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ON THE BINARY LOCATING–DOMINATION NUMBER OF REGULAR AND STRONGLY–REGULAR GRAPHS

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Abstract. Graphs possessing minimal dominating sets have potential applicability in computer science & engineering. In a graph G, a dominating set L meeting $N(x)\cap L \neq N(y)\cap L$ for any $x,y \in V\backslash L$ is known as a binary locating-dominating set. Minimizing the cardinality of such a set in G would be called the binary location-domination number $\gamma_{l-d}(G)$ of G. This paper considers regular and strongly-regular graphs to study their binary location-domination and global binary location-domination numbers. Being an NP-complete problem, it is natural to study this parameter for special families of graphs having combinatorial and geometrical importance. Exact values of $\gamma_{l-d}(G)$ have been evaluated for complete graphs, cycles, complete bipartite graphs and the generalized Petersen graphs $P(n,2)$, $n \geqslant 4$ and $P(n,4)$, $\big(5 \leqslant n \equiv 0 \pmod 3\big)$. Certain tight upper and lower bounds are shown for the path graphs, generalized Petersen graph $P(n,4)$, $\big(5 \leqslant n \equiv 1,2 \pmod 3\big)$, prism graphs and two infinite families of strongly regular graphs known as the triangular graphs and the square grid graphs. Moreover, an integer linear programming (ILP) model has been employed via CPLEX solver to show tightness in the upper bounds. By studying the binary locating-dominating sets in the complements of some of the above families, we also study their global location-domination number. Some open problems which naturally arise from the study have been proposed at the end.

1. Introduction

We refer the interested readers to the book by Henning et al. [8] which provides, till 1980, a brief overview of the results regarding domination in graphs. Haynes et al. [9] considered trees for their total domination & the binary location-domination numbers. Minimum ℓ-locating-dominating as well as ℓ-identifying sets/codes in chains & cyclic graphs have been constructed by Charon et al. [5]. Sharp upper & lower bounds on the minimality of ℓ-locating-dominating sets for general graph were also found.

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Further reading on these contemporary domination related parameters can be studied in [14, 15].

Salter [22] & Seo et al. [18, 19] introduced the concepts of the open-neighborhood location-domination number and the fault-tolerant location-domination number respectively and derived results on these parameters for trees. Location-domination number for certain classes of convex polytopes was investigated recently by Raza et al. [17] & Simić et al. [20]. They found its exact values for some families of convex polytopes and tight upper bounds were found for other families. The reader is suggested to read [12, 15, 21, 23] for more details on these domination related graph-theoretic parameters. Generalized Petersen graphs are of prime importance in graph theory and they provide counter examples to many graph-theoretic conjectures. They were considered by a number of researchers in [6, 25, 26, 29, 30, 31] for their parameters related to domination in graphs.

Decision problems for the complexities of binary locating-dominating sets as well as identifying codes have been shown [3, 4] to be NP-hard. On the other hand, Charon et al. [3] showed that the existence of ℓ-locating-dominating as well as ℓ-identifying sets/codes for an arbitrary graph is an NP-complete problem. A comprehensive list of articles on the binary locating-dominating/identifying sets/codes have been maintained by Lobstein [16]. A related problem of finding minimum locating sets in metacyclic & Cayley graphs have been investigated by Abas & Vetrik [1].

This paper considers regular graphs to study their minimal binary locating-dominating and global binary locating-dominating sets. In Section 2, we define some necessary terminologies and deliver preliminary results which are to be used in the subsequent sections. A modified ILP model for evaluating the binary location-domination number of an arbitrary graph is delivered in Section 3. The binary location-domination number and its version for classical families of complete multipartite graphs, paths, cycles has been studied in Section 4. Section 5 considers $P(n,k)$, where $k = 1,2,4$ i.e. the generalized Petersen graphs, to study their binary location-domination & global binary location-domination numbers. The same problems for two infinite families of strongly regular graphs have been studied in Section 6. Two constructions are presented to find upper bounds on binary location-domination number and their tightness of the triangular and square grid graphs. We finish the paper by raising certain open problems which naturally arise from our research.

2. Preliminaries

Let G be a graph, with vertex set $V(G)$ and edge set $E(G)$ having no isolated vertex. Denoted by $x \sim y$ (resp. $x \nsim y$) for $x,y \in V(G)$, the vertices x and y are said be adjacent (resp. non-adjacent). The *open neighborhood* of a vertex $x \in V$ is $N(x) = \{y \in V | (x,y) \in E\}$ and the *close neighborhood* is $N[x] = \{y \in V | (x,y) \in E\} \cup \{x\}$. The *degree* of x is $d(x) = |N(x)|$. If the graph G is clear from the context, we simply write V and E rather than $V(G)$ and $E(G)$. A set L of vertices in a graph G is called a *dominating set* of a graph G if every vertex in $V \backslash L$ is adjacent to some vertex in L. The *domination number*, $\gamma(G)$, of G is the minimum cardinality of dominating set in G. A dominating set of cardinality $\gamma(G)$ is called a dominating code. A dominating set is called *global* if it is a dominating set of both G and its complement graph, $\overline{G}$. The minimum cardinality of a global dominating set of G is the *global domination number* of G, denoted with $\gamma_g(G)$ [13]. If T is a subset of V and $x \in V$, we say that x dominates T if $T \subseteq N(x)$.

An alternative way of investigating a dominating set is by allocating 1 (resp. 0) to $x \in L$ (resp. $x \in V \setminus L$). In this case, L is said to be a dominating set of G if for every $x \in V$ the sum of weights for closed neighborhoods is at least 1, i.e. $|N[x] \cap L| \geqslant 1$. A *binary locating-dominating (BLD) set* in a graph G is a dominating set L of G such that for every two different vertices $x, y \in V \backslash L$, we have $N(x) \cap L \neq N(y) \cap L$. The minimum cardinality of an BLD set of G is called the *binary location-domination (BLD) number* denoted by $\gamma_{l-d}(G)$. An LD set of cardinality $\gamma_{l-d}(G)$ is called a binary location-domination code. Note that we have $\gamma(G) \leqslant \gamma_{l-d}(G)$. The concept of the binary location-domination number is also known as simply the LD number in graphs. Location-domination number has been extensively studied among the currently existing list of domination variants.

For a k-regular graph G, Slater [23] derived the following lower bound for $\gamma_{l-d}(G)$.

THEOREM 1. [23] *For an n-vertex, k-regular graph, we have*

$$\gamma_{l-d}(G) \geqslant \left\lceil \frac{2n}{k+3} \right\rceil.$$

Applying Salter's lower bound (Theorem 1) on the complement of a regular graph, we obtain the following result.

PROPOSITION 1. *Let G be an n-vertex, k-regular graph. If $\overline{G}$ is the complement of G, then*

$$\gamma_{l-d}(\overline{G}) \geqslant \left\lceil \frac{2n}{n-k+2} \right\rceil.$$

Proof. Note that $\overline{G}$ is an n-vertex regular graph with degree $\bar{k} = n-k-1$. Thus, applying Theorem 1 to $\overline{G}$ gets the required bound. $\square$

If a set $L \subset V(G)$ is simultaneously an LD set of both G and its complement $\overline{G}$, then L is said to be a *global LD set*. The minimum order of such a set in G is known as the *global binary location-domination number* $\lambda_g(G)$ of G. An LD code C in G is *global* if it is a global LD set, i.e. C is an LD set code of G and an LD set of $\overline{G}$.

Hernando et al. [13] showed the following result.

PROPOSITION 2. [13] *An LD set L is a global LD set if and only if there exists no $x \in V \setminus L$ such that element x dominates the set L.*

By using the above proposition, one may show the following result.

PROPOSITION 3. *If L is an LD set of a graph G, then L is a global LD set if and only if L is a global dominating set.*

3. An integer linear programming (ILP) formulation

This section delivers an ILP formulation for evaluating the LD number for an arbitrary graph, due to Simić et al. [20]. This ILP model is used in the subsequent sections by means of CPLEX solver.

An ILP model for the problem of minimality of identifying codes was put forward by Bange et al. [2]. For $G=(V,E)$, let v_ℓ (resp. $P\subset V$) be the decision variables (resp. identifying set) such that the following holds:

$$v_\ell=\begin{cases}1, & \ell\in P;\\ 0, & \ell\notin P.\end{cases}$$

The proposed ILP by Bange et al. [2] proceeds as follows:

$$\min\sum_{\ell\in V} v_\ell \tag{1}$$

following the subjective constraints

$$\sum_{p\in N[\ell]} v_p\geqslant 1,\qquad \ell\in V \tag{2}$$

$$\sum_{p\in(N[\ell]\nabla N[k])} v_p\geqslant 1,\qquad \ell,k\in V,\ \ell\neq k \tag{3}$$

$$v_\ell\in\{0,1\},\qquad \ell\in V \tag{4}$$

The symbol ∇ denotes the symmetric difference of two sets. The minimality of an identifying set in the aforementioned formulation is ensured by the objective function (1). Constraints (2 (resp. (3)) represent a dominating set P (resp. identifying feature). The binary nature of v_ℓ i.e. deciding variables is ensured by constraint (4).

In what follows, we modify the above ILP formulation for the problem of the LD number. This purpose is delivered by transforming constraint (3) to the following:

$$v_\ell+v_k+\sum_{p\in(N(\ell)\nabla N(k))} v_p\geqslant 1,\qquad \ell,k\in V,\ \ell\neq k \tag{5}$$

If the vertices $\ell\nsim k$ i.e. $N[\ell]\nabla N[k]=\{\ell,p\}\cup\big(N(\ell)\nabla N(k)\big)$, then constraints (3) & (5) are exactly similar. This implies that the changes between (3) & (5) appear only when $\ell\nsim k$ i.e. $\ell\in N(k)$. Thus, constraint (5) ensures that P must contain at least one of ℓ, k or some $p\in\big(N(\ell)\nabla N(k)\big)$. So, in case when $\ell\nsim k$, we have $N[\ell]\nabla N[k]=\{\ell,p\}\cup\big(N(\ell)\nabla N(k)\big)$ and that (3) & (5) are exactly similar.

Let $\lambda(x,y)$ be the set of common neighbors of both $x,y\in V$. A result by Sweigart et al. [24] that for vertices $x,y\in V$ at distance at least 3, we have $\lambda(x,y)=\phi$. Thus, we are not required to assess $N(x)\cap L\neq N(y)\cap L$ for equivalence. It also allows us to lessen the required constraints which an LD set requirement delivers. This also happens to be computationally significant for graphs on large number of vertices. Employment of this idea improves (5) as follows:

$$v_\ell+v_k+\sum_{p\in(N(\ell)\nabla N(k))} v_p\geqslant 1,\qquad \ell,k\in V,\ d(\ell,k)\leqslant 2,\ \ell\neq k \tag{6}$$

It is worth noticing that the ILP model with reduced constraints assists us in calculating the exact values for the binary location-domination problem for small graphs. For large dimension problems, metaheuristic approaches [7] could be employed in order to transform the problem and obtain optimal solutions.

4. Classical graph families

In this section, we focus on calculating the LD number of some basic structures such as the path graphs, the cycle graphs, and the complete multipartite graphs. We would like to add that these results might be known to researchers in this area as they provide the base for the binary location-domination number. However, they are not explicitly available in the literature, thereby, we are providing proofs for these results so that are readily available for the reader.

