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A NEW HILBERT–TYPE INTEGRAL INEQUALITY WITH THE GENERAL NONHOMOGENEOUS KERNEL AND APPLICATIONS

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Abstract. In this paper, by the use of weight functions and the technique of real analysis, a new Hilbert-type integral inequality with a general nonhomogeneous kernel as $H(xv(y))$ is given. A few equivalent statements related to the best possible constant factor and parameters are considered. As applications, the equivalent forms, the operator expressions and some corollaries are obtained.

1. Introduction

If $0<\int_0^\infty f^2(x)\,dx<\infty$ and $0<\int_0^\infty g^2(y)\,dy<\infty$, then we have the following Hilbert's integral inequality (cf. [9], Theorem 316):

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dxdy<\pi\left(\int_0^\infty f^2(x)dx\int_0^\infty g^2(y)dy\right)^{\frac{1}{2}}. \tag{1}$$

where, the constant factor π is the best possible.

In 1934, Hardy et al proved a Hilbert-type integral inequality with the general homogeneous kernel of degree -1 as $k(x,y)$ (cf. [9], Theorem 319): If $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $f(x),g(y)\geqslant 0$, $0<\int_0^\infty f^p(x)\,dx<\infty$ and $0<\int_0^\infty g^q(y)\,dy<\infty$, $k(x,y)\,(\geqslant 0)$, $k_p:=\int_0^\infty k(u,1)\,u^{-\frac{1}{p}}du\in\mathbf{R}_+=(0,\infty)$, then

$$\int_0^\infty\int_0^\infty k(x,y)f(x)g(y)dxdy$$
$$<k_p\left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty g^q(y)dy\right)^{\frac{1}{q}}, \tag{2}$$

where the constant factor k_p is the best possible. For $p=q=2$, $k(x,y)=\frac{1}{x+y}$, (2) reduces to (1).

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Additionally, the following Hilbert-type integral inequality with nonhomogeneous kernel was provided: assuming that $H(u)$ is a nonnegative measurable function in $\mathbf{R}_+$, $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\gamma=\sigma_1$, $\sigma\in\mathbf{R}=(-\infty,\infty)$, such that

$$k(\gamma):=\int_0^\infty H(u)\,u^{\gamma-1}du\in\mathbf{R}_+,$$

then we have

$$\begin{aligned}&\int_0^\infty\int_0^\infty H(xy)\,f(x)g(y)\,dxdy\\&<k\left(\frac{1}{p}\right)\left\{\int_0^\infty x^{p-2}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty g^q(y)dy\right\}^{\frac{1}{q}}.\end{aligned}\tag{3}$$

where the constant factor $K\left(\frac{1}{p}\right)$ is the best possible (cf. [9], Theorem 350).

In 1998, by introducing a parameter $\lambda>0$, Yang [25] gave an extension of (1) as follows:

$$\begin{aligned}&\int_0^\infty\int_0^\infty\frac{f(x)\,g(y)}{(x+y)^\lambda}dxdy\\&<B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left(\int_0^\infty x^{1-\lambda}f^2(x)dx\int_0^\infty y^{1-\lambda}g^2(y)dy\right)^{\frac{1}{2}},\end{aligned}\tag{4}$$

where, the constant factor $B(\frac{\lambda}{2},\frac{\lambda}{2})$ is the best possible, and

$$B(u,v):=\int_0^\infty\frac{t^{u-1}}{(1+t)^{u+v}}dt\quad(u,v>0)$$

is the beta function (cf. [20]). For $\lambda=1$, (4) also reduces to (1).

Inequalities (1)–(4) are important in analysis and its applications. We can find a number of extensions and improvements in the past mathematics literature (cf. [5]–[2], [6], [7], [21]–[23], [26], [27], [33]). In 2006, Yang [32] gave a discrete Hilbert-type inequality with the kernel as $\frac{1}{(1+u(m)u(n))^\lambda}$, the other similar results were provided by (cf. [4], [16]). In 2013, Yang [28] studied the equivalence of inequalities (2) and (3), by adding a condition that $H(u)=k_\lambda(u,1)$. In 2017, Hong et. al [10] considered an equivalent condition for (2) involving certain parameters. Some further related results were obtained in (cf. [3], [8], [11]–[13], [17]–[19], [24], [29]–[31]).

In this paper, making use of the way of real analysis and the weight functions, we study a new Hilbert-type integral inequality with the nonhomogeneous kernel as $H(xv(y))$ and the best possible constant factor. A few equivalent statements related to the best possible constant factor and parameters are provided. As applications, we also consider the equivalent forms, the operator expressions and some corollaries.

2. Some lemmas

In what follows, we assume that $p > 1,\ \frac{1}{p} + \frac{1}{q} = 1,\ v(x) > 0,\ v'(x) > 0\ (x \in \mathbf{R}_+ = (0,\infty))$, with $v(0^+) = 0,\ v(\infty) = \infty,\ H(u)$ is a nonnegative measurable functions in $\mathbf{R}_+,\ \sigma,\ \sigma_1 \in \mathbf{R} = (-\infty,\infty)$,

$$K(\gamma) := \int_0^\infty H(u)u^{\gamma-1}du \in \mathbf{R}_+\ (\gamma = \sigma, \sigma_1),$$

$f(x)$ and $g(y)$ are nonnegative measurable functions in $\mathbf{R}_+$, satisfying:

$$\begin{aligned} &0 < \int_0^\infty x^{p\left[1-\left(\frac{\sigma}{p}+\frac{\sigma_1}{q}\right)\right]-1} f^p(x)\,dx < \infty, \quad \text{and} \\ &0 < \int_0^\infty \frac{[v(y)]^{q\left[1-\left(\frac{\sigma}{p}+\frac{\sigma_1}{q}\right)\right]-1}}{(v'(y))^{q-1}} g^q(y)dy < \infty \end{aligned} \tag{5}$$

LEMMA 1. *We have the following Hilbert-type integral inequality with the non-homogeneous kernel:*

$$\begin{aligned} I :=& \int_0^\infty \int_0^\infty H(xv(y))\, f(x) g(y)\, dxdy \\ <& K^{\frac{1}{p}}(\sigma) K^{\frac{1}{q}}(\sigma_1) \left\{ \int_0^\infty x^{p\left[1-\left(\frac{\sigma}{p}+\frac{\sigma_1}{q}\right)\right]-1} f^p(x)dx \right\}^{\frac{1}{p}} \\ &\times \left\{ \int_0^\infty \frac{(v(y))^{q\left[1-\left(\frac{\sigma}{p}+\frac{\sigma_1}{q}\right)\right]-1}}{(v'(y))^{q-1}} g^q(y)dy \right\}^{\frac{1}{q}}. \end{aligned} \tag{6}$$

