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STABILITY OF ADDITIVE FUNCTIONAL INEQUALITY IN VARIOUS NORMED SPACES

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Abstract. In this paper, we establish the general solution of the following functional inequality

$$\|2f(x)+2f(y)+2f(z)-f(x+y)-f(y+z)\| \leqslant \|f(x+z)\|,$$

and then investigate the generalized Hyers-Ulam stability of this inequality in Banach spaces and in non-Archimedean Banach spaces by using two different approaches.

1. Introduction

The study of stability problems for functional equations is related to a question of Ulam [36] concerning the stability of group homomorphisms. A functional equation is called stable if any approximate solution to the functional equation is near a true solution of that functional equation. Hyers [15] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Subsequently, Hyers' result was generalized by Aoki [3] and Bourgin [4] for additive mappings and by Rassias [32] for linear mappings by considering the Cauchy difference operator $CDf(x,y) = f(x+y) - [f(x)+f(y)]$ to controlled by $\varepsilon(\|x\|^p + \|y\|^p)$ $(\varepsilon > 0,\ p \in [0,1))$. Gajda [9] answered the question for the case $p > 1$, which was raised by Rassias. This new concept is known as generalized Hyers-Ulam stability or Hyers-Ulam-Rassias stability of functional equations. In 1994, a generalization of the Rassias' theorem was obtained by Găvruţă [10] who permitted the Cauchy difference to become arbitrarily unbounded. Moslehian and Rassias [23] investigated the Hyers-Ulam stability of the Cauchy functional equation and the quadratic functional equation in non-Archimedean normed space. Later, the stability results of various functional equations in non-Archimedean normed spaces have been studied in [13, 14, 22, 24, 25]. During the last decades, the stability of several functional equations has been extensively studied and generalized by a number of authors, and there has been many interesting and applicable results obtained concerning this problem (see [1, 2, 6, 16, 17, 18, 19, 20, 21, 26, 29, 30, 31, 33, 35] and references therein).

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Gilányi [11] and Rätz [34] showed that if f satisfies the functional inequality

$$\|2f(x)+2f(y)-f(xy^{-1})\| \leqslant \|f(xy)\|, \tag{1.1}$$

then f satisfies the Jordan-Von Neumann functional equation

$$2f(x)+2f(y)=f(xy)+f(xy^{-1}). \tag{1.2}$$

Fechner [8] and Gilányi [12] considered the functional inequality (1.1) and proved the generalized Hyers-Ulam stability of this inequality. Park et al. [27] investigated the generalized Hyers-Ulam stability of functional inequalities associated with Jordon-Von Neumann type additive functional equations. In 2008, Park et al. [28] studied the A-linear mapping associated with the following functional inequality

$$\|2f(x)+2f(y)+2f(z)-f(x+y)-f(y+z)\| \leqslant \|f(x+z)\| \tag{1.3}$$

in Banach modules over a C^*-algebra, and then proved the generalized Hyers-Ulam stability of A-linear mappings (1.3) in Banach A-modules associated with the functional inequality (1.3) when f is an odd mapping. Furthermore, they applied these results to investigate homomorphisms in complex Banach algebras and prove the generalized Hyers-Ulam stability of homomorphisms in complex Banach algebras.

The main purpose of this paper is to determine the general solution of the functional inequality (1.3), and then prove the generalized Hyers-Ulam stability of the functional inequality (1.3) in Banach spaces and in non-Archimedean spaces by employing the fixed point and direct methods.

2. Stability of (1.3): Fixed point method

In this section, assume that X is a normed space and Y is a Banach space. We will prove the generalized Hyers-Ulam stability of the functional inequality (1.3) in Banach spaces by using the fixed point method. First, we give the definition of a generalized metric on a set E. A function $d : E\times E \to [0,\infty]$ is called a generalized metric on E if d satisfies the following:

(1) $d(x,y)=0$ if and only if $x=y$;

(2) $d(x,y)=d(y,x)$, $\forall x,y\in E$;

(3) $d(x,z)\leqslant d(x,y)+d(y,z)$, $\forall x,y,z\in E$.

In [7], Diaz and Margolis constructed a method using a fixed point theory, which is extensively applied to the stability theory of functional equations.