4.1. Complete multipartite graphs

A bipartite graph is called *complete bipartite*, if any vertex in one color class is adjacent to all the vertices in the other color class and vice versa. It is denoted as $K_{p,q}$, where p and q are the cardinalities of the two partite sets. The next result finds the exact value of $\gamma_{l-d}(K_{p,q})$.

PROPOSITION 4. *Let $K_{p,q}$ be the complete bipartite graph, where $p,q \geqslant 3$. Then $\gamma_{l-d}(K_{p,q}) = p+q-2$.*

Proof. Let (X,Y) be the bipartition of the vertex set of $K_{p,q}$ such that $|X| = p$ and $|Y| = q$. Note that the induced subgraphs on X and Y are the empty graphs $\overline{K_p}$ and $\overline{K_q}$ respectively. Let $x \in X$ and $y \in Y$ be the arbitrary vertices in X and Y respectively. Note that the set $L = X \setminus \{x\} \bigcup Y \setminus \{y\}$ is an LD set of $K_{p,q}$ as for both vertices x and y, the sets $L \cap N[x] = Y \setminus \{y\}$ and $L \cap N[y] = X \setminus \{x\}$ are mutually disjoint. Since $|L| = p+q-2$, we obtain that $\gamma_{l-d}(K_{p,q}) \leqslant p+q-2$.

Next, we prove the lower bound $\gamma_{l-d}(K_{p,q}) \geqslant p+q-2$. We will prove this by showing that $\gamma_{l-d}(K_{p,q})$ can not be $p+q-3$. On contrary, we suppose that $\gamma_{l-d}(K_{p,q}) = p+q-3$. This implies that there exists an LD set L of cardinality $p+q-3$. Let x, y and z be the three vertices in $V \setminus L$. We distinguish the following possible cases for the vertices x, y and z:

Case 1: All the three vertices x, y and z belong to one partite set.

Without loss of generality, we assume that x, y and z belong to X. Note that $L \cap N[x] = L \cap N[y] = L \cap N[z] = Y$ which arises contradiction to the fact any two such intersections must be disjoint. This suggests us the following case:

Case 2: One of the x, y or z belongs to the partite set different from other two vertices.

Without loss of generality, we assume that $x \in X$ and $y, z \in Y$. Then $L \cap N[y] = L \cap N[z] = X$, which again arises a contradiction that L preserves as an LD set.

All in all, we obtain that L must contain exactly $p+q-2$ vertices which completes the proof. □

Rather having two partite sets in complete bipartite graphs, if we have three partite sets, then we obtain the complete tripartite graphs. Next proposition computes the binary location-domination number of the complete tripartite graphs.

PROPOSITION 5. *Let $K_{p,q,r}$ be the complete tripartite graph with $1 \leqslant p \leqslant q \leqslant r$. Then $\gamma_{l-d}(K_{p,q,r}) = p+q+r-3$.*

Proof. Let (X,Y,Z) be the tripartition of $V(K_{p,q,r})$ such that $|X| = p$, $|Y| = q$ and $|Z| = r$. For arbitrary vertices $x \in X$, $y \in Y$ and $z \in Z$, let $L = X \setminus \{x\} \bigcup Y \setminus \{y\} \bigcup Z \setminus \{z\}$. Note that L is a binary locating-dominating set of $K_{p,q,r}$ since $L \bigcap N[x] = Y \setminus \{y\} \bigcup Z \setminus \{z\}$, $L \bigcap N[y] = X \setminus \{x\} \bigcup Z \setminus \{z\}$ and $L \bigcap N[z] = X \setminus \{x\} \bigcup Y \setminus \{y\}$ are non-empty and mutually disjoint. Thus, we obtain that $\gamma_{l-d}(K_{p,q,r}) \leqslant |L| = p+q+r-3$.

In order to complete the proof, we need to show that $\gamma_{l-d}(K_{p,q,r}) \geqslant p+q+r-3$. We show it by proving that $\gamma_{l-d}(K_{p,q,r}) > p+q+r-4$. On contrary, we assume that $\gamma_{l-d}(K_{p,q,r}) = p+q+r-4$. Thus there exists a binary locating-dominating set of order four and let L be that set. Let $V(K_{p,q,r}) \setminus L = \{w,x,y,z\}$. We distinguish the following possible cases for the vertices w, x, y and z as follows:

Case 1: All w, x, y and z belong to one partite set.

Without loss of generality, we may assume that $\{w,x,y,z\} \subset X$. Assuming it would generate $L \bigcap N[w] = L \bigcap N[x] = L \bigcap N[y] = L \bigcap N[z] = Y \bigcup Z$ which arises a contradiction to the fact that L is a binary locating-dominating set of $K_{p,q,r}$.

Case 2: All w, x, y and z belong to exactly two partite sets.

Subcase 2.1: One partite set is singleton.

Without loss of generality, we may assume that $\{w,x,y\} \subset X$ and $z \in Y$. Then same intersections $L \bigcap N[w] = L \bigcap N[x] = L \bigcap N[y] = Z \bigcup Y \setminus \{z\}$ arise a contradiction that L is a binary-locating dominating set.

Subcase 2.2: Both partite sets have cardinality 2.

Similar to the previous cases, without loss of generality, we may assume that $\{w,x\} \subset X$ and $\{y,z\} \subset Y$. In this case, we obtain that $L \bigcap N[w] = L \bigcap N[x] = Z \bigcup Y \setminus \{y,z\}$ and $L \bigcap N[y] = L \bigcap N[z] = Z \bigcup X \setminus \{w,x\}$, which is a contradiction again.

Case 3: All w, x, y and z belong to exactly three partite sets.

Without loss of generality, we may assume that $w \in X$, $x \in Y$ and $\{y,z\} \in Z$. Assuming it would generate $L \bigcap N[z] = X \setminus \{w\} \bigcup Y \setminus \{x\} = L \bigcap N[y]$. Thus, the two intersections are not mutually disjoint which arises a contradiction.

Combining all the possible case, we obtain that L is not a binary locating-dominating set of $K_{p,q,r}$. Since L was an arbitrary set of cardinality four, we obtain that $\gamma_{l-d}(K_{p,q,r}) > p+q+r-4 \geqslant p+q+r-3$. This completes the proof. □

Next, we apply induction on the number of partite sets to calculate the binary location-domination number of the complete r-partite graphs. This result has been proven recently by Hayat et al. [10].

THEOREM 2. *Let $K_{t_1,t_2,\ldots,t_r}$ be the complete r-partite graph with $1 \leqslant t_1 \leqslant t_2 \leqslant \ldots \leqslant t_r$ and $\sum_{i=1}^{r} t_i = n$. Then*

$$\gamma_{l-d}(K_{t_1,t_2,\ldots,t_r}) = \sum_{i=1}^{r} t_i - r. \tag{7}$$

Proof. We prove the result by applying induction on r which is the number of partite sets in $K_{t_1,t_2,\ldots,t_r}$. By Propositions 4 & 5, the result is true for $r=2$ and $r=3$ respectively. Now let the assertion of be valid for $r=k$ as an induction step. We show that the result is true for $r=k+1$ to complete the proof.

Let T_i $(1 \leqslant i \leqslant r)$ be the i^{th} partite set of $K_{t_1,t_2,\ldots,t_r}$ where $|T_i| := t_i$. Since (2) holds for $r=k$, we obtain that

$$\gamma_{l-d}(K_{t_1,t_2,\ldots,t_k}) = \sum_{i=1}^{k} t_i - k. \tag{8}$$

This implies that there exists a minimum binary locating-dominating set of cardinality $\sum_{i=1}^{k} t_i - k$ in $K_{t_1,t_2,\ldots,t_k}$. Let L be the minimum binary locating-dominating set of cardinality $\sum_{i=1}^{k} t_i - k$. in $K_{t_1,t_2,\ldots,t_k}$. Let $U = \{x_i \in T_i : 1 \leqslant i \leqslant k\}$ and $L = V(K_{t_1,t_2,\ldots,t_k}) \setminus U$, then L comprises $t_i - 1$ elements from every T_i where $1 \leqslant i \leqslant r$. Note that L is a binary locating-dominating set of $K_{t_1,t_2,\ldots,t_k}$ since $L \bigcap N[x_i] \neq L \bigcap N[x_j]$ for $1 \leqslant i < j \leqslant k$ and $L \bigcap N[x_i] = L \setminus T_i$ $1 \leqslant i \leqslant k$. Moreover, L is minimum as the inclusion of more than one vertices from any partite set T_i in L would arise a contradiction as the two vertices from a same partite set would have the same intersections. Moreover, the minimality of L follows from (8) as L is a binary locating-dominating set.

Now we add $(k+1)^{\text{th}}$ partite set, say T_{k+1}, to $K_{t_1,t_2,\ldots,t_k}$ to obtain $K_{t_1,t_2,\ldots,t_k,t_{k+1}}$. For any $x' \in T_{k+1}$, let $L' = L \bigcup \{x'\}$. Note that L' is a binary locating-dominating set of $K_{t_1,t_2,\ldots,t_{k+1}}$ since $L' \bigcap N[x'] \neq L' \bigcap N[x_i]$ for $1 \leqslant i \leqslant k$ and $L' \bigcap N[x'] = L' \setminus T_{k+1}$. Moreover, L' is minimum as L is minimum in $K_{t_1,t_2,\ldots,t_k}$. Since $|L| = \sum_{i=1}^{k+1} t_i - (k+1)$, we obtain that

$$\gamma_{l-d}(K_{t_1,t_2,\ldots,t_{k+1}}) = \sum_{i=1}^{k+1} t_i - (k+1).$$

By applying the induction hypothesis on r, the proof is finished. $\square$

As a corollary to Theorem 2, we have the following.

COROLLARY 1. *Let K_n be the complete graph on n vertices, where $n \geqslant 4$. Then $\gamma_{l-d}(K_n) = n-1$.*

4.2. Cycle graphs

This subsection studies the LD number and the global LD number of cycle graphs. Note that a connected graph is called cycle if it is regular of degree two.

The following result evaluates the exact value of $\gamma_{l-d}(C_n)$.