Proof. Since $v'(y) > 0$, with $v(0^+) = 0,\ v(y)$ is strictly increasing and then $v(y) > v(0^+) = 0\ (y \in \mathbf{R}_+)$. We define the following weight functions:

$$\begin{aligned} \omega(\sigma,x) &:= x^\sigma \int_0^\infty H(xv(y))\, v^{\sigma-1}(y)\, v'(y)\, dy \quad (x \in \mathbf{R}_+), \\ \varpi(\sigma_1,y) &:= (v(y))^{\sigma_1} \int_0^\infty H(xv(y))\, x^{\sigma_1-1} dx \quad (y \in \mathbf{R}_+). \end{aligned} \tag{7}$$

For fixed $x > 0$, setting $u = xv(y)$, we obtain

$$\omega(\sigma,x) = \int_0^\infty H(u)\, u^{\sigma-1} du = K(\sigma) \in \mathbf{R}_+. \tag{8}$$

In the same way, for fixed $y > 0$, setting $u = xv(y)$, we obtain

$$\varpi(\sigma_1,y) = \int_0^\infty H(u)\, u^{\sigma_1-1} du = K(\sigma_1) \in \mathbf{R}_+. \tag{9}$$

By Hölder's inequality (cf. [15]), Fubini theorem (cf. [14]) and (7), we have

$$\begin{aligned}
I &= \int_0^\infty \int_0^\infty H(xv(y)) f(x)g(y)dxdy \\
&= \int_0^\infty \int_o^\infty H(xv(y)) \left[\frac{(v(y))^{\frac{\sigma-1}{p}}(v'(y))^{\frac{1}{p}}}{x^{\frac{\sigma_1-1}{q}}} f(x) \right] \\
&\quad \times \left[\frac{x^{\frac{\sigma_1-1}{q}}}{(v(y))^{\frac{\sigma-1}{p}}(v'(y))^{\frac{1}{p}}} g(y) \right] dxdy \\
&\leqslant \left[\int_0^\infty \int_0^\infty H(xv(y)) \frac{(v(y))^{\sigma-1}v'(y)}{x^{(\sigma_1-1)(p-1)}} f^p(x)dydx \right]^{\frac{1}{p}} \\
&\quad \times \left[\int_0^\infty \int_0^\infty H(xv(y)) \frac{x^{\sigma_1-1}}{(v(y))^{(\sigma-1)(q-1)}(v'(y))^{q-1}} g^q(y)dxdy \right]^{\frac{1}{q}} \\
&= \left\{ \int_0^\infty \omega(\sigma,x) x^{p\left[1-\left(\frac{\sigma}{p}+\frac{\sigma_1}{q}\right)\right]-1} f^p(x)dx \right\}^{\frac{1}{p}} \\
&\quad \times \left\{ \int_0^\infty \varpi(\sigma_1,y) \frac{(v(y))^{q\left[1-\left(\frac{\sigma}{p}+\frac{\sigma_1}{q}\right)\right]-1}}{(v'(y))^{q-1}} g^q(y)dy \right\}^{\frac{1}{q}}. \qquad (10)
\end{aligned}$$

If (10) keeps the form of equality, then, there exist constants A and B, such that they are not both zero and (cf. [15]),

$$A\frac{[v(y)]^{\sigma-1}v'(y)}{x^{(\sigma_1-1)(p-1)}} f^p(x) = \frac{Bx^{\sigma_1-1}}{(v(y))^{(\sigma-1)(q-1)}(v'(y))^{q-1}} g^q(y)$$
$$\text{a.e. in } (0,\infty)\times(0,\infty).$$

We assume that $A \neq 0$. For fixed a.e. $y \in (0,\infty)$, we have

$$x^{p\left[1-\left(\frac{\sigma}{p}+\frac{\sigma_1}{q}\right)\right]-1} f^p(x) = \frac{Bg^q(y)}{A(v(y))^{(\sigma-1)q}(v'(y))^q} x^{\sigma_1-\sigma-1} \quad \text{a.e. in } (0,\infty).$$

Since $\int_0^\infty x^{\sigma_1-\sigma-1}dx = \infty$, the above expression contradicts the fact that

$$0 < \int_0^\infty x^{p\left[1-\left(\frac{\sigma}{p}+\frac{\sigma_1}{q}\right)\right]-1} f^p(x)\,dx < \infty.$$

Therefore, by (8) and (9), we have (6).

The lemma is proved. □

REMARK 1. If $\sigma_1 = \sigma$, then by (5) and (6), we have

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x)\,dx < \infty, \qquad 0 < \int_0^\infty \frac{(v(y))^{q(1-\sigma)-1}}{(v'(y))^{q-1}} g^q(y)dy < \infty$$

and the following inequality:

$$I=\int_0^\infty\int_0^\infty H(xv(y))f(x)g(y)\,dxdy<K(\sigma)$$
$$\times\left[\int_0^\infty x^{p(1-\sigma)-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty\frac{(v(y))^{q(1-\sigma)-1}}{(v'(y))^{q-1}}g^q(y)dy\right]^{\frac{1}{q}}. \tag{11}$$

LEMMA 2. *The constant factor $K(\sigma)$ in (11) is the best possible.*

Proof. For any $\varepsilon>0$, we set

$$\widetilde{f}(x):=\begin{cases}0, & 0<x<1\\ x^{\sigma-\frac{\varepsilon}{p}-1}, & x\geqslant 1\end{cases},$$
$$\widetilde{g}(y):=\begin{cases}(v(y))^{\sigma+\frac{\varepsilon}{q}-1}v'(y), & 0<y\leqslant 1\\ 0, & y>1\end{cases}.$$

If there exists a positive constant M $(\leqslant K(\sigma))$, such that (11) is valid when we replace $K(\sigma)$ by M, then in particular, we have

$$\begin{aligned}\widetilde{I}&:=\int_0^\infty\int_0^\infty H(xv(y))\widetilde{f}(x)\widetilde{g}(y)dxdy\\&<M\left[\int_0^\infty x^{p(1-\sigma)-1}\widetilde{f}^P(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty\frac{(v(y))^{q(1-\sigma)-1}}{(v'(y))^{q-1}}\widetilde{g}^q(x)dy\right]^{\frac{1}{q}}\\&=M\left(\int_1^\infty x^{-\varepsilon-1}dx\right)^{\frac{1}{p}}\left[\int_0^1(v(y))^{\varepsilon-1}v'(y)\,dy\right]^{\frac{1}{q}}=\frac{M}{\varepsilon}(v(1))^{\frac{\varepsilon}{q}}.\end{aligned}$$

