x)$. From $\lim_{x\to 0+0} f_1(x)=0$ and $\lim_{x\to\frac{\pi}{2}-0} f_1(x)=0$, we can get $f_1(x)<0$ for $0<x<\frac{\pi}{2}$. Next, we will show that

$$\begin{aligned}g_1(x) = &\left(\frac{64(\pi-2)(3\pi-2)}{\pi^9}+\frac{128\ln\frac{2}{\pi}}{\pi^7}\right)x^7-\ln\left(\frac{2}{\pi}+\frac{(\pi-2)(\pi^2-4x^2)}{\pi^3}\right)\\ &-\frac{4(\pi-2)x^2(\pi^3-4x^2+2\pi x^2)}{\pi^6}>0.\end{aligned}$$

The derivative of $g_1(x)$ is

$$\begin{aligned}g_1'(x) = x^6\Bigg(&\frac{64(\pi-2)\left(8\pi^3-8\pi^4+2\pi^5-14\pi^3x+21\pi^4x-112x^3+224\pi x^3-84\pi^2x^3\right)}{\pi^9\left(\pi^3+8x^2-4\pi x^2\right)x}\\ &+\frac{896\ln\frac{2}{\pi}}{\pi^7}\Bigg)=x^6 g_2(x).\end{aligned}\tag{2.2}$$

The derivative of $g_2(x)$ is

$$g_2'(x) = -\frac{128(\pi-2)^3\left(\pi^3+24x^2-12\pi x^2\right)}{\pi^6x^2\left(\pi^3+8x^2-4\pi x^2\right)^2}.$$

Since we have $\pi^3+24x^2-12\pi x^2>0$ for $0<x<\frac{\pi^{\frac{3}{2}}}{\sqrt{12\pi-24}}\cong 1.50445$ and $\pi^3+24x^2-12\pi x^2<0$ for $\frac{\pi^{\frac{3}{2}}}{\sqrt{12\pi-24}}<x<\frac{\pi}{2}$, $g_2(x)$ is strictly decreasing for $0<x<\frac{\pi^{\frac{3}{2}}}{\sqrt{12\pi-24}}$ and strictly increasing for $\frac{\pi^{\frac{3}{2}}}{\sqrt{12\pi-24}}<x<\frac{\pi}{2}$. From $\lim_{x\to 0+0} g_2(x)=+\infty$ and

$$\begin{aligned}\lim_{x\to\frac{\pi}{2}-0} g_2(x) &= \frac{64(\pi-2)\left(-6\pi^3+13\pi^4+2\pi^5\right)}{\pi^{12}}+\frac{896\ln\frac{2}{\pi}}{\pi^7}\\ &= \frac{64}{\pi^9}\left(12-32\pi+9\pi^2+2\pi^3+14\pi^3\ln\frac{2}{\pi}\right)\cong -\frac{64}{\pi^9}\times 0.0891691<0,\end{aligned}$$

there exists uniquely a number x_1 such that $g_2(x) > 0$ for $0 < x < x_1$ and $g_2(x) < 0$ for $x_1 < x < \frac{\pi}{2}$. Therefore, $g_1(x)$ is strictly increasing for $0 < x < x_1$ and strictly decreasing for $x_1 < x < \frac{\pi}{2}$. See (2.2) for the relationship between $g_1'(x)$ and $g_2(x)$. From $\lim_{x\to 0+0} g_1(x) = 0$ and $\lim_{x\to\frac{\pi}{2}-0} g_1(x) = 0$, we can get $g_1(x) > 0$ for $0 < x < \frac{\pi}{2}$. This completes the proof of Lemma 1. □

3. Proof of Theorem 5

Proof of Theorem 5. First, we will show that

$$f_1(x) = -\frac{x^2}{6} + \frac{2x}{\pi}\theta(x) - \ln\frac{\sin x}{x} = -\frac{x^2}{6} + \frac{4x^5}{3\pi^3} + \frac{32x^5\ln\frac{2}{\pi}}{\pi^5} - \ln\sin x + \ln x > 0.$$

The derivative of $f_1(x)$ is

$$f_1'(x) = x^4\left(\frac{\frac{1}{x} - \frac{x}{3} + \frac{20x^4}{3\pi^3} - \frac{1}{\tan x}}{x^4} + \frac{160\ln\frac{2}{\pi}}{\pi^5}\right) = x^4 f_2(x). \tag{3.1}$$