THEOREM 3. *Let C_n denote the n-vertex cycle graph having $n \geqslant 3$. Then*

$$\gamma_{l-d}(C_n) = \left\lceil \frac{2n}{5} \right\rceil.$$

Proof. Notice that C_n is an n-vertex 2-regular graph. By using Theorem 1, we find the lower bound, i.e.

$$\gamma_{l-d}(C_n) \geqslant \left\lceil \frac{2n}{5} \right\rceil.$$

To prove the upper bound, we introduce L such that

$$L = \begin{cases} \{y_{5k+3}, y_{5k}\} \mid k = 0, \ldots, \ell - 1 & n = 5\ell,\ n \equiv 0 \pmod 5; \\ \{y_{5\ell}\} \bigcup \{y_{5k+3}, y_{5k}\} \mid k = 0, \ldots, \ell - 1 & n = 5\ell + 1,\ n \equiv 1 \pmod 5; \\ \{y_{5\ell}\} \bigcup \{y_{5k}, y_{5k+3}\} \mid k = 0, \ldots, \ell - 1 & n = 5\ell + 2,\ n \equiv 2 \pmod 5; \\ \{y_{5\ell}, y_{5\ell+2}\} \bigcup \{y_{5k+3}, y_{5k}\} \mid k = 0, \ldots, \ell - 1 & n = 5\ell + 3,\ n \equiv 3 \pmod 5; \\ \{y_{5k+3}, y_{5k}\} \bigcup \{y_{5\ell+3}, y_{5\ell}\} \mid k = 0, \ldots, \ell - 1 & n = 5\ell + 4,\ n \equiv 4 \pmod 5. \end{cases}$$

Here we prove that L is an LD set.

n	$x \in V \setminus L$	$L \cap N[x]$	$x \in V \setminus L$	$L \cap N[x]$
5ℓ	y_{5k+4} y_{5k+1}	$\{y_{5(k+1)}, y_{5k+3}\}$ $\{y_{5k}\}$	y_{5k+2}	$\{y_{5k+3}\}$
$5\ell+1$	y_{5k+4} y_{5k+1}	$\{y_{5(k+1)}, y_{5k+3}\}$ $\{y_{5k}\}$	y_{5k+2}	$\{y_{5k+3}\}$
$5\ell+2$	y_{5k+4} y_{5k+1}	$\{y_{5k+3}, y_{5(k+1)}\}$ $\{y_{5k}\}$	$y_{5\ell+1}$ y_{5k+2}	$\{y_0, y_{5\ell}\}$ $\{y_{5k+3}\}$
$5\ell+3$	y_{5k+4} y_{5k+1}	$\{y_{5(k+1)}, y_{5k+3}\}$ $\{y_{5k}\}$	$y_{5\ell+1}$ y_{5k+2}	$\{y_{5\ell+2}, y_{5\ell}\}$ $\{y_{5k+3}\}$
$5\ell+4$	y_{5k+4} $y_{5\ell+2}$ y_{5k+1}	$\{y_{5(k+1), y_{5k+3}}\}$ $\{y_{5\ell+3}\}$ $\{y_{5k}\}$	$y_{5\ell+1}$ $y_{5(\ell-1)+4}$ y_{5k+2}	$\{y_{5\ell}\}$ $\{y_{5\ell-2}, y_0\}$ $\{y_{5k+3}\}$

Table 1: *Vertices in an LD set of C_n.*

In Table 1, the vertices $x \in V \setminus L$ and their corresponding $L \cap N[x]$ are shown. Since all intersections are simultaneously disjoint & nonempty, this implies L to be an LD set. Since $|L| = \left\lceil \frac{2n}{5} \right\rceil$, we obtain that $\gamma_{l-d}(C_n) \leqslant \left\lceil \frac{2n}{5} \right\rceil$. This completes the proof. □

Now we proceed to the global version of the LD number of cycles. First, we prove the following result on the LD number of the complements of the cycle graphs.

PROPOSITION 6. *Let C_n be the n-vertex cycle graph having $n \geqslant 6$. Then*

$$2 \leqslant \gamma_{l-d}(\overline{C_n}) \leqslant \left\lceil \frac{2n}{5} \right\rceil.$$

Moreover, the upper bound is tight.

Proof. The lower bound follows from Proposition 1.

Now we show that the set L from the proof of Theorem 3 is also an LD set for the complements of cycles $\overline{C_n}$. By Proposition 2, we only need to show that L is a dominating set of $\overline{C_n}$. By Table 1 from the proof of Theorem 3, there does not exist any vertex $v \in V \setminus L$, such that $L \cap N[v] = L$. This means that for any $v \in V \setminus L$, there exists at least one vertex $u \in L$, such $u \notin L \cap N[v]$. This implies that in the complement $\overline{C_n}$, for every $v \in V(\overline{C_n}) \setminus L$ we have $L \cap N[v] \neq \emptyset$. This shows that L is a dominating set of $\overline{C_n}$ and thus by Proposition 2 L satisfies the property of being an LD set of $\overline{C_n}$. This shows the upper bound.

The ILP formulation with (1), (2), (4) and (6) is used in CLPEX solver to show tightness. This, in turn, provides us with the following optimal solution for small cases: $\gamma_{l-d}(\overline{C_6}) = 3$, $\gamma_{l-d}(\overline{C_7}) = 3, \ldots,$ $\gamma_{l-d}(\overline{C_{15}}) = 6, \ldots, \gamma_{l-d}(\overline{C_{35}}) = 14$. This shows the tightness. □

By using the definition of the global LD code and then using Theorem 3 and Proposition 6, we obtain the following result.

THEOREM 4. *Let C_n denote n-vertex cycle graph having $n \geqslant 3$. Then*

$$\lambda_g(C_n) = \left\lceil \frac{2n}{5} \right\rceil.$$

Recently, Arockiaraj et al. [11] studied optimal wirelength of balanced complete multipartite graphs onto cartesian product of path, cycle and trees. These results could also be employed to study domination related parameters.

4.3. Path graphs

This subsection studies the problem of LD number and its global version for paths. A path graph is obtained by deleting an edge from a cycle graph.

Now we consider the binary location-domination number of path graphs. A path graph on n vertices is denoted by P_n, where $n \geqslant 2$. The following result provides an upper bound for P_n.

THEOREM 5. *Let P_n denote the n-vertex path graph having $n \geqslant 5$. Then*

$$\gamma_{l-d}(P_n) \leqslant \begin{cases} \left\lceil \frac{2n}{5} \right\rceil + 1, & n \equiv 0 \pmod 5; \\ \left\lceil \frac{2n}{5} \right\rceil, & n \equiv 1,2,3,4 \pmod 5). \end{cases}$$

Moreover, the upper bound is tight.

Proof. Let $L \subset V(P_n)$, where

$$L = \begin{cases} \{y_{5\ell-1}\} \bigcup \{y_{5k+3}, y_{5k}\} \mid k = 0,\dots,\ell-1 & n = 5\ell,\ n \equiv 0 \pmod 5; \\ \{y_{5\ell}\} \bigcup \{y_{5k+3}, y_{5k}\} \mid k = 0,\dots,\ell-1 & n = 5\ell+1,\ n \equiv 1 \pmod 5; \\ \{y_{5\ell}\} \bigcup \{y_{5k+3}, y_{5k}\} \mid k = 0,\dots,\ell-1 & n = 5\ell+2,\ n \equiv 2 \pmod 5; \\ \{y_{5\ell}, y_{5\ell+2}\} \bigcup \{y_{5k+3}, y_{5k}\} \mid k = 0,\dots,\ell-1 & n = 5\ell+3,\ n \equiv 3 \pmod 5; \\ \{y_{5\ell+3}, y_{5\ell}\} \bigcup \{y_{5k+3}, y_{5k}\} \mid k = 0,\dots,\ell-1 & n = 5\ell+4,\ n \equiv 4 \pmod 5. \end{cases}$$

Here we prove that L is an LD set of P_n. Table 2 shows the vertices $v \in V \setminus L$ and their corresponding intersections $L \cap N[v]$. Note that, for a fixed ℓ, any two intersections are mutually disjoint as well as nonempty. This implies that L is an LD set of P_n. Since

$$|L| = \begin{cases} \left\lceil \frac{2n}{5} \right\rceil + 1, & n \equiv 0 \pmod 5; \\ \left\lceil \frac{2n}{5} \right\rceil, & n \equiv 1,2,3,4 \pmod 5. \end{cases}$$

This shows the upper bound.

n	$x \in V \setminus L$	$L \cap N[x]$	$x \in V \setminus L$	$L \cap N[x]$
5ℓ	y_{5k+4} $(k < \ell-1)$ y_{5k+1}	$\{y_{5(k+1)}, y_{5k+3}\}$ $\{y_{5k}\}$	y_{5k+2}	$\{y_{5k+3}\}$
$5\ell+1$	y_{5k+4} y_{5k+1}	$\{y_{5(k+1)}, y_{5k+3}\}$ $\{y_{5k}\}$	y_{5k+2}	$\{y_{5k+3}\}$
$5\ell+2$	y_{5k+4} y_{5k+1}	$\{y_{5(k+1)}, y_{5k+3}\}$ $\{y_{5k}\}$	$y_{5\ell+1}$ y_{5k+2}	$\{y_{5\ell}\}$ $\{y_{5k+3}\}$
$5\ell+3$	y_{5k+4} y_{5k+1}	$\{y_{5(k+1)}, y_{5k+3}\}$ $\{y_{5k}\}$	$y_{5\ell+1}$ y_{5k+2}	$\{y_{5\ell+2}, y_{5\ell}\}$ $\{y_{5k+3}\}$
$5\ell+4$	y_{5k+4} y_{5k+1} $y_{5\ell+2}$	$\{y_{5(k+1)}, y_{5k+3}\}$ $\{y_{5k}\}$ $\{y_{5\ell+3}\}$	$y_{5\ell+1}$ y_{5k+2}	$\{y_{5\ell}\}$ $\{y_{5k+3}\}$

Table 2: *Locating-dominating vertices in the path graph P_n.*

The ILP formulation with (1), (2), (4) and (6) is used in CLPEX solver to show tightness. This, in turn, provides us with the following optimal solution for small cases: $\gamma_{l-d}(P_5) = 3$, $\gamma_{l-d}(P_6) = 3, \dots,$ $\gamma_{l-d}(P_{15}) = 7, \dots, \gamma_{l-d}(P_{33}) = 14$. This shows the tightness. □

5. Generalized Petersen graphs

Denoted by $P(n,k)$ where $n \geqslant 3$ satisfying $1 \leqslant k \leqslant \left\lfloor \frac{n-1}{2} \right\rfloor$, the vertex set of the generalized Petersen graph is

$$V = \{y_0, y_1, \dots, y_{n-1}, x_0, x_1, \dots, x_{n-1}\}$$

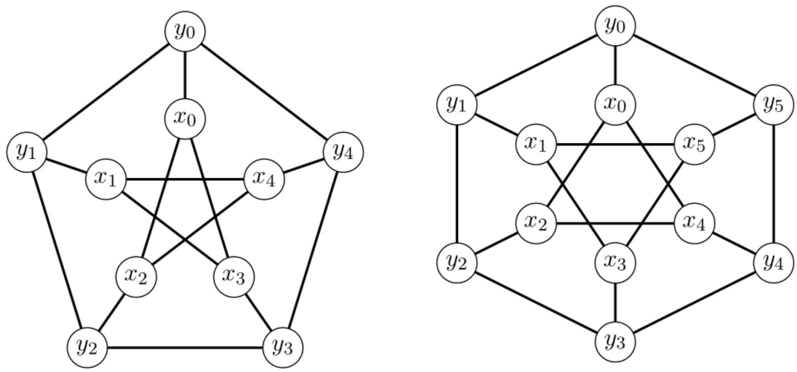
and an edge set

$$E = \{y_i y_{i+1}, y_i x_i, x_i x_{i+k} \mid \text{with indices taken as modulo } n-1\}.$$

Watkins [27] was the first researcher who introduced these combinatorially important graphs. Recently, Rajasingh et al. [28] studied the circular wirelength of generalized Petersen graphs. For $k = 1$, the generalized Petersen graph $P(n,1)$ is called prism graph, denoted by D_n.

5.1. Generalized Petersen graph $P(n,2)$

We consider the the problem of finding an LD set of generalized Petersen graph for $k = 2$. For the sake of simplicity we call the cycle induced by $\{y_0, y_1, \ldots, y_{n-1}\}$, outer cycle and the cycle induced by $\{x_0, x_1, \ldots, x_{n-1}\}$, inner cycle or cycles respectively. In Figure 1, the generalized Petersen graphs $P(5,2)$ and $P(6,2)$ are depicted.



(a) *(b)*

Figure 1: *(a): The generalized Petersen graph $P(5,2)$, (b): The generalized Petersen graph $P(6,2)$.*

The following theorem exhibits the exact value of $\gamma_{l-d}(G)$, where $G = P(n,2)$.