In view of Fubini theorem (cf. [14]), we have

$$\begin{aligned}\widetilde{I}&=\int_0^1\left(\int_1^\infty H(xv(y))x^{\sigma-\frac{\varepsilon}{p}-1}dx\right)(v(y))^{\sigma+\frac{\varepsilon}{q}-1}v'(y)\,dy\\&\overset{u=xv(y)}{=}\int_0^1\left(\int_{v(y)}^\infty H(u)u^{\sigma-\frac{\varepsilon}{p}-1}du\right)(v(y))^{\varepsilon-1}v'(y)\,dy\\&=\int_0^1\left(\int_{v(y)}^{v(1)}H(u)u^{\sigma-\frac{\varepsilon}{p}-1}du\right)(v(y))^{\varepsilon-1}v'(y)\,dy\\&\quad+\int_0^1\left(\int_{v(1)}^\infty H(u)u^{\sigma-\frac{\varepsilon}{p}-1}du\right)(v(y))^{\varepsilon-1}v'(y)\,dy\\&=\int_0^{v(1)}\left[\int_0^{v^{-1}(u)}(v(y))^{\varepsilon-1}v'(y)\,dy\right]H(u)u^{\sigma-\frac{\varepsilon}{p}-1}du\\&\quad+\frac{1}{\varepsilon}(v(1))^\varepsilon\int_{v(1)}^\infty H(u)u^{\sigma-\frac{\varepsilon}{p}-1}du\\&=\frac{1}{\varepsilon}\left[\int_0^{v(1)}H(u)u^{\sigma+\frac{\varepsilon}{q}-1}du+(v(1))^\varepsilon\int_{v(1)}^\infty H(u)u^{\sigma-\frac{\varepsilon}{p}-1}du\right].\end{aligned}$$

Hence, it follows that

$$\int_0^{v(1)} H(u)\, u^{\sigma+\frac{\varepsilon}{q}-1} du + (v(1))^{\varepsilon} \int_{v(1)}^{\infty} H(u)\, u^{\sigma-\frac{\varepsilon}{p}-1} du$$
$$= \varepsilon \widetilde{I} < M(v(1))^{\frac{\varepsilon}{q}}$$

For $\varepsilon \to 0^+$, by Fatou lemma (cf. [14]), we have

$$\begin{aligned} K(\sigma) &= \int_0^{v(1)} \lim_{\varepsilon\to 0^+} H(u)u^{\sigma+\frac{\varepsilon}{q}-1} du + \int_{v(1)}^{\infty} \lim_{\varepsilon\to 0^+} (v(1))^{\varepsilon} H(u)\, u^{\sigma-\frac{\varepsilon}{p}-1} du \\ &\leqslant \underline{\lim}_{\varepsilon\to 0^+} \left[\int_0^{v(1)} H(u)u^{\sigma+\frac{\varepsilon}{q}-1} du + (v(1))^{\varepsilon} \int_{v(1)}^{\infty} H(u)\, u^{\sigma-\frac{\varepsilon}{p}-1} du\right] \\ &\leqslant M. \end{aligned}$$

Hence, $M = K(\sigma)$ is the best possible constant factor in (11).

The lemma is proved. □

REMARK 2. Setting $\widehat{\sigma} := \frac{\sigma}{p} + \frac{\sigma_1}{q}$, we can rewrite (6) as follows:

$$\begin{aligned} &\int_0^{\infty}\int_0^{\infty} H(xv(y)) f(x) g(y)\, dx dy \\ &< K^{\frac{1}{p}}(\sigma) K^{\frac{1}{q}}(\sigma_1) \left[\int_0^{\infty} x^{p(1-\widehat{\sigma})-1} f^p(x) dx\right]^{\frac{1}{p}} \\ &\times \left[\int_0^{\infty} \frac{(v(y))^{q(1-\widehat{\sigma})-1}}{(v'(y))^{q-1}} g^q(y) dy\right]^{\frac{1}{q}}. \end{aligned} \tag{12}$$

By Hölder's inequality with weight (cf. [15]), we obtain the following inequality:

$$\begin{aligned} 0 < K(\widehat{\sigma}) &= K\left(\frac{\sigma}{p} + \frac{\sigma_1}{q}\right) = \int_0^{\infty} H(u)\, u^{\frac{\sigma}{p}+\frac{\sigma_1}{q}-1} du \\ &= \int_0^{\infty} H(u) \left(u^{\frac{\sigma-1}{p}}\right)\left(u^{\frac{\sigma_1-1}{q}}\right) du \\ &\leqslant \left(\int_0^{\infty} H(u)\, u^{\sigma-1} du\right)^{\frac{1}{p}} \left(\int_0^{\infty} H(u)\, u^{\sigma_1-1} du\right)^{\frac{1}{q}} \\ &= K^{\frac{1}{p}}(\sigma) K^{\frac{1}{q}}(\sigma_1) < \infty. \end{aligned} \tag{13}$$

LEMMA 3. *If the constant factor $K^{\frac{1}{p}}(\sigma) K^{\frac{1}{q}}(\sigma_1)$ in (12) (or (6)) is the best possible, then we have $\sigma = \sigma_1$.*

Proof. If the constant factor $K^{\frac{1}{p}}(\sigma) K^{\frac{1}{q}}(\sigma_1)$ in (12) is the best possible, then in view of (11) (for $\sigma = \widehat{\sigma}$), we have the following inequality:

$$K^{\frac{1}{p}}(\sigma) K^{\frac{1}{q}}(\sigma_1) \leqslant K(\widehat{\sigma}) \quad (\in \mathbf{R}_+),$$

namely, (13) keeps the form of equality.

We observe that (13) keeps the form of equality if and only if there exist constants A and B, such that they are not both zero and (cf. [15])

$$Au^{\sigma-1} = Bu^{\sigma_1-1} \text{ a.e. in } \mathbf{R}_+.$$

Assuming that $A \neq 0$, we have $u^{\sigma_1-1} = \frac{B}{A}$ a.e. in $\mathbf{R}_+$, which follows that $\sigma - \sigma_1 = 0$, namely, $\sigma_1 = \sigma$.