LEMMA 2.1. ([7]). *Let (E,d) be a complete generalized metric space. Further let $J : E\to E$ be a strictly contractive mapping with Lipschitz constant $L<1$. Then for each fixed element $x\in E$, either*

$$d(J^n x, J^{n+1}x)=\infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(i) $d(J^n x, J^{n+1} x) < \infty, \quad \forall n \geqslant n_0$;
(ii) *the sequence* $\{J^n x\}$ *is convergent to a fixed point* y^* *of* J;
(iii) y^* *is the unique fixed point of* J *in the set* $E^* := \{y \in E \mid d(J^{n_0} x, y) < +\infty\}$;
(iv) $d(y, y^*) \leqslant \frac{1}{1-L} d(y, Jy), \quad \forall y \in E^*$.

Now, we present the general solution of the functional inequality (1.3) in real vector spaces.

LEMMA 2.2. *Let* V *and* W *be real vector spaces. Let* $f : V \to W$ *be a mapping such that*

$$\|2f(x) + 2f(y) + 2f(z) - f(x+y) - f(y+z)\| \leqslant \|f(x+z)\| \tag{2.1}$$

for all $x, y, z \in V$. *Then the mapping* f *is Cauchy additive.*

Proof. Putting $x = y = z = 0$ in (2.1) yields $\|4f(0)\| \leqslant \|f(0)\|$. So, $f(0) = 0$. Letting $y = 0$ and $z = -x$ in (2.1), we obtain

$$\|f(x) + f(-x)\| \leqslant \|f(0)\| = 0 \tag{2.2}$$

for all $x \in V$. It imply that $f(-x) = -f(x)$ for all $x \in V$. Letting $z = -x$ in (2.1), we get

$$\|2f(y) - f(y+x) - f(y-x)\| \leqslant \|f(0)\| = 0, \tag{2.3}$$

which yields $f(y+x) + f(y-x) = 2f(y)$ for all $x, y \in V$. And we infer that

$$f(x+y) = f(x) + f(y) \tag{2.4}$$

for all $x, y \in V$. Thus the mapping f is additive. $\square$

THEOREM 2.1. *Let* $\varphi : X^3 \to [0, \infty)$ *be a function such that there exists an* $0 < L < 1$ *with*

$$\begin{gathered} \varphi(2x, 2y, 2z) \leqslant 2L\varphi(x, y, z) \\ \left(\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \leqslant \frac{L}{2}\varphi(x, y, z), resp.\right) \end{gathered} \tag{2.5}$$

for all $x, y, z \in X$. *Suppose that* $f : X \to Y$ *is a mapping with* $f(0) = 0$ *and satisfying the functional inequality*

$$\|2f(x) + 2f(y) + 2f(z) - f(x+y) - f(y+z)\| \leqslant \|f(x+z)\| + \varphi(x, y, z) \tag{2.6}$$

for all $x, y, z \in X$. *Then there exists a unique additive mapping* $A : X \to Y$ *such that*

$$\begin{gathered} \|f(x) - A(x)\| \leqslant \frac{1}{2(1-L)}\{\varphi(x, x, -x) + 2\varphi(x, 0, -x)\} \\ \left(\|f(x) - A(x)\| \leqslant \frac{L}{2(1-L)}\{\varphi(x, x, -x) + 2\varphi(x, 0, -x)\}, resp.\right) \end{gathered} \tag{2.7}$$

for all $x \in X$.

Proof. Letting $y=x$ and $z=-x$ in (2.6), we get

$$\|4f(x)+2f(-x)-f(2x)\| \leqslant \varphi(x,x,-x) \tag{2.8}$$

for all $x \in X$. Putting $y=0$ and $z=-x$ in (2.6), we have

$$\|f(x)+f(-x)\| \leqslant \varphi(x,0,-x) \tag{2.9}$$

for all $x \in X$. It follows from (2.8) and (2.9) that

$$\|2f(x)-f(2x)\| \leqslant \varphi(x,x,-x)+2\varphi(x,0,-x) :\equiv \Phi(x) \tag{2.10}$$

and so

$$\left\| f(x)-\frac{f(2x)}{2} \right\| \leqslant \frac{1}{2}\Phi(x) \tag{2.11}$$

for all $x \in X$.