From $\sin x > x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040}$ and $\cos x < 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320}$ for $0 < x < \frac{\pi}{2}$, we have

$$\begin{aligned}
f_2'(x) &= -\frac{5}{x^6} + \frac{4\cos x}{x^5\sin x} + \frac{1}{x^4} + \frac{1}{x^4\sin^2 x}\\
&< -\frac{5}{x^6} + \frac{4\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320}\right)}{x^5\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040}\right)} + \frac{1}{x^4} + \frac{1}{x^4\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040}\right)^2}\\
&= \frac{f_3(x)}{2x^2(5040 - 840x^2 + 42x^4 - x^6)^2},
\end{aligned} \tag{3.2}$$

where $f_3(x) = -1128960 + 483840x^2 - 52920x^4 + 2856x^6 - 80x^8 + x^{10}$. Here we have

$$\begin{aligned}
f_3(x) &= -1128960 + 483840x^2 - 52920x^4 + 2856x^6 + (-80 + x^2)x^8\\
&< -1128960 + 483840x^2 - 52920x^4 + 2856\left(\frac{\pi}{2}\right)^6\\
&= -1128960 + \frac{357\pi^6}{8} + 483840x^2 - 52920x^4 = f_4(x).
\end{aligned} \tag{3.3}$$

Since the derivative of $f_4(x)$ is

$$f_4'(x) = x(967680 - 211680x^2) > x\left(967680 - 211680\left(\frac{\pi}{2}\right)^2\right) > 0$$

for $0 < x < \frac{\pi}{2}$, $f_4(x)$ is strictly increasing for $0 < x < \frac{\pi}{2}$. From

$$f_4(x) < f_4\left(\frac{\pi}{2}\right) = -1128960 + 120960\pi^2 - \frac{6615\pi^4}{2} + \frac{357\pi^6}{8} \cong -214411 < 0$$

for $0<x<\frac{\pi}{2}$, we have $f_3(x)<0$ for $0<x<\frac{\pi}{2}$ and $f_2(x)$ is strictly decreasing for $0<x<\frac{\pi}{2}$. See (3.2) and (3.3) for the relationship between $f_2'(x)$, $f_3(x)$ and $f_4(x)$. From $\tan x > x+\frac{x^3}{3}+\frac{2x^5}{15}$ for $0<x<\frac{\pi}{2}$, we have

$$\begin{aligned}\lim_{x\to 0+0} f_2(x) &> \lim_{x\to 0+0}\left(\frac{1-\frac{x^2}{3}-\frac{x}{x+\frac{x^3}{3}+\frac{2x^5}{15}}}{x^5}+\frac{20}{3\pi^3}+\frac{160\ln\frac{2}{\pi}}{\pi^5}\right)\\ &=\lim_{x\to 0+0}\left(\frac{1-2x^2}{3x(15+5x^2+2x^4)}+\frac{20}{3\pi^3}+\frac{160\ln\frac{2}{\pi}}{\pi^5}\right)=+\infty\end{aligned}$$

and

$$\lim_{x\to\frac{\pi}{2}-0} f_2(x)=\frac{32}{\pi^5}+\frac{4}{\pi^3}+\frac{160\ln\frac{2}{\pi}}{\pi^5}=\frac{4}{\pi^5}\left(8+\pi^2+40\ln\frac{2}{\pi}\right)\cong-\frac{4}{\pi^5}\times 0.193704<0.$$

Thus, there exists uniquely a number x_0 such that $f_2(x)>0$ for $0<x<x_0$ and $f_2(x)<0$ for $x_0<x<\frac{\pi}{2}$. Therefore, $f_1(x)$ is strictly increasing for $0<x<x_0$ and strictly decreasing for $x_0<x<\frac{\pi}{2}$. See (3.1) for the relationship between $f_1'(x)$ and $f_2(x)$. From $\lim_{x\to 0+0} f_1(x)=0$ and $\lim_{x\to\frac{\pi}{2}-0} f_1(x)=0$, we can get $f_1(x)>0$ for $0<x<\frac{\pi}{2}$. Next, we will show that

$$g_1(x)=-\frac{x^2}{6}+\theta(x)-\ln\frac{\sin x}{x}=-\frac{x^2}{6}+\frac{2x^4}{3\pi^2}+\frac{16x^4}{\pi^4}\ln\frac{2}{\pi}-\ln\sin x+\ln x<0.$$