The lemma is proved. $\square$

3. Main results

THEOREM 1. *Inequality (12) (or (6)) is equivalent to the following inequality:*

$$\begin{aligned} J &:= \left[\int_0^\infty (v(y))^{p\widehat{\sigma}-1} v'(y) \left(\int_0^\infty H(xv(y)) f(x)dx\right)^p dy\right]^{\frac{1}{p}} \\ &< K^{\frac{1}{p}}(\sigma) K^{\frac{1}{q}}(\sigma_1) \left[\int_0^\infty x^{p(1-\widehat{\sigma})-1} f^p(x)dx\right]^{\frac{1}{p}}. \end{aligned} \tag{14}$$

The constant factor $K^{\frac{1}{p}}(\sigma)K^{\frac{1}{q}}(\sigma_1)$ in (14) is the best possible if and only if the same constant factor in (12) is the best possible. In particular, for $\sigma_1 = \sigma$, we have the following inequality equivalent to (11) with the same best possible constant factor $K(\sigma)$:

$$\begin{aligned} &\left[\int_0^\infty (v(y))^{p\sigma-1} v'(y) \left(\int_0^\infty H(xv(y)) f(x)dx\right)^p dy\right]^{\frac{1}{p}} \\ &< K(\sigma) \left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}}. \end{aligned} \tag{15}$$

Proof. If (14) is valid, then by Hölder's inequality, we have

$$\begin{aligned} I &= \int_0^\infty \left[(v(y))^{\frac{-1}{p}+\widehat{\sigma}} (v'(y))^{\frac{1}{p}} \int_0^\infty H(xv(y)) f(x)dx\right] \\ &\quad \times \left[(v(y))^{\frac{1}{p}-\widehat{\sigma}} (v'(y))^{-\frac{1}{p}} g(y)\right] dy \\ &\leqslant J \left[\int_0^\infty \frac{(v(y))^{q(1-\widehat{\sigma})-1}}{(v'(y))^{q-1}} g(y)\, dy\right]^{\frac{1}{q}}. \end{aligned} \tag{16}$$

By (14), we have (12).

On the other hand, assuming that (12) is valid, we set

$$g(y) := (v(y))^{p\widehat{\sigma}-1} v'(y) \left(\int_0^\infty H(xv(y)) f(x)dx\right)^{p-1} dy \quad (y > 0).$$

Then it follows that

$$J^{P}=\int_{0}^{\infty}\frac{(v(y))^{q(1-\widehat{\sigma})-1}}{(v'(y))^{q-1}}g^{q}(y)dy=I. \tag{17}$$

If $J=0,$ then (14) is naturally valid; if $J=\infty,$ then it is impossible to make (14) valid, namely, $J<\infty.$ Suppose that $0<J<\infty.$ By (12), we have

$$\begin{aligned}
0<J^{P}&=\int_{0}^{\infty}\frac{(v(y))^{q(1-\widehat{\sigma})-1}}{(v'(y))^{q-1}}g^{q}(y)dy=I\\
&<K^{\frac{1}{P}}(\sigma)K^{\frac{1}{q}}(\sigma_1)\left[\int_{0}^{\infty}x^{p(1-\widehat{\sigma})-1}f^{p}(x)dx\right]^{\frac{1}{p}}J^{P-1}<\infty,\\
J&=\left[\int_{0}^{\infty}\frac{(v(y))^{q(1-\widehat{\sigma})-1}}{(v'(y))^{q-1}}g^{q}(y)dy\right]^{\frac{1}{P}}\\
&<K^{\frac{1}{P}}(\sigma)K^{\frac{1}{q}}(\sigma_1)\left[\int_{0}^{\infty}x^{p(1-\widehat{\sigma})-1}f^{p}(x)dx\right]^{\frac{1}{p}},
\end{aligned}$$

namely, (14) follows, which is equivalent to (12).

If the constant factor in (12) is the best possible, then the same constant factor in (14) is also the best possible. Otherwise, by (16) (for $\sigma_1=\sigma$), we would reach a contradiction that the same constant factor in (11) is the best possible. If the constant factor in (14) is not the best possible, then by (17), we would reach a contradiction that the same constant factor in (12) is not the best possible.

The Theorem is proved. □

THEOREM 2. *The following statements* (i), (ii), (iii) *and* (iv) *are equivalent:*

(i) *Both* $K^{\frac{1}{P}}(\sigma)K^{\frac{1}{q}}(\sigma_1)$ *and* $K\left(\frac{\sigma}{p}+\frac{\sigma_1}{q}\right)$ *are independent of* p,q;

(ii) $K^{\frac{1}{P}}(\sigma)K^{\frac{1}{q}}(\sigma_1)\leqslant K\left(\frac{\sigma}{p}+\frac{\sigma_1}{q}\right)$;

(iii) $\sigma_1=\sigma$;

(iv) *The constant factor* $K^{\frac{1}{P}}(\sigma)K^{\frac{1}{q}}(\sigma_1)$ *in* (12) *(resp.* (14)*) is the best possible.*

Proof. $(i)\Rightarrow(ii)$. we have

$$K^{\frac{1}{P}}(\sigma)K^{\frac{1}{q}}(\sigma_1)=\lim_{p\to1^{+}}\lim_{q\to\infty}K^{\frac{1}{P}}(\sigma)K^{\frac{1}{q}}(\sigma_1)=K(\sigma).$$

By Fatou lemma, we find

$$\begin{aligned}
K\left(\frac{\sigma}{p}+\frac{\sigma_1}{q}\right)&=\lim_{p\to1^{+}}\lim_{q\to\infty}K\left(\frac{\sigma}{p}+\frac{\sigma_1}{q}\right)\\
&\geqslant K(\sigma)=K^{\frac{1}{P}}(\sigma)K^{\frac{1}{q}}(\sigma_1).
\end{aligned}$$

$(ii) \Rightarrow (iii)$ If $K^{\frac{1}{p}}(\sigma)K^{\frac{1}{q}}(\sigma_1) \leqslant K\left(\frac{\sigma}{p}+\frac{\sigma_1}{q}\right)$, then (13) keeps the form of equality. In view of the proof in Lemma 4, we have $\sigma_1 = \sigma$.

$(iii) \Rightarrow (iv)$ By Lemma 3 (resp. Theorem 1), the constant factor

$$K^{\frac{1}{p}}(\sigma)K^{\frac{1}{q}}(\sigma_1)(=K(\sigma))$$

is the best possible in (12) (resp. (14)).