THEOREM 6. *Let G be a generalized Petersen graph $P(n,2)$, where $n \geqslant 3$. Then*

$$\gamma_{l-d}(G) = \left\lceil \frac{2n}{3} \right\rceil.$$

Proof. Since G is a 3-regular graph, by Theorem 1, we obtain

$$\gamma_{l-d}(G) \geqslant \left\lceil \frac{2n}{3} \right\rceil.$$

Let

$$L = \begin{cases} \{x_{3k}, y_{3k+1}\} \mid k = 0, \dots, \ell-1 & n = 3\ell,\ n \equiv 0 \pmod 3; \\ \{x_{3k}, y_{3k+1}\} \bigcup \{x_{3\ell}\} \mid k = 0, \dots, \ell-1 & n = 3\ell+1,\ n \equiv 1 \pmod 3; \\ \{x_{3k+1}, y_{3k}\} \bigcup \{x_{3\ell+1}, y_{3\ell}\} \mid k = 0, \dots, \ell-1 & n = 3\ell+2,\ n \equiv 2 \pmod 3; \end{cases}$$

n	$x \in V \setminus L$	$L \cap N[x]$	$x \in V \setminus L$	$L \cap N[x]$
3ℓ	x_{3k+1} y_{3k}	$\{x_{3(k+1)}, y_{3k+1}\}$ $\{x_{3k}, y_{3k+1}\}$	x_{3k+2} y_{3k+2}	$\{x_{3k}\}$ $\{y_{3k+1}\}$
$3\ell+1$	x_{3k+1} y_{3k} $y_{3\ell}$	$\{x_{3(k+1)}, y_{3k+1}\}$ $\{x_{3k}, y_{3k+1}\}$ $\{x_{3\ell}\}$	x_{3k+2} y_{3k+2}	$\{x_{3(k-1)+3}\}$ $\{y_{3k+1}\}$
$3\ell+2$	x_0 $x_{3(k+1)}$ y_{3k+1}	$\{y_0\}$ $\{x_{3k+1}, y_{3(k+1)}\}$ $\{y_{3k}, x_{3k+1}\}$	$y_{3\ell+1}$ x_{3k+2} y_{3k+2}	$\{x_{3\ell+1}, y_0, y_{3\ell}\}$ $\{x_{3k+4}\}$ $\{y_{3(k+1)}\}$

Table 3: *Vertices belonging to an LD set in* $P(n,2)$.

By Table 3, it is observed that L does satisfy to be an LD set. Note that $|L| = \lceil \frac{2n}{3} \rceil$. This implies that $\gamma_{l-d}(G) \leqslant \lceil \frac{2n}{3} \rceil$. This shows the theorem. □

Next, we study the global LD number of the generalized Petersen graphs $P(n,2)$. First, we show the following result on the LD number of the complement of $P(n,2)$.

PROPOSITION 7. *Let G be generalized Petersen graph $P(n,2)$, where $n \geqslant 5$. Then*

$$4 \leqslant \gamma_{l-d}(\overline{G}) \leqslant \left\lceil \frac{2n}{3} \right\rceil.$$

Moreover, the upper bound is tight.

Proof. The lower bound follows from Proposition 1.

Now we show that the set L from the proof of Theorem 6 for G is also an LD set for the complements of G. By Proposition 2, we only need to show that L is a dominating set of $\overline{G}$. By Table 3 from the proof of Theorem 6, there does not exist any vertex $v \in V \setminus L$, such that $L \cap N[v] = L$. This means that for any $v \in V \setminus L$, there exists at least one vertex $u \in L$, such $u \notin L \cap N[v]$. This implies that in the complement $\overline{G}$, for every $v \in V(\overline{G}) \setminus L$ we have $L \cap N[v] \neq \emptyset$. This shows that L is a dominating set of $\overline{G}$ and thus by Proposition 2 L is also a binary locating-dominating set of $\overline{G}$. This shows the upper bound.

The ILP formulation with (1), (2), (4) and (6) is used in CLPEX solver to show tightness. This, in turn, provides us with the following optimal solution for small cases: $\gamma_{l-d}(\overline{P(5,2)}) = 4$, $\gamma_{l-d}(\overline{P(6,2)}) = 4, \dots$, $\gamma_{l-d}(\overline{P(11,2)}) = 8, \dots, \gamma_{l-d}(\overline{P(28,2)}) = 19$. This shows the tightness. □

By using the definition of the global LD code and then using Theorem 6 and Proposition 7, we obtain the following result.

THEOREM 7. *Let G be the generalized Petersen graph $P(n,2)$, where $n \geqslant 5$. Then*

$$\lambda_g(G) = \left\lceil \frac{2n}{3} \right\rceil.$$

5.2. Generalized Petersen graph $P(n,4)$

In this subsection, we consider the problem of finding the LD number of generalized Petersen graphs $P(n,4)$. For the sake of simplicity we call the cycle induced by $\{y_0, y_1, \ldots, y_{n-1}\}$, outer cycle and the cycle induced by $\{x_0, x_1, \ldots, x_{n-1}\}$, inner cycle or cycles respectively. In Figure 2, the generalized Petersen graphs $P(9,4)$ and $P(10,4)$ are depicted.

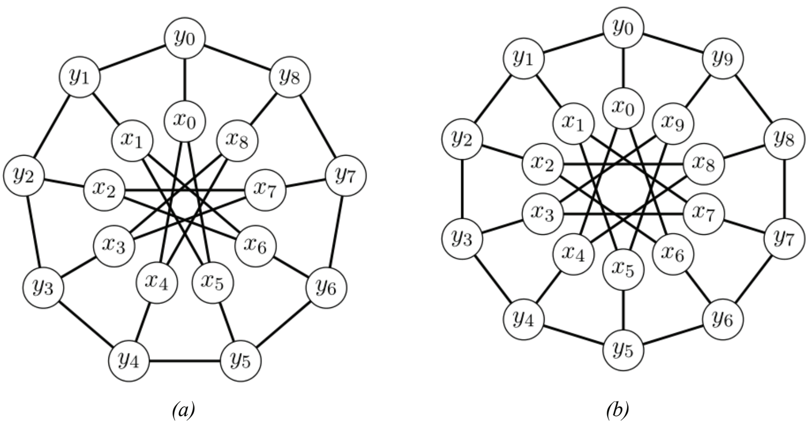


Figure 2: *(a): The generalized Petersen graph $P(9,4)$, (b): The generalized Petersen graph $P(10,4)$.*

In the following theorem, we present the LD number of $P(n,4)$.

THEOREM 8. *Let G be a generalized Petersen graph $P(n,4)$, where $n \geqslant 12$. Then*

$$\left\lceil \frac{2n}{3} \right\rceil \leqslant \gamma_{l-d}(G) \begin{cases} = \lceil \frac{2n}{3} \rceil, & n \equiv 0 \pmod 3; \\ \leqslant \lceil \frac{2n}{3} \rceil + 2, & n \equiv 1 \pmod 3; \\ \leqslant \lceil \frac{2n}{3} \rceil + 1, & n \equiv 2 \pmod 3. \end{cases}$$

Moreover, the upper bounds are tight.

Proof. Note that generalized Petersen graph $P(n,4)$ is a regular graph of degree 3, with $2n$ vertices. By Theorem 1, we obtain $\gamma_{l-d}(G) \geqslant \lceil \frac{2(2n)}{3+3} \rceil = \lceil \frac{2n}{3} \rceil$. Let

$$L=\begin{cases}\{y_{6i},y_{6i+3},x_{6i+1},x_{6i+4}\}\bigcup\{y_{6(\ell-1)},y_{6(\ell-1)+3},\\ x_{6(\ell-1)+1},x_{6\ell-2}\}\mid i=0,\ldots,\ell-2\}, \\ \qquad\qquad n=6\ell,\ \ n\equiv 0 \pmod 6;\\ \{y_{6i},y_{6i+3},x_{6i+1},x_{6i+4}\}\bigcup\{y_{6(\ell-1)},y_{6\ell-3},y_{6\ell-1},\\ x_{6(\ell-1)+1},x_{6(\ell-1)+2},x_{6\ell-2},x_{6\ell-1}\}\mid i=0,\ldots,\ell-2\},\\ \qquad\qquad n=6\ell+1,\ \ n\equiv 1 \pmod 6;\\ \{y_{6i+1},y_{6i+4},x_{6i},x_{6i+3}\}\bigcup\{y_{6(\ell-1)+1},y_{6\ell-2},y_{6\ell},\\ x_{6(\ell-1)},x_{6(\ell-1)+3},x_{6\ell-1},x_{6\ell}\}\mid i=0,\ldots,\ell-2\},\\ \qquad\qquad n=6\ell+2,\ \ n\equiv 2 \pmod 6;\\ \{y_{6i},y_{6i+3},x_{6i+1},x_{6i+4}\}\bigcup\{y_{6(\ell-1)},y_{6\ell-3},y_{6\ell},\\ x_{6(\ell-1)+1},x_{6\ell-2},x_{6\ell+1}\}\mid i=0,\ldots,\ell-2\},\\ \qquad\qquad n=6\ell+3,\ \ n\equiv 3 \pmod 6;\\ \{y_{6i},y_{6i+3},x_{6i+1},x_{6i+4}\}\bigcup\{y_{6(\ell-1)},y_{6\ell-3},y_{6\ell},\\ y_{6\ell+3},x_{6(\ell-1)+1},x_{6\ell-2},x_{6\ell},x_{6\ell+1},x_{6\ell+2}\}\mid i=0,\ldots,\ell-2\},\\ \qquad\qquad n=6\ell+4,\ \ n\equiv 4 \pmod 6;\\ \{y_{6i},y_{6i+3},x_{6i+1},x_{6i+4}\}\bigcup\{y_{6(\ell-1)},y_{6\ell-3},y_{6\ell},\\ y_{6\ell+3},x_{6(\ell-1)+1},x_{6\ell-2},x_{6\ell+1},x_{6\ell+3},x_{6\ell+4}\}\mid i=0,\ldots,\ell-2\},\\ \qquad\qquad n=6\ell+5,\ \ n\equiv 5 \pmod 6;\end{cases}$$

The following cases are distinguished to prove L to be an LD set.

Case 1: When $n\equiv 0 \pmod 6$.

Table 4 presents the required intersections for vertices in $V\setminus L$. It can be observed that all the intersections are simultaneously disjoint and nonempty. Thus, we get $x,y\in V\setminus L$, it is satisfied that $L\bigcap N[x]\neq L\bigcap N[y]\neq\emptyset$. Thus, L meets the requirements to an LD set.

Case 2: When $n\equiv 1 \pmod 6$.

Employing the same reasoning as we have in Case 1 and since the intersections in Table 4 are disjoint and nonempty, we obtain that L is an LD set.

Case 3: When $n\equiv 2 \pmod 6$.

By using the same reasoning as in previous two cases and Table 4, we find that L is an LD set, if $n\equiv 2 \pmod 6$.

The cases when $n\equiv 3,4,5 \pmod 6$ are exactly similar to the previous cases.

Note that

$$|L|=\begin{cases}\left\lceil\frac{2n}{3}\right\rceil, & n\equiv 0 \pmod 3;\\ \left\lceil\frac{2n}{3}\right\rceil+2, & n\equiv 1 \pmod 3;\\ \left\lceil\frac{2n}{3}\right\rceil+1, & n\equiv 2 \pmod 3.\end{cases}$$

This shows the upper bound.