The derivative of $g_1(x)$ is

$$g_1'(x)=x^3\left(\frac{\frac{1}{x}-\frac{x}{3}+\frac{8x^3}{3\pi^2}-\frac{1}{\tan x}}{x^3}+\frac{64\ln\frac{2}{\pi}}{\pi^4}\right)=x^3 g_2(x). \tag{3.4}$$

From $\sin x < x-\frac{x^3}{6}+\frac{x^5}{120}$ and $\cos x > 1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}$ for $0<x<\frac{\pi}{2}$, we have

$$\begin{aligned}g_2'(x)&=-\frac{4}{x^5}+\frac{2}{3x^3}+\frac{3\cos x}{x^4\sin x}+\frac{1}{x^3\sin^2 x}\\ &>-\frac{4}{x^5}+\frac{2}{3x^3}+\frac{3\left(1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}\right)}{x^4\left(x-\frac{x^3}{6}+\frac{x^5}{120}\right)}+\frac{1}{x^3\left(x-\frac{x^3}{6}+\frac{x^5}{120}\right)^2}\\ &=\frac{x(20-x^2)(14-x^2)}{6(120-20x^2+x^4)^2}>0\end{aligned}$$

for $0 < x < \frac{\pi}{2}$. Therefore, $g_2(x)$ is strictly increasing for $0 < x < \frac{\pi}{2}$. Since we have

$$\begin{aligned}
\lim_{x\to 0+0} g_2(x) &= \lim_{x\to 0+0}\left(\frac{1-\frac{x^2}{3}-\frac{x\cos x}{\sin x}}{x^4}+\frac{8}{3\pi^2}+\frac{64\ln\frac{2}{\pi}}{\pi^4}\right)\\
&< \lim_{x\to 0+0}\left(\frac{1-\frac{x^2}{3}-x\frac{1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}}{x-\frac{x^3}{6}+\frac{x^5}{120}}}{x^4}+\frac{8}{3\pi^2}+\frac{64\ln\frac{2}{\pi}}{\pi^4}\right)\\
&= \lim_{x\to 0+0}\left(\frac{(4-x)(4+x)}{6(120-20x^2+x^4)}+\frac{8}{3\pi^2}+\frac{64\ln\frac{2}{\pi}}{\pi^4}\right)\\
&= \frac{1}{45}+\frac{8}{3\pi^2}+\frac{64\ln\frac{2}{\pi}}{\pi^4}\\
&= \frac{1}{45\pi^4}\left(120\pi^2+\pi^4+2880\ln\frac{2}{\pi}\right)\\
&\cong -\frac{1}{45\pi^4}\times 18.7966 < 0
\end{aligned}$$

and

$$\begin{aligned}
\lim_{x\to\frac{\pi}{2}-0} g_2(x) &= \frac{16}{\pi^4}+\frac{4}{3\pi^2}+\frac{64\ln\frac{2}{\pi}}{\pi^4}\\
&= \frac{4}{3\pi^4}\left(12+\pi^2+48\ln\frac{2}{\pi}\right)\\
&\cong \frac{4}{3\pi^4}\times 0.193635 > 0,
\end{aligned}$$

there exists uniquely a number x_1 such that $g_2(x) < 0$ for $0 < x < x_1$ and $g_2(x) > 0$ for $x_1 < x < \frac{\pi}{2}$. Therefore, $g_1(x)$ is strictly decreasing for $0 < x < x_1$ and strictly increasing for $x_1 < x < \frac{\pi}{2}$. See (3.4) for the relationship between $g_1'(x)$ and $g_2(x)$. From $\lim_{x\to 0+0} g_1(x) = 0$ and $\lim_{x\to\frac{\pi}{2}-0} g_1(x) = 0$, we can get $g_1(x) < 0$ for $0 < x < \frac{\pi}{2}$. This completes the proof of Theorem 5. □

4. Proof of Theorem 6

Proof of Theorem 6. First, we will show that

$$f_1(x) = \ln\frac{\sin x}{x} - \vartheta(x)\ln\left(\frac{2}{\pi}+\frac{\pi-2}{\pi^3}(\pi^2-4x^2)\right) < 0.$$