Consider the set $S := \{g|\ g : X \to Y, g(0) = 0\}$, and introduce a generalized metric d on S as follows:

$$d(g,h) := \inf\left\{ \delta \in \mathbb{R}_+ \middle| \|g(x)-h(x)\| \leqslant \delta\Phi(x), \forall x \in X \right\}.$$

It is easy to prove that (S,d) is a complete generalized metric space (cf. [5]). Now we define the mapping $\mathcal{J} : S \to S$ given by

$$\mathcal{J}g(x) := \frac{1}{2}g(2x), \text{ for all } g \in S \text{ and } x \in X. \tag{2.12}$$

Let $g,h \in S$ and let $\delta \in \mathbb{R}_+$ be an arbitrary constant with $d(g,h) \leqslant \delta$. From the definition of d, we obtain

$$\|g(x)-h(x)\| \leqslant \delta\Phi(x)$$

for all $x \in X$. Hence

$$\|\mathcal{J}g(x)-\mathcal{J}h(x)\| \leqslant \frac{\delta}{2}\Phi(2x) \leqslant \delta L\Phi(x). \tag{2.13}$$

for some $L<1$ and for all $x \in X$. Hence, it holds that $d(\mathcal{J}g,\mathcal{J}h) \leqslant \delta L$, that is, $d(\mathcal{J}g,\mathcal{J}h) \leqslant Ld(g,h)$ for all $g,h \in S$.

It follows from (2.11) that $d(f,\mathcal{J}f) \leqslant \frac{1}{2}$ holds. Hence, by Lemma 2.1, the sequence $\mathcal{J}^n f$ converges to a fixed point A of $\mathcal{J}$, that is,

$$A : X \to Y, \quad \lim_{n\to\infty} \frac{1}{2^n} f(2^n x) = A(x)$$

and

$$A(2x) = 2A(x) \tag{2.14}$$

for all $x \in X$. And the mapping A is the unique fixed point of $\mathcal{J}$ in the set $S^* = \{g \in S : d(f,g) < \infty\}$. This implies that A is a unique mapping satisfying (2.14) such that there exists a $\delta \in \mathbb{R}_+$ such that

$$\|f(x) - A(x)\| \leqslant \delta \Phi(x)$$

for all $x \in X$. Hence, we have

$$d(f,A) \leqslant \frac{1}{1-L} d(f,\mathcal{J}f) \leqslant \frac{1}{2(1-L)}.$$

This means that the inequality (2.7) holds.

Next, we verify that the mapping A is additive. It follows from (2.5) and (2.6) that

$$\begin{aligned}
&\|2A(x) + 2A(y) + 2A(z) - f(x+y) - f(y+z)\| \\
&= \lim_{n\to\infty} \frac{1}{2^n} \|2f(2^n x) + 2f(2^n y) + 2f(2^n z) \\
&\quad - f(2^n(x+y)) - f(2^n(y+z))\| \\
&\leqslant \lim_{n\to\infty} \frac{1}{2^n} \|f(2^n(x+z))\| + \lim_{n\to\infty} \frac{1}{2^n} \varphi(2^n x, 2^n y, 2^n z) \\
&\leqslant \lim_{n\to\infty} \frac{1}{2^n} \|f(2^n(x+z))\| + \lim_{n\to\infty} L^n \varphi(x,y,z) \\
&= \|A(x+z)\|
\end{aligned} \tag{2.15}$$

for all $x,y,z \in X$. Thus, by Lemma 2.2, the mapping $A : X \to Y$ is additive, as desired. This completes the proof. $\square$

COROLLARY 2.1. *Let $\theta \geqslant 0$ be a real number and r be a positive real number with $r \neq 1$. If a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\begin{aligned}
&\|2f(x) + 2f(y) + 2f(z) - f(x+y) - f(y+z)\| \\
&\qquad \leqslant \|f(x+z)\| + \theta(\|x\|^r + \|y\|^r + \|z\|^r)
\end{aligned} \tag{2.16}$$

for all $x,y,z \in X$, then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leqslant \frac{7\theta}{|2-2^r|} \|x\|^r \tag{2.17}$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.1 by taking $\varphi(x,y,z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ for all $x,y,z \in X$. Then we can choose $L = 2^{r-1}$ or $L = 2^{1-r}$, and we get the desired result. $\square$

3. Stability of (1.3): Direct method

In this section, we suppose that X is a normed space and Y is a Banach space. We will investigate the generalized Hyers-Ulam stability of the functional inequality (1.3) in Banach spaces by using the direct method.