The ILP formulation with (1), (2), (4) and (6) is used in CLPEX solver to show tightness. This, in turn, provides us with the following optimal solution for small cases: $\gamma_{l-d}\big(P(13,4)\big)=11$, $\gamma_{l-d}\big(P(14,4)\big)=11,\ldots,\ \gamma_{l-d}\big(P(22,4)\big)=17,\ldots,\gamma_{l-d}\big(P(35,4)\big)=25$. This shows the tightness. $\square$

n	$x \in V \setminus L$	$L \cap N[x]$	$x \in V \setminus L$	$L \cap N[x]$
6ℓ	y_{6i+1}	$\{y_{6i}, x_{6i+1}\}$	y_{6i+2}	$\{y_{6i+3}\}$
	y_{6i+4}	$\{y_{6i+3}, x_{6i+4}\}$	y_{6i+5}	$\{y_{6(i+1)}\}$
	x_{6i}	$\{y_{6i}, x_{6i+4}\}$	x_{6i+2}	$\{x_{6(i-1)+4}\}$
	x_{6i+3}	$\{y_{6i+3}, x_{6i+7}\}$	x_{6i+5}	$\{x_{6i+1}\}$
	$y_{6(\ell-1)+1}$	$\{y_{6(\ell-1)}, x_{6(\ell-1)+1}\}$	$y_{6(\ell-1)+4}$	$\{y_{6(\ell-1)+3}, x_{6(\ell-1)+4}\}$
	$y_{6\ell-1}$	$\{y_0\}$	$y_{6(\ell-1)+2}$	$\{y_{6(\ell-1)+3}\}$
	$x_{6(\ell-1)}$	$\{y_{6(\ell-1)}, x_{6\ell-2}\}$	$x_{6(\ell-1)+2}$	$\{x_{6(\ell-1)-2}\}$
	$x_{6\ell-3}$	$\{y_{6\ell-3}, x_1\}$	$x_{6\ell-1}$	$\{x_{6(\ell-1)+1}\}$
$6\ell+1$	y_{6i+1}	$\{y_{6i}, x_{6i+1}\}$	y_{6i+2}	$\{y_{6i+3}\}$
	y_{6i+4}	$\{y_{6i+3}, x_{6i+4}\}$	y_{6i+5}	$\{y_{6(i+1)}\}$
	x_{6i}	$\{y_{6i}, x_{6i+4}\}$	x_{6i+2}	$\{x_{6(i-1)+4}\}$
	x_{6i+3}	$\{y_{6i+3}, x_{6i+7}\}$	x_{6i+5}	$\{x_{6i+1}\}$
	$y_{6(\ell-1)+1}$	$\{y_{6(\ell-1)}, x_{6(\ell-1)+1}\}$	$y_{6(\ell-1)+2}$	$\{y_{6\ell-3}, x_{6(\ell-1)+2}\}$
	$y_{6\ell}$	$\{y_{6\ell-1}, y_0\}$	$y_{6\ell-2}$	$\{y_{6\ell-3}, y_{6\ell-1}, x_{6\ell-2}\}$
	$x_{6(\ell-1)}$	$\{y_{6(\ell-1)}, x_{6\ell-2}\}$	$x_{6\ell-3}$	$\{y_{6\ell-3}\}$
	$x_{6\ell}$	$\{x_{6(\ell-1)+2}\}$		
$6\ell+2$	y_{6i}	$\{y_{6i+1}, x_{6i}\}$	y_{6i+2}	$\{y_{6i+1}\}$
	y_{6i+3}	$\{y_{6i+4}, x_{6i+3}\}$	y_{6i+5}	$\{y_{6i+4}\}$
	x_{6i+1}	$\{y_{6i+1}, x_{6(i-1)+3}\}$	x_{6i+2}	$\{x_{6(i+1)}\}$
	x_{6i+4}	$\{y_{6i+4}, x_{6(i-1)+6}\}$	x_{6i+5}	$\{y_{6i+9}\}$
	$y_{6(\ell-1)}$	$\{y_{6(\ell-1)+1}, x_{6(\ell-1)}\}$	$y_{6(\ell-1)+2}$	$\{y_{6(\ell-1)+1}\}$
	$y_{6(\ell-1)+3}$	$\{y_{6(\ell-1)+4}, x_{6(\ell-1)+3}\}$	$y_{6\ell-1}$	$\{y_{6\ell-2}, y_{6\ell}, x_{6\ell-1}\}$
	$y_{6\ell+1}$	$\{y_{6\ell}\}$	$x_{6\ell-4}$	$\{x_{6\ell}\}$
	$x_{6\ell-2}$	$\{y_{6\ell-2}, x_{6(\ell-1)}, x_0\}$	$x_{6\ell+1}$	$\{x_{6\ell-3}, x_3\}$
	$x_{6(\ell-1)+1}$	$\{y_{6(\ell-1)+1}, x_{6(\ell-2)+3}, x_{6\ell-1}\}$		
$6\ell+3$	y_{6i+1}	$\{y_{6i}, x_{6i+1}\}$	y_{6i+2}	$\{y_{6i+3}\}$
	y_{6i+4}	$\{y_{6i+3}, x_{6i+4}\}$	y_{6i+5}	$\{y_{6(i+1)}\}$
	x_{6i}	$\{y_{6i}, x_{6i+4}\}$	x_{6i+2}	$\{x_{6(i-1)+4}\}$
	x_{6i+3}	$\{y_{6i+3}, x_{6i+7}\}$	x_{6i+5}	$\{x_{6i+1}\}$
	$y_{6(\ell-1)+1}$	$\{y_{6(\ell-1)}, x_{6(\ell-1)+1}\}$	$y_{6\ell-4}$	$\{y_{6\ell-3}\}$
	$y_{6\ell-2}$	$\{y_{6\ell-3}, x_{6\ell-2}\}$	$y_{6\ell-1}$	$\{y_{6\ell}\}$
	$y_{6\ell+1}$	$\{y_{6\ell}, x_{6\ell+1}\}$	$y_{6\ell+2}$	$\{y_0\}$
	$x_{6\ell-3}$	$\{y_{6\ell-3}, x_{6\ell+1}\}$	$x_{6\ell-1}$	$\{x_{6(\ell-1)+1}\}$
	$x_{6\ell}$	$\{y_{6\ell}, x_1\}$	$x_{6\ell+2}$	$\{x_{6\ell-2}\}$
	$x_{6(\ell-1)}$	$\{y_{6(\ell-1)}, x_{6\ell-2}\}$		
$6\ell+4$	y_{6i+1}	$\{y_{6i}, x_{6i+1}\}$	y_{6i+2}	$\{y_{6i+3}\}$
	y_{6i+4}	$\{y_{6i+3}, x_{6i+4}\}$	y_{6i+5}	$\{y_{6(i+1)}\}$
	x_{6i}	$\{y_{6i}, x_{6i+4}\}$	x_{6i+3}	$\{y_{6i+3}, x_{6i+7}\}$
	x_{6i+5}	$\{x_{6i+1}\}$	x_{6i+2}	$\{x_{6(i-1)+4}\}$
	$y_{6(\ell-1)+1}$	$\{y_{6(\ell-1)}, x_{6(\ell-1)+1}\}$	$y_{6(\ell-1)+2}$	$\{y_{6(\ell-1)+3}\}$
	$y_{6\ell-2}$	$\{y_{6\ell-3}, x_{6\ell-2}\}$	$y_{6\ell-1}$	$\{y_{6\ell}\}$
	$y_{6\ell+1}$	$\{y_{6\ell}, x_{6\ell+1}\}$	$y_{6\ell+2}$	$\{y_{6\ell+3}, x_{6\ell+2}\}$
	$x_{6(\ell-1)}$	$\{y_{6(\ell-1)}, x_{6\ell-2}\}$	$x_{6(\ell-1)+2}$	$\{x_{6\ell}\}$
	$x_{6\ell-3}$	$\{y_{6\ell-3}, x_{6\ell+1}\}$	$x_{6\ell-1}$	$\{x_{6(\ell-1)+1}\}$
	$x_{6\ell+3}$	$\{y_{6\ell+3}\}$		
$6\ell+5$	y_{6i+1}	$\{y_{6i}, x_{6i+1}\}$	y_{6i+2}	$\{y_{6i+3}\}$
	y_{6i+4}	$\{y_{6i+3}, x_{6i+4}\}$	y_{6i+5}	$\{y_{6(i+1)}\}$
	x_{6i}	$\{y_{6i}, x_{6i+4}\}$	x_{6i+2}	$\{x_{6(i-1)+4}\}$
	x_{6i+3}	$\{y_{6i+3}, x_{6i+7}\}$	x_{6i+5}	$\{x_{6i+1}\}$
	$y_{6(\ell-1)+1}$	$\{y_{6(\ell-1)}, x_{6(\ell-1)+1}\}$	$y_{6(\ell-1)+2}$	$\{y_{6(\ell-1)+3}\}$
	$y_{6\ell-2}$	$\{y_{6\ell-3}, x_{6\ell-2}\}$	$y_{6\ell-1}$	$\{y_{6\ell}\}$
	$y_{6\ell+1}$	$\{y_{6\ell}, x_{6\ell+1}\}$	$y_{6\ell+2}$	$\{y_{6\ell+3}\}$
	$y_{6\ell+4}$	$\{y_{6\ell+3}, x_{6\ell+4}, y_0\}$	$x_{6(\ell-1)}$	$\{y_{6(\ell-1)}, x_{6\ell-2}\}$
	$x_{6(\ell-1)+2}$	$\{x_{6(\ell-1)-2}\}$	$x_{6\ell-3}$	$\{y_{6\ell-3}, x_{6\ell+1}\}$
	$x_{6\ell-1}$	$\{x_{6\ell+3}, x_{6(\ell-1)+1}\}$	$x_{6\ell}$	$\{y_{6\ell}, x_{6\ell+4}\}$
	$x_{6\ell+2}$	$\{x_{6\ell-2}, x_1\}$		

Table 4: *Vertices belonging to an LD set in* $P(n,4)$.

Next, we study the global LD number of the generalized Petersen graphs $P(n,4)$. First, we show the following result on the LD number of the complement of $P(n,4)$.

PROPOSITION 8. *Let G be a generalized Petersen graph $P(n,4)$, where $n \geqslant 12$. Then*

$$4 \leqslant \gamma_{l-d}(\overline{G}) \begin{cases} = \lceil \frac{2n}{3} \rceil, & n \equiv 0 \pmod 3; \\ \leqslant \lceil \frac{2n}{3} \rceil + 2, & n \equiv 1 \pmod 3; \\ \leqslant \lceil \frac{2n}{3} \rceil + 1, & n \equiv 2 \pmod 3. \end{cases}$$

Moreover, the upper bounds are tight.

Proof. The lower bound follows from Proposition 1.

Now we show that the set L from the proof of Theorem 8 for G is also an LD set of $\overline{G}$. By Proposition 2, we only need to show that L is a dominating set of $\overline{G}$. By Table 4 from the proof of Theorem 8, there does not exist any vertex $v \in V \setminus L$, such that $L \cap N[v] = L$. This means that for any $v \in V \setminus L$, there exists at least one vertex $u \in L$, such $u \notin L \cap N[v]$. This implies that in the complement $\overline{G}$, for every $v \in V(\overline{G}) \setminus L$ we have $L \cap N[v] \neq \emptyset$. This shows that L is a dominating set of $\overline{G}$ and thus by Proposition 2 L is also a binary locating-dominating set of $\overline{G}$. This shows the upper bound.