By Theorem 5 and Lemma 1, we have

$$\begin{aligned}
f_1(x) <& -\frac{x^2}{6}+\frac{2x}{\pi}\left(\frac{2x^4}{3\pi^2}+\frac{16x^4}{\pi^4}\ln\frac{2}{\pi}\right)\\
&-\vartheta(x)\left(\left(\frac{32(\pi-2)(3\pi-2)}{\pi^8}+\frac{64\ln\frac{2}{\pi}}{\pi^6}\right)x^6-\frac{4(\pi-2)x^2(\pi^3-4x^2+2\pi x^2)}{\pi^6}\right)\\
=&-\frac{x^2}{6}+\frac{4x^5}{3\pi^3}+\frac{(\pi-2x)x^2(\pi+2x)(\pi^4-96x+48\pi x-2\pi^3x)(\pi^3-4x^2+6\pi x^2)}{6\pi^9}\\
&-\frac{8(\pi-2x)x^5(24\pi-12\pi^2+48x-24\pi x+\pi^3x)\ln\frac{2}{\pi}}{3(\pi-2)\pi^7}\\
=&8(\pi-2x)x^5(24\pi-12\pi^2+48x-24\pi x+\pi^3x)f_2(x) \qquad (4.1)
\end{aligned}$$

for $0<x<\frac{\pi}{2}$, where

$$\begin{aligned}
f_2(x) =&\frac{1}{24\pi^9x^2(24\pi-12\pi^2+48x-24\pi x+\pi^3x)}\\
&\times\Big(-48\pi^4+24\pi^5-\pi^7-96\pi^3x+48\pi^4x-2\pi^5x-\pi^6x+192\pi x^2-384\pi^2x^2\\
&+144\pi^3x^2+384x^3-768\pi x^3+288\pi^2x^3+8\pi^3x^3-12\pi^4x^3\Big)-\frac{\ln\frac{2}{\pi}}{3(\pi-2)\pi^7}.
\end{aligned}$$

Since we have

$$\begin{aligned}
24\pi-12\pi^2+48x-24\pi x+\pi^3x &= 24\pi-12\pi^2+\left(48-24\pi+\pi^3\right)x\\
&<24\pi-12\pi^2+\left(48-24\pi+\pi^3\right)\left(\frac{\pi}{2}\right)=-37.3695<0
\end{aligned}$$

for $0<x<\frac{\pi}{2}$, it suffices to show that $f_2(x)>0$. See (4.1) for the relationship between $f_1(x)$ and $f_2(x)$. The derivative of $f_2(x)$ is

$$f_2'(x)=\frac{f_3(x)}{24\pi^6x^3(24\pi-12\pi^2+48x-24\pi x+\pi^3x)^2}, \qquad (4.2)$$

where

$$\begin{aligned}
f_3(x) =&2304\pi^2-2304\pi^3+576\pi^4+48\pi^5-24\pi^6+9216\pi x-9216\pi^2x\\
&+2352\pi^3x+288\pi 4x-156\pi^5x+3\pi^7x+9216x^2-9216\pi x^2+2496\pi^2x^2\\
&+192\pi^3x^2-144\pi^4x^2+4\pi^5x^2+2\pi^6x^2.
\end{aligned}$$

The derivative of $f_3(x)$ is

$$\begin{aligned}
f_3'(x) =&9216\pi-9216\pi^2+2352\pi^3+288\pi^4-156\pi^5+3\pi^7\\
&+4(4-\pi)(24-6\pi-\pi^2)(48-24\pi+\pi^3)x.
\end{aligned}$$

By $24-6\pi-\pi^2<0$ and $48-24\pi+\pi^3>0$, $f_3'(x)$ is strictly decreasing for $0<x<\frac{\pi}{2}$. Here, we have

$$f_3'(x) > f_3'\left(\frac{\pi}{2}\right) = 18432\pi - 18432\pi^2 + 4848\pi^3 + 480\pi^4 - 300\pi^5 + 4\pi^6 + 5\pi^7 \\ \cong 205.198 > 0$$

for $0<x<\frac{\pi}{2}$. Therefore, $f_3(x)$ is strictly increasing for $0<x<\frac{\pi}{2}$. From

$$f_3(x) < f_3\left(\frac{\pi}{2}\right) = 9216\pi^2 - 9216\pi^3 + 2376\pi^4 + 240\pi^5 - 138\pi^6 + \pi^7 + 2\pi^8 \\ \cong -581.201 < 0$$

for $0<x<\frac{\pi}{2}$, $f_2(x)$ is strictly decreasing for $0<x<\frac{\pi}{2}$. See (4.2) for the relationship between $f_2'(x)$ and $f_3(x)$. From