THEOREM 3.1. *Let $\varphi : X^3 \to [0,\infty)$ be a function such that*

$$\begin{gathered}\sum_{j=0}^{\infty}\frac{1}{2^j}\varphi(2^jx,2^jy,2^jz)<\infty,\\ \Big(\sum_{j=1}^{\infty}2^j\varphi\Big(\frac{x}{2^j},\frac{y}{2^j},\frac{z}{2^j}\Big)<\infty, resp.\Big)\end{gathered}\tag{3.1}$$

for all $x,y,z \in X$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$\|2f(x)+2f(y)+2f(z)-f(x+y)-f(y+z)\| \leqslant \|f(x+z)\|+\varphi(x,y,z) \tag{3.2}$$

for all $x,y,z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\begin{gathered}\|f(x)-A(x)\| \leqslant \frac{1}{2}\sum_{j=0}^{\infty}\frac{1}{2^j}\{\varphi(2^jx,2^jx,-2^jx)+2\varphi(2^jx,0,-2^jx)\},\\ \Big(\|f(x)-A(x)\| \leqslant \frac{1}{2}\sum_{j=1}^{\infty}2^j\Big\{\varphi\Big(\frac{x}{2^j},\frac{x}{2^j},-\frac{x}{2^j}\Big)+2\varphi\Big(\frac{x}{2^j},0,-\frac{x}{2^j}\Big)\Big\}, resp.\Big)\end{gathered}\tag{3.3}$$

for all $x \in X$.

Proof. According to (2.10), we obtain

$$\|f(x)-\frac{f(2x)}{2}\| \leqslant \frac{1}{2}\{\varphi(x,x,-x)+2\varphi(x,0,-x)\} \tag{3.4}$$

for all $x \in X$. Then, It follows from (3.4) that for all nonnegative integers n and m with $n > m$

$$\begin{aligned}\left\|\frac{f(2^mx)}{2^m}-\frac{f(2^nx)}{2^n}\right\| &\leqslant \sum_{j=m}^{n-1}\frac{1}{2^j}\left\|f(2^jx)-\frac{f(2^{j+1}x)}{2}\right\|\\ &\leqslant \frac{1}{2}\sum_{j=m}^{n-1}\frac{1}{2^j}\{\varphi(2^jx,2^jx,-2^jx)+2\varphi(2^jx,0,-2^jx)\}\end{aligned}\tag{3.5}$$

for all $x \in X$. It means that the sequence $\{\frac{f(2^nx)}{2^n}\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, it follows from that the sequence $\{\frac{f(2^nx)}{2^n}\}$ converges in Y. Therefore, one can define a mapping $A : X \to Y$ by $A(x) := \lim_{n\to\infty}\frac{f(2^nx)}{2^n}$ for all $x \in X$. Moreover, letting $m = 0$ and taking the limit $n \to \infty$ in (3.5), we get the approximation (3.3) of f by A, as desired.

Next, we claim that the mapping $A: X \to Y$ is additive. In fact, it follows from (3.1) and (3.2) that

$$\begin{aligned}
&\|2A(x)+2A(y)+2A(z)-A(x+y)-f(y+z)\| \\
&= \lim_{n\to\infty} \frac{1}{2^n}\|2f(2^n x)+2f(2^n y)+2f(2^n z) \\
&\quad - f(2^n(x+y)) - f(2^n(y+z))\| \\
&\leqslant \lim_{n\to\infty} \frac{1}{2^n}\|f(2^n(x+z))\| + \lim_{n\to\infty}\frac{1}{2^n}\varphi(2^n x, 2^n y, 2^n z) \\
&= \|A(x+z)\|. \qquad (3.6)
\end{aligned}$$

Thus, the mapping $A: X \to Y$ is additive by Lemma 2.2.

Finally, we show that the uniqueness of A. Let $A': X \to Y$ is another additive mapping satisfying (3.3). Then, we obtain

$$\begin{aligned}
\|A(x)-A'(x)\| &= \frac{1}{2^n}\|A(2^n x)-A'(2^n x)\| \\
&\leqslant \frac{1}{2^n}(\|A(2^n x)-f(2^n x)\| + \|A'(2^n x)-f(2^n x)\|) \\
&\leqslant \sum_{j=0}^{\infty} \frac{1}{2^{j+n}}\{\varphi(2^{j+n}x, 2^{j+n}x, -2^{j+n}x) + 2\varphi(2^{j+n}x, 0, -2^{j+n}x)\} \\
&= \sum_{j=n}^{\infty}\frac{1}{2^j}\{\varphi(2^j x, 2^j x, -2^j x) + 2\varphi(2^j x, 0, -2^j x)\} \qquad (3.7)
\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. Hence $A(x) = A'(x)$ for all $x \in X$. This completes the proof of the theorem. □