The ILP formulation with (1), (2), (4) and (6) is used in CLPEX solver to show tightness. This, in turn, provides us with the following optimal solution for small cases: $\gamma_{l-d}\big(\overline{P(5,2)}\big) = 5$, $\gamma_{l-d}\big(\overline{P(6,2)}\big) = 4, \ldots, \gamma_{l-d}\big(\overline{P(11,2)}\big) = 9, \ldots, \gamma_{l-d}\big(\overline{P(28,2)}\big) = 20$. This shows the tightness. $\square$

By using the definition of the global LD code and then using Theorem 8 and Proposition 8, we obtain the following result.

THEOREM 9. *Let G be the generalized Petersen graph $P(n,4)$, where $n \geqslant 12$. Then*

$$\lambda_g(G) \begin{cases} = \lceil \frac{2n}{3} \rceil, & n \equiv 0 \pmod 3; \\ \leqslant \lceil \frac{2n}{3} \rceil + 2, & n \equiv 1 \pmod 3; \\ \leqslant \lceil \frac{2n}{3} \rceil + 1, & n \equiv 2 \pmod 3. \end{cases}$$

5.3. Prism graphs

This subsection studies the problems of LD and its global version for the generalized Petersen graph $P(n,1)$, also known as the prism graphs.

A general prism is a polyhedron possessing two congruent polygonal faces and with all remaining faces parallelograms. A prism is a graph corresponding to the skeleton of an n-prism. An n-prism graph has $2n$ nodes and $3n$ edges, and is equivalent to the generalized Petersen graph $P(n,1)$. Figure 3 shows the prism graph $P(7,1)$ and $P(8,1)$ are shown.

The following result exhibits a lower bound and a tight upper bound on $\gamma_{l-d}(G)$, where $G = P(n,1)$.

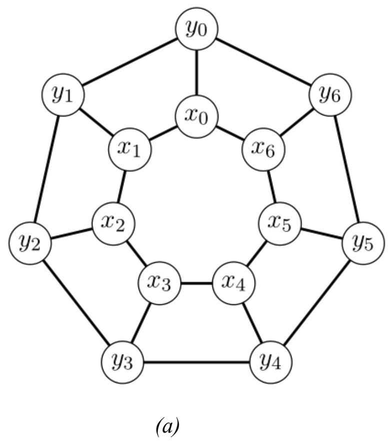


Figure 3: *(a): The generalized Petersen graph $P(7,1)$, (b): The generalized Petersen graph $P(8,1)$.*

THEOREM 10. *Let G be the prism graph $P(n,1)$, where $n \geqslant 15$. Then*

$$\left\lceil \frac{2n}{3} \right\rceil \leqslant \gamma_{l-d}(G) \leqslant \begin{cases} \left\lceil \frac{4n}{5} \right\rceil + 1, & n \equiv 0,1,2,3 \pmod 5; \\ \left\lceil \frac{4n}{5} \right\rceil, & n \equiv 4 \pmod 5. \end{cases}$$

Moreover, the upper bound is tight.

Proof. Firstly, we see that G is a regular graph of degree 3, with $2n$ vertices. Then by Theorem 1, we obtain $\gamma_{l-d}(G) \geqslant \left\lceil \frac{2(2n)}{3+3} \right\rceil = \left\lceil \frac{2n}{3} \right\rceil$.

Next, we show the upper bound. Let

$$L = \begin{cases} \{y_{5i}, y_{5i+3}, x_{5i}, x_{5i+3}\} \cup \\ \{y_{5(\ell-1)}, y_{5\ell-2}, x_{5(\ell-1)}, x_{5\ell-3}, x_{5\ell-1}\} \mid i = 0,\ldots,\ell-2 \\ \qquad\qquad n \equiv 0 \pmod 5,\ \ n = 5\ell; \\ \{y_{5i}, y_{5i+3}, x_{5i}, x_{5i+3}\} \cup \\ \{y_{5(\ell-1)}, y_{5\ell-2}, y_{5\ell}, x_{5(\ell-1)}, x_{5\ell-3}, x_{5\ell-1}\} \mid i = 0,\ldots,\ell-2 \\ \qquad\qquad n \equiv 1 \pmod 5,\ \ n = 5\ell+1; \\ \{y_{5i}, y_{5i+3}, x_{5i}, x_{5i+3}\} \cup \\ \{y_{5(\ell-1)}, y_{5\ell-2}, y_{5\ell}, x_{5(\ell-1)}, x_{5\ell-3}, x_{5\ell-1}, x_{5\ell+1}\} \mid i = 0,\ldots,\ell-2 \\ \qquad\qquad n \equiv 2 \pmod 5,\ \ n = 5\ell+2; \\ \{y_{5i}, y_{5i+3}, x_{5i}, x_{5i+3}\} \cup \\ \{y_{5(\ell-1)}, y_{5\ell-2}, y_{5\ell}, y_{5\ell+2}, x_{5(\ell-1)}, x_{5\ell-3}, x_{5\ell-1}, x_{5\ell+2}\} \mid i = 0,\ldots,\ell-2 \\ \qquad\qquad n \equiv 3 \pmod 5,\ \ n = 5\ell+3; \\ \{y_{5i}, y_{5i+3}, x_{5i}, x_{5i+3}\} \cup \\ \{y_{5(\ell-1)}, y_{5\ell-2}, y_{5\ell}, y_{5\ell+3}, x_{5(\ell-1)}, x_{5\ell-3}, x_{5\ell-1}, x_{5\ell+2}\} \mid i = 0,\ldots,\ell-2 \\ \qquad\qquad n \equiv 4 \pmod 5,\ \ n = 5\ell+4. \end{cases}$$

Here we prove L to be an LD set. Note that Table 5 shows the LD vertices in $P(n,1)$. By distinguishing the cases as we have done in Theorem 5 for the case of path graphs and using Table 5, it follows that L is an LD set. Note that

$$|L| = \begin{cases} \lceil \frac{4n}{5} \rceil + 1, & n \equiv 0,1,2,3 \pmod 5; \\ \lceil \frac{4n}{5} \rceil, & n \equiv 4 \pmod 5). \end{cases}$$

This shows the upper bound.

The ILP formulation with (1), (2), (4) and (6) is used in CLPEX solver to show tightness. This, in turn, provides us with the following optimal solution for small cases: $\gamma_{l-d}\big(P(15,1)\big) = 13$, $\gamma_{l-d}\big(P(16,1)\big) = 14, \ldots,$ $\gamma_{l-d}\big(P(23,1)\big) = 20, \ldots, \gamma_{l-d}\big(P(50,1)\big) = 41$. This shows the tightness. $\square$

6. Strongly regular graphs

Let G be a connected k-regular graphs on n vertices. Then G is said to *strongly regular*, if any two distinct pairs of adjacent (resp. non-adjacent) vertices in G has λ (resp. μ) common neighbors. A graph of this kind with parameter n, k, λ and μ is written as srg(n,k,λ,μ). Trivially, the regular complete multipartite graphs are strongly regular. Other non-trivial examples include the cycle of length 5 with parameters srg(5,2,0,1) and the generalized Petersen graph $P(5,2)$ (see Section 5) with parameters srg(10,3,0,1).

Strongly regular graphs have interesting algebraic and combinatorial properties. For example, their adjacency matrix has always three distinct eigenvalues and they are characterized by this property. Also, they are contained in more general combinatorial and geometric structure such as association schemes and partial geometries.

6.1. Triangular graphs

In this section, we study the LD number of an important infinite family of strongly regular graphs known as the triangular graphs.

An m-dimensional triangular graph T_m is the line graph of the complete graph K_m. The vertices of T_m may be identified with the 2-subsets of $\{1,2,\ldots,m\}$ that are adjacent if and only if the 2-subsets have a nonempty intersection. It is a special class of so-called Johnson graphs $J(m,n)$ with $n=2$. The triangular graph T_m is a strongly regular graph with parameters srg$\big(\frac{m(m-1)}{2}, 2(m-2), m-2, 4\big)$. For the sake of simplicity, we label an arbitrary vertex of T_m by y_i $\big(i = 0,1,2,\ldots,\frac{m(m-1)}{2}-1\big)$.

The following theorem is the main result of this subsection.

THEOREM 11. *Let T_m, where $m \geqslant 9$ be the triangular graph of dimension m. Then*

$$\left\lceil \frac{m(m-1)}{2m-1} \right\rceil \leqslant \gamma_{l-d}(T_m) \leqslant m-2.$$

Moreover, the upper bound is tight.