$(iv) \Rightarrow (i)$ By Lemma 4, we have $\sigma_1 = \sigma$ and then both $K^{\frac{1}{p}}(\sigma)K^{\frac{1}{q}}(\sigma_1)$ and $K\left(\frac{\sigma}{p}+\frac{\sigma_1}{q}\right)$ equal to $K(\sigma)$, which are independent of p,q.

Hence, the statements $(i), (ii), (iii)$ and (iv) are equivalent.

The Theorem is proved. $\square$

If $k_\lambda(x,y)(\geqslant 0)$ is a homogeneous function of degree $-\lambda$ in $\mathbf{R}_+$, satisfying

$$K_\lambda(ux,uy) = u^{-\lambda}K_\lambda(x,y) \quad (u,x,y \in \mathbf{R}_+).$$

then setting $H(u) = K_\lambda(1,u)$, replacing x by $\frac{1}{x}$, and $x^{\lambda-2}f\left(\frac{1}{x}\right)$ by $f(x)$ in Lemma 2, Theorem 1 and Theorem 2, for $\sigma_1 = \lambda - \mu$,

$$K_\lambda(\gamma) := \int_0^\infty K_\lambda(1,u)u^{\sigma-1}du \in \mathbf{R}_+ \, (\gamma = \sigma, \lambda-\mu),$$

we have

COROLLARY 1. *For $f(x), g(y) \geqslant 0$,*

$$0 < \int_0^\infty x^{p\left[1-\left(\frac{\lambda-\sigma}{p}+\frac{\mu}{q}\right)\right]-1} f^p(x)\,dx < \infty, \ \text{and}$$

$$0 < \int_0^\infty \frac{(v(y))^{q\left[1-\left(\frac{\sigma}{p}+\frac{\lambda-\sigma}{q}\right)\right]-1}}{(v'(y))^{q-1}} g^q(y)dy < \infty,$$

we have the following equivalent Hilbert-type integral inequalities with the homogeneous kernel:

$$\int_0^\infty \int_0^\infty K_\lambda(x,v(y))f(x)g(y)\,dxdy$$
$$< K_\lambda^{\frac{1}{p}}(\sigma)K_\lambda^{\frac{1}{q}}(\lambda-\mu)\left\{\int_0^\infty x^{p\left[1-\left(\frac{\lambda-\sigma}{p}+\frac{\mu}{q}\right)\right]-1} f^p(x)dx\right\}^{\frac{1}{p}}$$
$$\times\left\{\int_0^\infty \frac{(v(y))^{q\left[1-\left(\frac{\sigma}{p}+\frac{\lambda-\mu}{q}\right)\right]-1}}{(v'(y))^{q-1}} g^q(y)dy\right\}^{\frac{1}{q}}, \tag{18}$$

$$\left[\int_0^\infty (v(y))^{p\left(\frac{\sigma}{p}+\frac{\lambda-\mu}{q}\right)-1} v'(y)\left(\int_0^\infty K_\lambda(x,v(y))f(x)dx\right)^p dy\right]^{\frac{1}{p}}$$
$$< K_\lambda^{\frac{1}{p}}(\sigma)K_\lambda^{\frac{1}{q}}(\lambda-\mu)\left\{\int_0^\infty x^{p\left[1-\left(\frac{\lambda-\sigma}{p}+\frac{\mu}{q}\right)\right]-1} f^p(x)dx\right\}^{\frac{1}{p}}. \tag{19}$$

For $\mu+\sigma=\lambda$, we have the following equivalent inequalities with the best possible constant factor.

$$\int_0^\infty\int_0^\infty K_\lambda\left(x,v\left(y\right)\right)f(x)g\left(y\right)dxdy$$
$$<K_\lambda\left(\sigma\right)\left[\int_0^\infty x^{p(1-\mu)-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty\frac{\left(v\left(y\right)\right)^{q(1-\sigma)-1}}{\left(v'\left(y\right)\right)^{q-1}}g^q(y)dy\right]^{\frac{1}{q}}, \qquad (20)$$

$$\left[\int_0^\infty\left(v\left(y\right)\right)^{p\sigma-1}v'\left(y\right)\left(\int_0^\infty K_\lambda\left(x,v\left(y\right)\right)f(x)dx\right)^p dy\right]^{\frac{1}{p}}$$
$$<K_\lambda\left(\sigma\right)\left[\int_0^\infty x^{p(1-\mu)-1}f^p(x)dx\right]^{\frac{1}{p}}. \qquad (21)$$

COROLLARY 2. *For $p>1$, the following statements $(i),(ii),(iii)$ and (iv) are equivalent:*

(i) *Both $K^{\frac{1}{p}}\left(\sigma\right)K^{\frac{1}{q}}\left(\lambda-\mu\right)$ and $K_\lambda\left(\frac{\sigma}{p}+\frac{\lambda-\mu}{q}\right)$ are independent of p,q;*

(ii) $K^{\frac{1}{p}}\left(\sigma\right)K^{\frac{1}{q}}\left(\lambda-\mu\right)\leqslant K_\lambda\left(\frac{\sigma}{p}+\frac{\lambda-\mu}{q}\right)$;

(iii) $\mu+\sigma=\lambda$;

(iv) *The constant factor $K^{\frac{1}{p}}\left(\sigma\right)K^{\frac{1}{q}}\left(\lambda-\mu\right)$ in (18) (resp. (19)) is the best possible.*

4. Operator expressions

(a) We set the following functions

$$\varphi\left(x\right):=x^{p\left[1-\left(\frac{\sigma}{p}+\frac{\sigma_1}{q}\right)\right]-1},\qquad \psi\left(y\right):=\frac{\left(v\left(y\right)\right)^{q\left[1-\left(\frac{\sigma}{p}+\frac{\sigma_1}{q}\right)\right]-1}}{\left(v'\left(y\right)\right)^{q-1}},$$

wherefrom,

$$\psi^{1-p}\left(y\right):=\left(v\left(y\right)\right)^{p\left(\frac{\sigma}{p}+\frac{\sigma_1}{q}\right)-1}v'\left(y\right)\quad \left(x,y\in\mathbf{R}_+\right).$$