$$\begin{aligned}\lim_{x\to\frac{\pi}{2}-0} f_2(x) &= \frac{32-96\pi+40\pi^2-\pi^4}{\pi^9(96-48\pi+\pi^3)} - \frac{\ln\frac{2}{\pi}}{3(\pi-2)\pi^7} \\ &= \frac{3(\pi-2)(32-96\pi+40\pi^2-\pi^4)-\pi^2(96-48\pi+\pi^3)\ln\frac{2}{\pi}}{3(\pi-2)\pi^9(96-48\pi+\pi^3)} \\ &\cong \frac{-10.8836}{3(\pi-2)\pi^9(96-48\pi+\pi^3)} > 0,\end{aligned}$$

we can get $f_2(x)>0$ for $0<x<\frac{\pi}{2}$. Next, we will show that

$$g_1(x) = \ln\frac{\sin x}{x} - \theta(x)\ln\left(\frac{2}{\pi} + \frac{\pi-2}{\pi^3}(\pi^2-4x^2)\right) > 0.$$

By Theorem 5 and Lemma 1, we have

$$\begin{aligned} g_1(x) > &-\frac{x^2}{6} + \frac{2x^4}{3\pi^2} + \frac{16x^4}{\pi^4}\ln\frac{2}{\pi} \\ &- \theta(x)\left(\left(\frac{64(\pi-2)(3\pi-2)}{\pi^9} + \frac{128\ln\frac{2}{\pi}}{\pi^7}\right)x^7 - \frac{4(\pi-2)x^2(\pi^3-4x^2+2\pi x^2)}{\pi^6}\right) \\ = &-\frac{x^2}{6} + \frac{2x^4}{3\pi^2} + \frac{(\pi-2x)x^2(\pi^6-384x^3+192\pi x^3-8\pi^3x^3)}{6\pi^{12}} \\ &\times(\pi^5+2\pi^4x-4\pi^2x^2+6\pi^3x^2-8\pi x^3+12\pi^2x^3-16x^4+24\pi x^4) \\ &+ \frac{16(\pi-2x)x^4(\pi^2+2\pi x+4x^2)(-6\pi^3+3\pi^4-48x^3+24\pi x^3-\pi^3x^3)\ln\frac{2}{\pi}}{3(\pi-2)\pi^{10}} \\ = &(\pi-2x)x^4(\pi^2+2\pi x+4x^2)(-6\pi^3+3\pi^4-48x^3+24\pi x^3-\pi^3x^3)g_2(x) \end{aligned} \tag{4.3}$$

for $0 < x < \frac{\pi}{2}$, where

$$\begin{aligned} g_2(x) =& \frac{1}{3\pi^{12}(\pi^2+2\pi x+4x^2)(-6\pi^3+3\pi^4-48x^3+24\pi x^3-\pi^3x^3)} \\ &\times \Big(-2\pi^8+3\pi^9-192\pi^5x+96\pi^6x-4\pi^7x+2\pi^8x-384\pi^4x^2+192\pi^5x^2 \\ &-8\pi^6x^2+4\pi^7x^2+768\pi^2x^3-1536\pi^3x^3+576\pi^4x^3+16\pi^5x^3-24\pi^6x^3 \\ &+1536\pi x^4-3072\pi^2x^4+1152\pi^3x^4+32\pi^4x^4-48\pi^5x^4+3072x^5 \\ &-6144\pi x^5+2304\pi^2x^5+64\pi^3x^5-96\pi^4x^5\Big) + \frac{16\ln\frac{2}{\pi}}{3(\pi-2)\pi^{10}}. \end{aligned}$$

Since we have

$$\begin{aligned} &-6\pi^3+3\pi^4-48x^3+24\pi x^3-\pi^3x^3 = -6\pi^3+3\pi^4-(48-24\pi+\pi^3)x^3 \\ &> -6\pi^3+3\pi^4-(48-24\pi+\pi^3)\left(\frac{\pi}{2}\right)^3 = \frac{1}{8}\pi^3\left(-96+48\pi-\pi^3\right) > 0 \end{aligned}$$

for $0 < x < \frac{\pi}{2}$, it suffices to show that $g_2(x) > 0$. See (4.3) for the relationship between $g_1(x)$ and $g_2(x)$. The derivative of $g_2(x)$ is

$$g_2'(x) = \frac{(48-24\pi+\pi^3)g_3(x)}{3\pi^9(\pi^2+2\pi x+4x^2)^2(-6\pi^3+3\pi^4-48x^3+24\pi x^3-\pi^3x^3)^2}, \tag{4.4}$$