n	$x \in V \setminus L$	$L \cap N[x]$	$x \in V \setminus L$	$L \cap N[x]$
5ℓ	y_{5i+1}	$\{y_{5i}\}$	y_{5i+2}	$\{y_{5i+3}\}$
	y_{5i+4}	$\{y_{5i+3}, y_{5(i+1)}\}$	x_{5i+1}	$\{x_{5i}\}$
	x_{5i+2}	$\{x_{5i+3}\}$	x_{5i+4}	$\{x_{5i+3}, x_{5(i+1)}\}$
	$y_{5(\ell-1)+1}$	$\{y_{5(\ell-1)}\}$	$y_{5\ell-3}$	$\{y_{5\ell-2}, x_{5\ell-3}\}$
	$y_{5\ell-1}$	$\{y_{5\ell-2}, x_{5\ell-1}, y_0\}$	$x_{5(\ell-1)+1}$	$\{x_{5(\ell-1)}, x_{5\ell-3}\}$
	$x_{5\ell-2}$	$\{x_{5\ell-3}, y_{5\ell-2}, x_{5\ell-1}\}$		
$5\ell+1$	y_{5i+1}	$\{y_{5i}\}$	y_{5i+2}	$\{y_{5i+3}\}$
	y_{5i+4}	$\{y_{5i+3}, y_{5(i+1)}\}$	x_{5i+1}	$\{x_{5i}\}$
	x_{5i+2}	$\{x_{5i+3}\}$	x_{5i+4}	$\{x_{5i+3}, x_{5(i+1)}\}$
	$y_{5(\ell-1)+1}$	$\{y_{5(\ell-1)}\}$	$y_{5\ell-3}$	$\{y_{5\ell-2}, x_{5\ell-3}\}$
	$y_{5\ell-1}$	$\{y_{5\ell-2}, x_{5\ell-1}, y_{5\ell}\}$	$x_{5(\ell-1)+1}$	$\{x_{5(\ell-1)}, x_{5\ell-3}\}$
	$x_{5\ell-2}$	$\{x_{5\ell-3}, y_{5\ell-2}, x_{5\ell-1}\}$	$x_{5\ell}$	$\{x_{5\ell-1}, y_{5\ell-1}, x_0\}$
$5\ell+2$	y_{5i+1}	$\{y_{5i}\}$	y_{5i+2}	$\{y_{5i+3}\}$
	y_{5i+4}	$\{y_{5i+3}, y_{5(i+1)}\}$	x_{5i+1}	$\{x_{5i}\}$
	x_{5i+2}	$\{x_{5i+3}\}$	x_{5i+4}	$\{x_{5i+3}, x_{5(i+1)}\}$
	$y_{5(\ell-1)+1}$	$\{y_{5(\ell-1)}\}$	$y_{5\ell-3}$	$\{y_{5\ell-2}, x_{5\ell-3}\}$
	$y_{5\ell-1}$	$\{y_{5\ell-2}, x_{5\ell-1}, y_{5\ell}\}$	$y_{5\ell+1}$	$\{y_{5\ell}, x_{5\ell+1}, x_0\}$
	$x_{5(\ell-1)+1}$	$\{x_{5(\ell-1)}, x_{5\ell-3}\}$	$x_{5\ell-2}$	$\{x_{5\ell-3}, y_{5\ell-2}, x_{5\ell-1}\}$
	$x_{5\ell}$	$\{x_{5\ell-1}, y_{5\ell}, x_{5\ell+1}\}$		
$5\ell+3$	y_{5i+1}	$\{y_{5i}\}$	y_{5i+2}	$\{y_{5i+3}\}$
	y_{5i+4}	$\{y_{5i+3}, y_{5(i+1)}\}$	x_{5i+1}	$\{x_{5i}\}$
	x_{5i+2}	$\{x_{5i+3}\}$	x_{5i+4}	$\{x_{5i+3}, x_{5(i+1)}\}$
	$y_{5(\ell-1)+1}$	$\{y_{5(\ell-1)}\}$	$y_{5\ell-3}$	$\{y_{5\ell-2}, x_{5\ell-3}\}$
	$y_{5\ell-1}$	$\{y_{5\ell-2}, x_{5\ell-1}, y_{5\ell}\}$	$y_{5\ell+1}$	$\{y_{5\ell}, x_{5\ell+1}, y_{5\ell+2}\}$
	$x_{5(\ell-1)+1}$	$\{x_{5(\ell-1)}, x_{5\ell-3}\}$	$x_{5\ell-2}$	$\{x_{5\ell-3}, y_{5\ell-2}, x_{5\ell-1}\}$
	$x_{5\ell}$	$\{x_{5\ell-1}, y_{5\ell}\}$	$x_{5\ell+1}$	$\{x_{5\ell+2}\}$
$5\ell+4$	y_{5i+1}	$\{y_{5i}\}$	y_{5i+2}	$\{y_{5i+3}\}$
	y_{5i+4}	$\{y_{5i+3}, y_{5(i+1)}\}$	x_{5i+1}	$\{x_{5i}\}$
	x_{5i+2}	$\{x_{5i+3}\}$	x_{5i+4}	$\{x_{5i+3}, x_{5(i+1)}\}$
	$y_{5(\ell-1)+1}$	$\{y_{5(\ell-1)}\}$	$y_{5\ell-3}$	$\{y_{5\ell-2}, x_{5\ell-3}\}$
	$y_{5\ell-1}$	$\{y_{5\ell-2}, x_{5\ell-1}, y_{5\ell}\}$	$y_{5\ell+1}$	$\{y_{5\ell}\}$
	$y_{5\ell+2}$	$\{y_{5\ell+3}, x_{5\ell+2}\}$	$x_{5(\ell-1)+1}$	$\{x_{5(\ell-1)}, x_{5\ell-3}\}$
	$x_{5\ell-2}$	$\{x_{5\ell-3}, y_{5\ell-2}, x_{5\ell-1}\}$	$x_{5\ell}$	$\{x_{5\ell-1}, y_{5\ell}\}$
	$x_{5\ell+1}$	$\{x_{5\ell+2}\}$	$x_{5\ell+3}$	$\{x_{5\ell+2}, y_{5\ell+3}, x_0\}$

Table 5: *Vertices belonging to an LD set in* $P(n,1)$.

Proof. Note that T_m is $(2m-4)$-regular graph on $\frac{m(m-1)}{2}$ number of vertices. The lower bound follows directly from Theorem 1. For the upper bound, we define

$$L = \bigcup_{i=1}^{m-2} y_\alpha, \text{ where } \alpha = \frac{(i-1)(2m-i)}{2}.$$

For instance, for $n=8$ and $n=9$, the above formula generates $L=\{y_0,y_7,y_{13},y_{18},y_{22},y_{25}\}$ and $L=\{y_0,y_8,y_{15},y_{21},y_{26},y_{30},y_{33}\}$ respectively. Now we focus on showing L to be an LD set.

$x\in V\setminus L$	$L\cap N[x]$
y_{m-2}	$\{y_0\}$
$y_{\frac{m(m-1)}{2}-1}$	$\{y_{\frac{m(m-1)}{2}-3}\}$
y_{m-3}	$\{y_0,y_{\frac{m(m-1)}{2}-3}\}$
y_α, where $\alpha=(i-2)m-\left(\frac{1}{2}i^2-\frac{3}{2}i+2\right)$, $i\leqslant m$ and $i=4,5,6\dots$	$\{y_{\alpha_1},y_{\alpha_2}\}$, where $\alpha_1=(i-4)m-\left(\frac{1}{2}i^2-\frac{7}{2}i+6\right)$ and $\alpha_2=(i-3)m-\left(\frac{1}{2}i^2-\frac{5}{2}i+3\right)$
y_α, where $\alpha=(i-2)m-\left(\frac{i^2}{2}-\frac{3}{2}i+3\right)$, $i\leqslant m-1$ and $i=4,5,6,\dots$	$\{y_{\alpha_1},y_{\alpha_2},y_{\frac{m(m-1)}{2}-3}\}$, where $\alpha_1=(i-4)m-\left(\frac{i^2}{2}-\frac{7}{2}i+6\right)$ and $\alpha_2=(i-3)m-\left(\frac{i^2}{2}-\frac{5}{2}i+3\right)$
y_{i-4}, where $i\leqslant m$ and $i=5,6,7,\dots$	$\{y_0,y_{\alpha_1},y_{\alpha_2}\}$, where $\alpha_1=(i-4)m-\left(\frac{i^2}{2}-\frac{7}{2}i+6\right)$ and $\alpha_2=(i-3)m-\left(\frac{i^2}{2}-\frac{5}{2}i+3\right)$
y_α, where $\alpha=(j-5)m+i-\left(\frac{j^2}{2}-\frac{7}{2}j+9\right)$, $j\leqslant m$, $j=6,7,8,\dots$, $i\leqslant m$ and $i=j,j+1,j+2,\dots$	$\{y_{\alpha_1},y_{\alpha_2},y_{\alpha_3},y_{\alpha_4}\}$, where $\alpha_1=(j-6)m-\left(\frac{j^2}{2}-\frac{11}{2}j+15\right)$, $\alpha_2=(j-5)m-\left(\frac{j^2}{2}-\frac{9}{2}j+10\right)$, $\alpha_3=(i-4)m-\left(\frac{i^2}{2}-\frac{7}{2}i+6\right)$ and $\alpha_4=(i-3)m-\left(\frac{i^2}{2}-\frac{5}{2}j+3\right)$

Table 6: *Vertices belonging to an LD set in T_m.*

n	$x\in V\setminus L$	$L\cap N[x]$	$x\in V\setminus L$	$L\cap N[x]$
8	y_1	$\{y_0,y_7,y_{13}\}$	y_2	$\{y_0,y_{13},y_{18}\}$
	y_3	$\{y_0,y_{18},y_{22}\}$	y_4	$\{y_0,y_{22},y_{25}\}$
	y_5	$\{y_0,y_{25}\}$	y_6	$\{y_0\}$
	y_8	$\{y_0,y_7,y_{13},y_{18}\}$	y_9	$\{y_0,y_7,y_{18},y_{22}\}$
	y_{10}	$\{y_0,y_7,y_{22},y_{25}\}$	y_{11}	$\{y_0,y_7,y_{25}\}$
	y_{12}	$\{y_0,y_7\}$	y_{14}	$\{y_7,y_{13},y_{18},y_{22}\}$
	y_{15}	$\{y_7,y_{13},y_{22},y_{25}\}$	y_{16}	$\{y_7,y_{13},y_{25}\}$
	y_{17}	$\{y_7,y_{13}\}$	y_{19}	$\{y_{13},y_{18},y_{22},y_{25}\}$
	y_{20}	$\{y_{13},y_{18},y_{25}\}$	y_{21}	$\{y_{13},y_{18}\}$
	y_{23}	$\{y_{18},y_{22},y_{25}\}$	y_{24}	$\{y_{18},y_{22}\}$
	y_{26}	$\{y_{22},y_{25}\}$	y_{27}	$\{y_{25}\}$

Table 7: *Vertices belonging to an LD set in T_8.*

Table 6 depicts the vertices $x\in V\setminus L$ and their corresponding intersections $L\cap$

n	$x \in V \setminus L$	$L \cap N[x]$	$x \in V \setminus L$	$L \cap N[x]$
9	y_1	$\{y_0, y_8, y_{15}\}$	y_2	$\{y_0, y_{15}, y_{21}\}$
	y_3	$\{y_0, y_{21}, y_{26}\}$	y_4	$\{y_0, y_{26}, y_{30}\}$
	y_5	$\{y_0, y_{30}, y_{33}\}$	y_6	$\{y_0, y_{33}\}$
	y_7	$\{y_0\}$	y_9	$\{y_0, y_8, y_{15}, y_{21}\}$
	y_{10}	$\{y_0, y_8, y_{21}, y_{26}\}$	y_{11}	$\{y_0, y_8, y_{26}, y_{30}\}$
	y_{12}	$\{y_0, y_8, y_{30}, y_{33}\}$	y_{13}	$\{y_0, y_8, y_{33}\}$
	y_{14}	$\{y_0, y_8\}$	y_{16}	$\{y_8, y_{15}, y_{21}, y_{26}\}$
	y_{17}	$\{y_8, y_{15}, y_{26}, y_{30}\}$	y_{18}	$\{y_8, y_{15}, y_{30}, y_{33}\}$
	y_{19}	$\{y_8, y_{15}, y_{33}\}$	y_{20}	$\{y_8, y_{15}\}$
	y_{22}	$\{y_{15}, y_{21}, y_{26}, y_{30}\}$	y_{23}	$\{y_{15}, y_{21}, y_{30}, y_{33}\}$
	y_{24}	$\{y_{15}, y_{21}, y_{33}\}$	y_{25}	$\{y_{15}, y_{21}\}$
	y_{27}	$\{y_{21}, y_{26}, y_{30}, y_{33}\}$	y_{28}	$\{y_{21}, y_{26}, y_{33}\}$
	y_{29}	$\{y_{21}, y_{26}\}$	y_{31}	$\{y_{26}, y_{30}, y_{33}\}$
	y_{32}	$\{y_{26}, y_{30}\}$	y_{34}	$\{y_{30}, y_{33}\}$
	y_{35}	$\{y_{33}\}$		

Table 8: *Vertices belonging to an LD set in T_9.*

$N[x]$. Note that, for any fixed m, all the intersections are nonempty and mutually distinct. For instance, for $m = 7$ and $m = 8$, Table 6 generates Tables 7 and 8 respectively. Notice that the intersections are not empty and for any $x, y \in V \setminus L$, we have $L \cap N[x] \neq L \cap N[y]$. This shows that L is an LD set of T_m.

Note that $|L| = m - 2$, therefore, we obtain that $\gamma_{l-d}(T_m) \leqslant m - 2$. The ILP formulation with (1), (2), (4) and (6) is used in CLPEX solver to show tightness. This, in turn, provides us with the following optimal solution for small cases: $\gamma_{l-d}(T_{10}) = 8, \ldots,$ $\gamma_{l-d}(T_{15}) = 13, \ldots,$ $\gamma_{l-d}(T_{33}) = 31, \ldots, \gamma_{l-d}(T_{50}) = 48$. This shows the tightness. $\square$

Next, we study the global LD number of the triangular graph T_m. First, we show the following result on the LD number of the complement of T_m.