Define some real normed linear spaces as follows:

$$L_{p,\varphi}\left(\mathbf{R}_+\right):=\left\{f=f\left(x\right):\|f\|_{p,\varphi}:=\left(\int_0^\infty\varphi(x)\left|f(x)\right|^p dx\right)^{\frac{1}{p}}<\infty\right\},$$
$$L_{q,\psi}\left(\mathbf{R}_+\right):=\left\{g=g\left(y\right):\|g\|_{q,\psi}:=\left(\int_0^\infty\psi(y)\left|g(y)\right|^q dy\right)^{\frac{1}{q}}<\infty\right\},$$
$$L_{p,\psi^{1-p}}\left(\mathbf{R}_+\right):=\left\{h=h\left(y\right):\|h\|_{p,\psi^{1-p}}:=\left(\int_0^\infty\psi^{1-p}\left(y\right)\left|h(y)\right|^p dy\right)^{\frac{1}{p}}<\infty\right\}.$$

For $f \in L_{p,\varphi}(\mathbf{R}_+)$, setting

$$h(y) := \int_0^\infty H(xv(y)) f(x)\,dx \quad (y \in \mathbf{R}_+).$$

by (14), we have

$$\|h\|_{p,\psi^{1-p}} < K^{\frac{1}{p}}(\sigma) K^{\frac{1}{q}}(\sigma_1) \|f\|_{p,\varphi} < \infty. \tag{22}$$

namely, $h \in L_{p,\psi^{1-p}}(\mathbf{R}_+)$.

DEFINITION 1. Define a Hilbert-type integral operator $T : L_{p,\varphi}(\mathbf{R}_+) \to L_{p,\psi^{1-p}}(\mathbf{R}_+)$ with the nonhomogeneous kernel as follows: For any $f \in L_{p,\varphi}(\mathbf{R}_+)$, there exists a unique representation $h = Tf \in L_{p,\psi^{1-p}}(\mathbf{R}_+)$, satisfying for any $y \in \mathbf{R}_+$, $Tf(y) = h(y)$.

Definition the formal inner product of Tf and $g \in L_{q,\psi}(\mathbf{R}_+)$ and the normed of T as follows:

$$(Tf,g) := \int_0^\infty \left(\int_0^\infty H(xv(y)) f(x)\,dx \right) g(y)\,dy = I,$$

$$\|T\| = \sup_{f(\neq 0) \in L_{p,\varphi}(\mathbf{R}_+)} \frac{\|Tf\|_{p,\psi^{1-p}}}{\|f\|_{p,\varphi}}.$$

In view of (22), it follows that

$$\|Tf\|_{p,\psi^{1-p}} = \|h\|_{p,\psi^{1-p}} \leqslant K^{\frac{1}{p}}(\sigma) K^{\frac{1}{q}}(\sigma_1) \|f\|_{p,\varphi}.$$

and then the operator T is bounded satisfying $\|T\| \leqslant K^{\frac{1}{p}}(\sigma) K^{\frac{1}{q}}(\sigma_1)$.

By Theorem 1 and Theorem 2, we have

THEOREM 3. *If $f(\geqslant 0) \in L_{p,\varphi}(\mathbf{R}_+)$, $g(\geqslant 0) \in L_{p,\varphi}(\mathbf{R}_+)$, $||f|| > 0$, $||g|| > 0$, then we have the following equivalent inequalities:*

$$(Tf,g) < K^{\frac{1}{p}}(\sigma) K^{\frac{1}{q}}(\sigma_1) \|f\|_{p,\varphi}, \tag{23}$$

$$\|Tf\|_{p,\psi^{1-p}} < K^{\frac{1}{p}}(\sigma) K^{\frac{1}{q}}(\sigma_1) \|f\|_{p,\varphi}. \tag{24}$$

Moreover, $\sigma_1 = \sigma$ if and only if the constant factor $K^{\frac{1}{p}}(\sigma) K^{\frac{1}{q}}(\sigma_1)$ in (23) (resp. (24)) is the best possible, namely, $\|T\| = K(\sigma)$.

(b) We set $\Phi(x) := x^{p\left[1-\left(\frac{\lambda-\sigma}{p}+\frac{\mu}{q}\right)\right]-1}$, $\Psi(y) := \frac{(v(y))^{q\left[1-\left(\frac{\sigma}{p}+\frac{\lambda-\mu}{q}\right)\right]-1}}{(v'(y))^{q-1}}$, wherefrom,

$$\Psi^{1-p}(y) := (v(y))^{p\left(\frac{\sigma}{p}+\frac{\lambda-\mu}{q}\right)-1} v'(y) \quad (x,y \in \mathbf{R}_+).$$

Define some real normed linear spaces as follows:

$$L_{p,\Phi}(\mathbf{R}_+) := \left\{ f = f(x) : \|f\|_{p,\Phi} := \left(\int_0^\infty \Phi(x) |f(x)|^p dx \right)^{\frac{1}{p}} < \infty \right\},$$

$$L_{q,\Psi}(\mathbf{R}_+) = \left\{ g = g(y) : \|g\|_{q,\Psi} := \left(\int_0^\infty \Psi(y) |g(y)|^q dy \right)^{\frac{1}{q}} < \infty \right\},$$

$$L_{p,\Psi^{1-p}}(\mathbf{R}_+) := \left\{ h = h(y) : \|h\|_{p,\Psi^{1-p}} := \left(\int_0^\infty \Psi^{1-p}(y) |h(y)|^p dy \right)^{\frac{1}{p}} < \infty \right\}.$$

For $f \in L_{p,\Phi}(\mathbf{R}_+)$, setting

$$H(y) := \int_0^\infty K_\lambda(x, v(y)) f(x) dx \quad (y \in \mathbf{R}_+).$$

We may rewrite (20) as follows:

$$\|H\|_{p,\Psi^{1-p}} < K^{\frac{1}{p}}(\sigma) K^{\frac{1}{q}}(\lambda - \mu) \|f\|_{p,\Phi} < \infty. \tag{25}$$

namely, $H \in L_{p,\Psi^{1-p}}(\mathbf{R}_+)$.

DEFINITION 2. Define a Hilbert-type integral operator $T_1 : L_{p,\Phi}(\mathbf{R}_+) \to L_{p,\Psi^{1-p}}(\mathbf{R}_+)$ with the homogeneous kernel as follows: For any $f \in L_{p,\Phi}(\mathbf{R}_+)$, there exists a unique representation $H = T_1 f \in L_{p,\Psi^{1-p}}(\mathbf{R}_+)$, satisfying for any $y \in \mathbf{R}_+, T_1 f(y) = H(y)$.