where

$$\begin{aligned} g_3(x) =& 24\pi^7-12\pi^8+96\pi^6x-48\pi^7x-288\pi^4x^2+576\pi^5x^2-216\pi^6x^2-6\pi^7x^2 \\ &+9\pi^8x^2-1152\pi^3x^3+1920\pi^4x^3-672\pi^5x^3-24\pi^6x^3+28\pi^7x^3 \\ &-3456\pi^2x^4+5376\pi^3x^4-1824\pi^4x^4-72\pi^5x^4+76\pi^6x^4-4608\pi x^5 \\ &+4608\pi^2x^5-1152\pi^3x^5-96\pi^4x^5+48\pi^5x^5-4608x^6+4608\pi x^6 \\ &-1152\pi^2x^6-96\pi^3x^6+48\pi^4x^6. \end{aligned}$$

The derivatives of $g_3(x)$ are

$$\begin{aligned} g_3'(x) =& 48\pi^6(2-\pi)-6\pi^4(2-3\pi)(48-24\pi+\pi^3)x \\ &-12\pi^3(6-7\pi)(48-24\pi+\pi^3)x^2-16\pi^2(18-19\pi)(48-24\pi+\pi^3)x^3 \\ &-240(2-\pi)\pi(48-24\pi+\pi^3)x^4-288(2-\pi)(48-24\pi+\pi^3)x^5 \end{aligned}$$

and

$$\begin{aligned} g_3''(x) =& 6(48-24\pi+\pi^3)\Big(\pi^4(3\pi-2)+4\pi^3(7\pi-6)x+8\pi^2(19\pi-18)x^2 \\ &+160\pi(\pi-2)x^3+240(\pi-2)x^4\Big) \\ >& 6\pi^4(48-24\pi+\pi^3)(3\pi-2) > 0 \end{aligned}$$

for $0<x<\frac{\pi}{2}$. Thus, $g_3'(x)$ is strictly increasing for $0<x<\frac{\pi}{2}$. From $\lim_{x\to 0+0} g_3'(x) = 48\pi^6(2-\pi)<0$ and

$$\lim_{x\to\frac{\pi}{2}-0} g_3'(x) = 4\pi^5(-1296+1776\pi-564\pi^2-27\pi^3+23\pi^4) \cong 147197>0,$$

there exists uniquely a number x_0 such that $g_3'(x)<0$ for $0<x<x_0$ and $g_3'(x)>0$ for $x_0<x<\frac{\pi}{2}$. Therefore, $g_3(x)$ is strictly decreasing for $0<x<x_0$ and strictly increasing for $x_0<x<\frac{\pi}{2}$. Hence, we have $g_3(x)<\max\{g_3(0), g_3(\frac{\pi}{2})\}$ for $0<x<\frac{\pi}{2}$. By $\lim_{x\to 0+0} g_3(x) = 24\pi^7-12\pi^8<0$ and

$$\lim_{x\to\frac{\pi}{2}-0} g_3(x) = \frac{3}{4}\pi^6(-864+1344\pi-456\pi^2-18\pi^3+17\pi^4) \cong -32012.5<0,$$

we obtain $g_3(x)<0$ for $0<x<\frac{\pi}{2}$ and $g_2(x)$ is strictly decreasing for $0<x<\frac{\pi}{2}$. See (4.4) for the relationship between $g_2'(x)$ and $g_3(x)$. From

$$\begin{aligned}\lim_{x\to\frac{\pi}{2}-0} g_2(x) &= \frac{32(-72+192\pi-78\pi^2+\pi^4)}{9\pi^{12}(96-48\pi+\pi^3)} + \frac{16\ln\frac{2}{\pi}}{3(\pi-2)\pi^{10}} \\ &= \frac{16\left(2(\pi-2)(-72+192\pi-78\pi^2+\pi^4)+3\pi^2(96-48\pi+\pi^3)\ln\frac{2}{\pi}\right)}{9(\pi-2)\pi^{12}(96-48\pi+\pi^3)} \\ &\cong \frac{-16\times 4.36971}{9(\pi-2)\pi^{12}(96-48\pi+\pi^3)} > 0,\end{aligned}$$

we can get $g_2(x)>0$ for $0<x<\frac{\pi}{2}$. This completes the proof of Theorem 6. $\square$

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