PROPOSITION 9. *Let G be a triangular graph T_m, where $m \geqslant 9$. Then*

$$\left\lceil \frac{m(m-1)}{m(m-3)+6} \right\rceil \leqslant \gamma_{l-d}(\overline{G}) \leqslant m - 2.$$

Moreover, the upper bounds are tight.

Proof. The lower bound follows from Proposition 1.

Now we show that the set L from the proof of Theorem 11 for G is also an LD set for the complements of G. By Proposition 2, we only need to show that L is a dominating set of $\overline{G}$. By Table 6 from the proof of Theorem 11, there does not exist any vertex $x \in V \setminus L$, such that $L \cap N[x] = L$. This means that for any $x \in V \setminus L$, there exists at least one vertex $y \in L$, such $y \notin L \cap N[x]$. This implies that in the complement

$\overline{G}$, for every $x \in V(\overline{G}) \setminus L$ we have $L \cap N[x] \neq \emptyset$. This shows that L is a dominating set of $\overline{G}$ and thus by Proposition 2 L is an LD. This shows the upper bound.

The ILP formulation with (1), (2), (4) and (6) is used in CLPEX solver to show tightness. This, in turn, provides us with the following optimal solution for small cases: $\gamma_{l-d}(\overline{T_{10}}) = 8, \ldots,\ \gamma_{l-d}(\overline{T_{15}}) = 13, \ldots,\ \gamma_{l-d}(\overline{T_{33}}) = 31, \ldots, \gamma_{l-d}(\overline{T_{50}}) = 48$. This shows the tightness. □

By using the definition of the global LD code and then using Theorem 11 and Proposition 9, we get:

THEOREM 12. *Assume G is an m-dimensional triangular graph T_m, where $m \geqslant 9$. Then*

$$\lambda_g(G) \leqslant m-2.$$

6.2. Square grid graphs

This section studies the binary location-domination number of another important infinite family of strongly regular graphs known as the square grid graphs or the lattice square graphs. They are constructed by taking the Cartesian product of two complete graphs with same sizes. The $(n \times n)$-grid graph is strong regular with parameters srg$(n^2, 2n-2, n-2, 2)$. We denote by $S(n)$ the $(n \times n)$-grid graph. Note that the Hamming graph $H(d,q)$ is the Cartesian product of d-copies of the complete graph K_q. Since we have considered $H(2,q)$ which is a subfamily of the general Hamming graph, it is standard to call it the square grid graph.

The following result exhibits the tight upper & lower bound on the LD number for $S(n)$.

THEOREM 13. *For square grid graphs $S(n)$, where $n \geqslant 14$, we have*

$$n \leqslant \gamma_{l-d}(S(n)) \leqslant \left\lceil \frac{3n-4}{2} \right\rceil.$$

Moreover, the upper bound is tight.

Proof. By Theorem 1, we obtain

$$\begin{aligned} \gamma_{l-d}(S(n)) &\geqslant \left\lceil \frac{2n^2}{2n+1} \right\rceil, \\ &\geqslant n, \qquad \text{as } \lim_{n\to\infty}\left(\frac{1}{n}\right) = 0. \end{aligned}$$

This shows the lower bound.

Let $x_{i,j}$ be the vertex of $S(n)$ in i^{th} row and j^{th} column. Assume

$$L_1 = \{x_{1,1};x_{1,5};x_{2,9};x_{3,2};x_{3,4};x_{4,3};x_{4,8};x_{5,6};x_{5,7};x_{7,11};x_{8,10}\},$$
$$L_2 = \bigcup_{12\leqslant i\leqslant n-1} \{x_{i,i}\},$$
$$L_3' = \bigcup_{\substack{i\equiv 0 \pmod 2\\ 12\leqslant i\leqslant n-1}} \{x_{i,i+1}\},$$
$$L_3'' = \bigcup_{\substack{i\equiv 0 \pmod 2\\ 12\leqslant i\leqslant n-2}} \{x_{i,i+1}\},$$

where,

$$L_3 = \begin{cases} L_3', \text{ if} & n \nmid 2 \\ L_3'', \text{ if} & n \mid 2. \end{cases}$$

Let $L = \bigcup_{i=1}^{3} L_i$ such that $L \subset V(S(n))$. For $n \geqslant 14$, $L = L_1 \cup L_2 \cup L_3$ where L_i $(i = 1,2,3)$ is chosen accordingly.

Now we prove L to be an LD set. By looking at the defining structure of $S(n)$, we can see that for any vertex $x_{m,n} \in V \setminus L$, we have

$$N[x_{m,n}] \cap L = L_{m,*} \cup L_{*,n}.$$

Note that, at least one of the two sets $L_{m,*}$ and $L_{*,n}$ is non-empty. This shows that any such intersection is not empty. Let $x_{m,n}, x_{p,q} \in V \setminus L$ where $m \neq p$ and $n \neq q$. Then

$$N[x_{m,n}] \cap L \neq N[x_{p,q}] \cap L,$$

because $L^{mn} \cap L^{pq} = 0$, where $L^{mn} = L_{m,*} \cup L_{*,n}$ and $L^{pq} = L_{p,*} \cup L_{*,q}$. This shows that any two such intersections are distinct implying that L is an LD set. Note that $|L| = n + \lceil \frac{n-4}{2} \rceil = \lceil \frac{3n-4}{2} \rceil$. Thus we have $\gamma_{l-d}\big(S(n)\big) \leqslant \lceil \frac{3n-4}{2} \rceil$.

The ILP formulation with (1), (2), (4) and (6) is used in CLPEX solver to show tightness. This, in turn, provides us with the following optimal solution for small cases: $\gamma_{l-d}\big(S(11)\big) = 15$, $\gamma_{l-d}\big(S(12)\big) = 16$, $\gamma_{l-d}\big(S(13)\big) = 18,\ldots,$ $\gamma_{l-d}\big(S(20)\big) = 28,\ldots,\gamma_{l-d}\big(S(33)\big) = 48, \ldots, \gamma_{l-d}\big(S(50)\big) = 73$. This shows the tightness. □

Next, we study the global LD number of the square grid graphs $S(n)$. First, we show the following result on the LD number of $\overline{S(n)}$.

PROPOSITION 10. *Let G be a n-dimensional square grid graphs $S(n)$, where $n \geqslant 11$. Then*

$$\left\lceil \frac{2n^2}{n^2 - 2n + 4} \right\rceil \leqslant \gamma_{l-d}\big(\overline{G}\big) \leqslant \left\lceil \frac{3n-4}{2} \right\rceil.$$

Moreover, the upper bounds are tight.

Proof. The lower bound follows from Proposition 1.

Now we show that the set L from the proof of Theorem 13 for G is also an LD set for the complements of G. By Proposition 2, we only need to show that L is a dominating set of $\overline{G}$. By Table 6 from the proof of Theorem 13, there does not exist any vertex $x \in V \setminus L$, such that $L \cap N[x] = L$. This means that for any $x \in V \setminus L$, there exists at least one vertex $y \in L$, such $y \notin L \cap N[x]$. This implies that in the complement $\overline{G}$, for every $x \in V(\overline{G}) \setminus L$ we have $L \cap N[x] \neq \emptyset$. This shows that L is a dominating set of $\overline{G}$ and thus by Proposition 2 L is also an LD set. This shows the upper bound.

The ILP formulation with (1), (2), (4) and (6) is used in CLPEX solver to show tightness. This, in turn, provides us with the following optimal solution for small cases: $\gamma_{l-d}\big(S(11)\big) = 15$, $\gamma_{l-d}\big(S(12)\big) = 16$, $\gamma_{l-d}\big(S(13)\big) = 18, \ldots,$ $\gamma_{l-d}\big(S(20)\big) = 28, \ldots, \gamma_{l-d}\big(S(33)\big) = 48, \ldots, \gamma_{l-d}\big(S(50)\big) = 73$. This shows the tightness. $\square$

By using the definition of the global LD code and then using Theorem 13 and Proposition 10, we obtain the following result.

THEOREM 14. *Let G be the square grid graphs $S(n)$, where $n \geqslant 11$. Then*

$$\lambda_g(G) \leqslant \left\lceil \frac{3n-4}{2} \right\rceil.$$

7. Conclusion

7.1. Contributions

In this paper, we study the binary location-domination number of graphs. In particular, the following are the main contributions of this paper:

- Exact values of the binary location-domination number of the complete multipartite graphs and cycle graphs were computed, whereas, an upper bound was given for the path graphs.
- Exact values for the generalized Petersen graphs $P(n,2), n \geqslant 4$ and $P(n,4)$, $\big(5 \leqslant n \equiv 0 \pmod 3\big)$ were also proven.
- Certain upper & lower bounds for the prism graph, the generalized Petersen graph $P(n,4)$, $\big(5 \leqslant n \equiv 0 \pmod 3\big)$ and two infinite families of strongly-regular graphs were provided.
- Using a modified ILP model, tightness in the obtained upper bounds was shown.
- By studying the binary locating-dominating sets in the complements of all these families, the global binary location-domination number was also studied.

7.2. Implications

The following are some direct implications of this study:

- The results contribute towards a broader domination theory of graphs.

- Constructions in this paper contribute in developing new proof techniques.
- Using CPLEX solver to prove tightness of bounds is novel.
- The results might have strong industrial applications in coding theory and, in general, computer science.

7.3. Limitations

Although, there are no direct limitation of the study. However, the methods have limitations in finding tight lower bounds on the binary location-domination number of non-regular graphs.

7.4. Future study

Based on the study conducted in this paper, we believe that the following conjectures are true:

CONJECTURE 1. *Let P_n denote the n-vertex path graph satisfying $n \geqslant 5$. Then*

$$\gamma_{l-d}(P_n) = \begin{cases} \lceil \frac{2n}{5} \rceil + 1, & n \equiv 0 \pmod 5; \\ \lceil \frac{2n}{5} \rceil, & n \equiv 1,2,3,4 \pmod 5). \end{cases}$$

CONJECTURE 2. *Let P_n denote the n-vertex path graph satisfying $n \geqslant 5$. Then*

$$\gamma_{l-d}(\overline{P_n}) \leqslant \begin{cases} \lceil \frac{2n}{5} \rceil, & n \equiv 0,1,2,4 \pmod 5; \\ \lceil \frac{2n}{5} \rceil - 1, & n \equiv 3 \pmod 5). \end{cases}$$

Moreover, the upper bound is tight.

CONJECTURE 3. *Let G be a generalized Petersen graph $P(n,4)$ and $n \geqslant 5$, then*

$$\gamma_{l-d}(G) = \begin{cases} \lceil \frac{2n}{3} \rceil, & n \equiv 0 \pmod 3; \\ \lceil \frac{2n}{3} \rceil + 2, & n \equiv 1 \pmod 3; \\ \lceil \frac{2n}{3} \rceil + 1, & n \equiv 2 \pmod 3). \end{cases}$$

We also raise the following open problems:

PROBLEM 1. *(i) Study the LD number of the generalized Petersen graphs $P(n,k)$, with $k = 3$ or $k \geqslant 5$.*

(ii) Study the LD number of the circulant graphs.

(iii) Find the LD number of the triangular and square grid graphs.

(iv) Study the LD number of other families of strongly regular graphs such as the Paley graphs and the Johnson graph $J(n,2)$ etc.

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