Define the formal inner product of $T_1 f$ and $g \in L_{q,\Psi}(\mathbf{R}_+)$ and the normed of T_1 as follows:

$$(T_1 f, g) := \int_0^\infty \left(\int_0^\infty K_\lambda(x, v(y)) f(x) dx \right) g(y) dy,$$

$$\|T_1\| = \sup_{f(\neq 0) \in L_{p,\Phi}(\mathbf{R}_+)} \frac{\|T_1 f\|_{p,\Psi^{1-p}}}{\|f\|_{P,\Phi}}.$$

In view of (25), it follows that

$$\|T_1 f\|_{p,\Psi^{1-p}} = \|H\|_{p,\Psi^{1-p}} \leqslant K_\lambda^{\frac{1}{p}}(\sigma) K_\lambda^{\frac{1}{q}}(\lambda - \mu) \|f\|_{p,\Phi}.$$

and then the operator T_1 is bounded satisfying $\|T_1\| \leqslant K_\lambda^{\frac{1}{p}}(\sigma) K_\lambda^{\frac{1}{q}}(\lambda - \mu)$.

By Corollary 1 and Corollary 2, we have

COROLLARY 3. *For $p > 1$, if $f(\geqslant 0) \in L_{p,\Phi}(\mathbf{R}_+)$, $g(\geqslant 0) \in L_{p,\Psi}(\mathbf{R}_+)$, $||f|| > 0$, $||g|| > 0$, then we have the following equivalent inequalities:*

$$(T_1 f, g) < K_\lambda^{\frac{1}{p}}(\sigma) K_\lambda^{\frac{1}{q}}(\lambda - \mu) \|f\|_{p,\Phi} \|g\|_{q,\Psi}, \tag{26}$$

$$\|T_1 f\|_{p,\Psi^{1-p}} < K_\lambda^{\frac{1}{p}}(\sigma) K_\lambda^{\frac{1}{q}}(\lambda - \mu) \|f\|_{p,\Phi}. \tag{27}$$

Moreover, $\mu+\sigma=\lambda$ *if and only if the constant factor* $K_{\lambda}^{\frac{1}{p}}(\sigma)K_{\lambda}^{\frac{1}{q}}(\lambda-\mu)$ *in* (26) *(resp.* (27)*) is the best possible, namely,* $\|T_1\|=K_{\lambda}(\sigma)$.

EXAMPLE 1. (i) We set $H(u)=K_{\lambda}(1,u)=\frac{1}{(1+u)^{\lambda}}$ $(u>0;\ \lambda>0)$. Then

$$H(xv(y)) = \frac{1}{(1+xv(y))^{\lambda}},$$
$$K_{\lambda}(x,v(y)) = \frac{1}{(x+v(y))^{\lambda}} \quad (x,y>0).$$

For $\gamma=\sigma,\ \sigma_1,\mu\in(0,\lambda)$,

$$K(\gamma)=K_{\lambda}(\gamma)=\int_0^{\infty}\frac{u^{\gamma-1}}{(1+u)^{\lambda}}du=B(\gamma,\lambda-\gamma)\in\mathbf{R}_+.$$

In view of Theorem 3, $\sigma=\sigma_1$ if and only if $||T||=B(\sigma,\lambda-\sigma)$. In view of Corollary 3, $\mu+\sigma=\lambda$ if and only if $||T_1||=B(\mu,\sigma)$.

(ii) We set $H(u)=K_{\lambda}(1,u)=\frac{\ln u}{u^{\lambda}-1}$ $(u>0;\lambda>0)$. Then

$$H(xv(y)) = \frac{\ln xv(y)}{(xv(y))^{\lambda}-1},$$
$$K_{\lambda}(x,v(y)) = \frac{\ln\frac{x}{v(y)}}{x^{\lambda}-(v(y))^{\lambda}} \quad (x,y>0).$$

For $\gamma=\sigma,\ \sigma_1,\mu\in(0,\lambda)$,

$$\begin{aligned}K(\gamma) &= K_{\lambda}(\gamma)=\int_0^{\infty}\frac{u^{\gamma-1}\ln u}{u^{\lambda}-1}du\\ &= \frac{1}{\lambda^2}\int_0^{\infty}\frac{v^{(\gamma/\lambda)-1}\ln v}{v-1}dv=\left[\frac{\pi}{\lambda\sin(\pi\gamma/\lambda)}\right]^2\in\mathbf{R}_+.\end{aligned}$$

In view of Theorem 3, $\sigma=\sigma_1$ if and only if $||T||=\left[\frac{\pi}{\lambda\sin(\pi\sigma/\lambda)}\right]^2$. In view of Corollary 3, $\mu+\sigma=\lambda$ if and only if $||T_1||=\left[\frac{\pi}{\lambda\sin(\pi\mu/\lambda)}\right]^2$.

(iii) We set $H(u)=K_{\lambda}(1,u)=\frac{1}{(\max\{1,u\})^{\lambda}}$ $(u>0;\ \lambda>0)$. Then

$$H(xv(y)) = \frac{1}{(\max\{1,xv(y)\})^{\lambda}},$$
$$K_{\lambda}(x,v(y)) = \frac{1}{(\max\{x,v(y)\})^{\lambda}} \quad (x,y>0).$$

For $\gamma=\sigma,\ \sigma_1,\mu\in(0,\lambda)$,

$$\begin{aligned}K(\gamma) &= K_{\lambda}(\gamma)=\int_0^{\infty}\frac{u^{\gamma-1}}{(\max\{1,u\})^{\lambda}}du\\ &= \int_0^1 u^{\gamma-1}du+\int_1^{\infty}\frac{u^{\gamma-1}}{u^{\lambda}}du=\frac{\lambda}{\gamma(\lambda-\gamma)}\in\mathbf{R}_+.\end{aligned}$$

In view of Theorem 3, $\sigma = \sigma_1$ if and only if $||T|| = \frac{\lambda}{\sigma(\lambda-\sigma)}$. In view of Corollary 3, $\mu + \sigma = \lambda$ if and only if $||T_1|| = \frac{\lambda}{\mu\sigma}$.

5. Conclusions

In this paper, making use of the technique of real analysis and the weight functions, we build some useful lemmas and study a new Hilbert-type integral inequality with the nonhomogeneous kernel as $H(xv(y))$ as well as the equivalent forms in Theorem 1. A few equivalent statements related to the best possible constant factor and parameters are provided in Theorem 2. As applications, we also consider the operator expressions in Theorem 3 and several particular kernels in Example 1. Some corollaries are also obtained. The lemmas and theorems provide an extensive account of this type of inequalities.

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REFERENCES

[1] V. ADIYASUREN, T. BATBOLD, M. KRNIĆ, *Multiple Hilbert-type inequalities involving some differential operators*, Banach Journal of Mathematical Analysis, **10**, 320–337 (2016).