COROLLARY 3.1. *Let $\theta_i \geqslant 0$ be a real number and r_i be a positive real numbers with $r_i < 1$ or $r_i > 1$ for all $i = 1,2,3$. If a mapping $f: X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\begin{aligned}
&\|2f(x)+2f(y)+2f(z)-f(x+y)-f(y+z)\| \\
&\qquad \leqslant \|f(x+z)\| + \theta_1\|x\|^{r_1} + \theta_2\|y\|^{r_2} + \theta_3\|z\|^{r_3} \qquad (3.8)
\end{aligned}$$

for all $x,y,z \in X$, then there exists a unique additive mapping $A: X \to Y$ such that

$$\|f(x)-A(x)\| \leqslant \frac{3\theta_1}{|2-2^{r_1}|}\|x\|^{r_1} + \frac{\theta_2}{|2-2^{r_2}|}\|x\|^{r_2} + \frac{3\theta_3}{|2-2^{r_3}|}\|x\|^{r_3} \qquad (3.9)$$

for all $x \in X$.

4. Stability of (1.3) in non-Archimedean spaces: Fixed point method

In this section, we will prove the stability of the functional inequality (1.3) in non-Archimedean Banach spaces by using the fixed point method. Now, we first recall some basic facts concerning non-Archimedean Banach space and some preliminary results.

By a non-Archimedean field we mean a field $\mathbb{K}$ equipped with a function (valuation) $|\cdot|$ from $\mathbb{K}$ into $[0,\infty)$ such that $|r|=0$ if and only if $r=0$, $|rs|=|r||s|$, and $|r+s|\leqslant \max\{|r|,|s|\}$ for $r,s\in\mathbb{K}$. Clearly $|1|=|-1|=1$ and $|n|\leqslant 1$ for all $n\in\mathbb{N}$. By the trivial valuation we mean the function $|\cdot|$ taking everything but 0 into 1 and $|0|=0$ (i.e., the function $|\cdot|$ is called the trivial valuation if $|r|=1,\forall r\in\mathbb{K}, r\neq 0$, and $|0|=0$).

DEFINITION 4.1. ([13, 23]). Let X be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$. A function $\|\cdot\|: X\to\mathbb{R}$ is called a non-Archimedean norm (valuation) if it satisfies the following conditions:

(i) $\|x\|=0$ if and only if $x=0$;

(ii) $\|rx\|=|r|\|x\|$ for all $r\in\mathbb{K}$ and $x\in X$;

(iii) The strong triangle inequality; namely,

$$\|x+y\|\leqslant \max\{\|x\|,\|y\|\}$$

for all $x,y\in X$.

Then $(X,\|\cdot\|)$ is called a non-Archimedean normed space.

Due to the fact that

$$\|x_n-x_m\|\leqslant \max\{\|x_{j+1}-x_j\|: m\leqslant j\leqslant n-1\},\ (n>m),$$

a sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1}-x_n\}$ converges to zero in a non-Archimedean normed space. By a complete non-Archimedean normed space we mean one in which every Cauchy sequence is convergent.

From now on, unless otherwise stated, we suppose that X is a non-Archimedean normed space and that Y is a non-Archimedean Banach space.

THEOREM 4.1. *Let $\varphi: X^3\to[0,\infty)$ be a function such that there exists an $0<L<1$ with*

$$\begin{gathered}\varphi(2x,2y,2z)\leqslant |2|L\varphi(x,y,z)\\ \left(\varphi\left(\frac{x}{2},\frac{y}{2},\frac{z}{2}\right)\leqslant \frac{L}{|2|}\varphi(x,y,z), resp.\right)\end{gathered}\tag{4.1}$$

for all $x,y,z\in X$. Suppose that $f: X\to Y$ is a mapping with $f(0)=0$ and satisfying the functional inequality

$$\|2f(x)+2f(y)+2f(z)-f(x+y)-f(y+z)\|\leqslant \|f(x+z)\|+\varphi(x,y,z)\tag{4.2}$$

for all $x,y,z\in X$. Then there exists a unique additive mapping $A: X\to Y$ such that

$$\begin{gathered}\|f(x)-A(x)\|\leqslant \frac{1}{|2|(1-L)}\Phi(x)\\ \left(\|f(x)-A(x)\|\leqslant \frac{L}{|2|(1-L)}\Phi(x), resp.\right)\end{gathered}\tag{4.3}$$

for all $x\in X$, where $\Phi(x):=\max\{\varphi(x,x,-x),|2|\varphi(x,0,-x)\}$ for all $x\in X$.