[2] V. ADIYASUREN, T. BATBOLD, M. KRNIĆ, *Hilbert-type inequalities involving differential operators, the best constants and applications*, Math. Inequal. Appl., **18**, 111–124 (2015).

[3] V. ADIYASUREN, T. BATBOLD, L. E. AZAR, *A new discrete Hilbert-type inequality involving partial sums*, Journal of Inequalities and Applications, 2019: 127, 2019.

[4] V. ADIYASUREN, T. BATBOLD, Y. SAWANO, *A multidimensional inte gral inequality related to Hilbert-type inequality* [J], Mediterranean Journal of Mathematics, 2016, 13: 3837–3848.

[5] L. E. AZAR, *The connection between Hilbert and Hardy inequalities*, Journal of Inequalities and Applications, 2013: 452, 2013.

[6] T. BATBOLD, Y. SAWANO, *Sharp bounds for m-linear Hilbert-type operators on the weighted Morrey spaces*, Math. Inequal. Appl. **20**, 263–283 (2017).

[7] T. BATBOLD, L. E. AZAR, *A new form of Hilbert integral inequality*, Math. Inequal. Appl., **12**, 379–390 (2018).

[8] Q. CHEN, B. HE, Y. HONG, L. C. ZHEN, *Equivalent paprameter conditions for the validity of half-discrete Hilbert-type multiple integral inequality with generalized homogeneous kernel*, Journal of Function Space, vol. 2020, Article ID 7414861.

[9] G. H. HARDY, J. E. LITTLEWOOD, G. PÓLYA, *Inequalities*, Cambridge University Press, Cambridge, USA (1934).

[10] Y. HONG, Q. L. HUANG, B. C. YANG, J. Q. LIAO, *The necessary and sufficient conditions for the existence of a kind of Hilbert-type multiple integral inequality with the non-homogeneous kernel and its applications*, Journal of Inequalities and Applications (2017), 2017: 316.

[11] Y. HONG, Q. L. HUANG, B. C. YANG, J. Q. LIAO, *The necessary and sufficient conditions for the existence of a kind of Hilbert-type multiple integral inequality with the non-homogeneous kernel and its applications*, Journal of Inequalities and Applications (2017), 2017: 316.

[12] Y. HONG, B. HE, B. C. YANG, *Necessary and sufficient conditions for the validity of Hilbert type integral inequalities with a class of quasi-homogeneous kernels and its application in operator theory*, Journal of Mathematics Inequalities, **12** (3), 777–788 (2018).

[13] Y. HONG, Y. WEN, *A necessary and sufficient condition of that Hilbert type series inequality with homogeneous kernel has the best constant factor*, Annals Mathematica, **37A** (3), 329–336 (2016).

[14] J. C. KUANG, *Real and functional analysis (Continuation)* (sec. vol.), Higher Education Press, Beijing (2015).

[15] J. C. KUANG, *Applied inequalities*, Shangdong Science and Technology Press, Jinan, China (2004).

[16] M. KRNIĆ, P. VUKOVIC, *On a more accurate class of discrete Hilbert-type inequalities* [J], Applied Mathematics and Computation, **234** (2014) 543–547.

[17] J. Q. LIAO, S. H. WU, B. C. YANG, *On a new half-discrete Hilbert-type inequality involving the variable upper limit integral and the partial sum*, Mathematics, 2020, 8, 229, doi:10.3390/math8020229.

[18] M. T. RASSIAS, B. C. YANG, A. RAIGORODSKII, *A new Hardy–Mulholland-type inequality with a mixed kernel*, Advances in Operator Theory (2021) 6: 27, https://doi.org/10.1007/s43036-020-00123-0.

[19] M. T. RASSIAS, B. C. YANG, A. RAIGORODSKII, *Equivalent conditions of a multiple Hilbert-type integral inequality with the non homogeneous kernel*, Rev. Real Acad. Cienc. Exact as Fis. Nat. Ser. A-Mat. (2022) 116: 107.

[20] Z. X. WANG, D. R. GUO, *Introduction to special functions*, Science Press, Beijing (1979).

[21] J. S. XU, *Hardy-Hilbert's inequalities with two parameters*, Advances in Mathematics, **36** (2), 63–76 (2007).

[22] Z. T. XIE, Z. ZENG, Y. F. SUN, *A new Hilbert-type inequality with the homogeneous kernel of degree* -2, Advances and Applications in Mathematical Sciences, **12** (7), 391–401 (2013).

[23] D. M. XIN, *A Hilbert-type integral inequality with the homogeneous kernel of zero degree*, Mathematical Theory and Applications, **30** (2), 70–74 (2010).

[24] D. M. XIN, B. C. YANG, A. Z. WANG, *Equivalent property of a Hilbert-type integral inequality related to the beta function in the whole plane*, Journal of Function Spaces, vol. 2018, Article ID2691816, 8 pages.

[25] B. C. YANG, *On Hilbert's integral inequality*, J. Math. Anal. & Appl., **220**, 778–785 (1998).

[26] B. C. YANG, *The norm of operator and Hilbert-type inequalities*, Science Press, Beijing, China (2009).

[27] B. C. YANG, *Hilbert-type integral inequalities*, Bentham Science Publishers Ltd., The United Arab Emirates (2009).

[28] B. C. YANG, *On Hilbert-type Integral Inequalities and their opeator expressions*, Journal of Guanggaong University of Education, 33 (5), 1–17 (2013).

[29] B. C. YANG, M. T. RASSIAS, A. RAIGORODSKII, *On an extension of a Hardy-Hilbert-type inequality with multi-parameters*, Mathematics, 2021, 9, 2432, https://doi.org/10.3390/math9192432.

[30] B. C. YANG, D. ANDRICA, O. BAGDASAR, M. T. RASSIAS, *An equivalent property of a Hilbert-type integral inequality and its applications*, Appl. Anal. Discrete Math. **16** (2022), 548–563.

[31] B. C. YANG, D. ANDRICA, O. BAGDASAR, M. T. RASSIAS, *On a Hilbert-type integral inequality in the whole plane with the equivalent forms*, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. (2023) 117: 57.

[32] B. C. YANG, *A new Hilbert-type inequality*, Bull. Belg. Math. Soc. **13** (2006) 479–487.
[33] Z. ZENG, K. RAJA RAMA GANDHI, Z. T. XIE, *A new Hilbert-type inequality with the homogeneous kernel of degree* -2 *and with the integral*, Bulletin of Mathematical Sciences and Applications, **3** (1), 11–20 (2014).

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