Proof. It follows from (2.8) and (2.9) that

$$\begin{aligned}\|f(2x)-2f(x)\| &\leqslant \max\{\|4f(x)+2f(-x)-f(2x)\|, |2|\|f(x)+f(-x)\|\} \\ &\leqslant \max\{\varphi(x,x,-x), |2|\varphi(x,0,-x)\|\} :\equiv \Phi(x)\end{aligned} \tag{4.4}$$

for all $x \in X$. Therefore, we get

$$\|f(x)-\frac{1}{2}f(2x)\| \leqslant \frac{1}{|2|}\Phi(x) \tag{4.5}$$

for all $x \in X$. Applying the similar argument to the corresponding proof of Theorem 2.1 on the complete generalized metric space (S,d), we obtain the desired result. □

COROLLARY 4.1. *Let $\theta \geqslant 0$ be a real number and r be a positive real number with $r \neq 1$. If a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\begin{aligned}\|2f(x)+2f(y)+&2f(z)-f(x+y)-f(y+z)\| \\ &\leqslant \|f(x+z)\|+\theta(\|x\|^r+\|y\|^r+\|z\|^r)\end{aligned} \tag{4.6}$$

for all $x,y,z \in X$, then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x)-A(x)\| \leqslant \frac{\max\{3,2|2|\}}{||2|-|2|^r|}\theta\|x\|^r \tag{4.7}$$

for all $x \in X$.

5. Stability of (1.3) in non-Archimedean spaces: Direct method

In this section, we will prove the stability of the functional inequality (1.3) in non-Archimedean Banach spaces by the direct method.

THEOREM 5.1. *Let $\varphi : X^3 \to [0,\infty)$ be a function such that*

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{|2|^n}\varphi(2^n x, 2^n y, 2^n z) = 0, \\ &\left(\lim_{n\to\infty}|2|^n\varphi\left(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\right) = 0, resp.\right)\end{aligned} \tag{5.1}$$

for all $x,y \in X$ and the limit

$$\begin{aligned}&\tilde{\varphi}(x) = \lim_{n\to\infty}\max\left\{\frac{1}{|2|^k}\Phi(2^k x) : 0 \leqslant k < n\right\}, \\ &\left(\tilde{\varphi}(x) = \lim_{n\to\infty}\max\left\{|2|^k\Phi\left(\frac{x}{2^k}\right) : 1 \leqslant k < n+1\right\}, resp.\right)\end{aligned} \tag{5.2}$$

exists for all $x \in X$, where

$$\Phi(x) := \max\{\varphi(x,x,-x), |2|\varphi(x,0,-x)\} \tag{5.3}$$

for all $x \in X$. *Suppose that* $f : X \to Y$ *is a mapping with* $f(0) = 0$ *and satisfying the functional inequality*

$$\|2f(x)+2f(y)+2f(z)-f(x+y)-f(y+z)\| \leqslant \|f(x+z)\|+\varphi(x,y,z) \tag{5.4}$$

for all $x,y,z \in X$. *Then there exists an additive mapping* $A : X \to Y$ *such that*

$$\|f(x)-A(x)\| \leqslant \frac{1}{|2|}\tilde{\varphi}(x) \tag{5.5}$$

for all $x \in X$. *Moreover, if*

$$\begin{gathered}\lim_{i\to\infty}\lim_{n\to\infty}\max\left\{\frac{1}{|2|^k}\Phi(2^k x) : i \leqslant k < n+i\right\} = 0,\\ \left(\lim_{i\to\infty}\lim_{n\to\infty}\max\left\{|2|^k\Phi(\frac{x}{2^k}) : i+1 \leqslant k < n+i+1\right\} = 0, resp.\right)\end{gathered} \tag{5.6}$$

for all $x,y \in X$, *then* A *is the unique additive mapping satisfying* (5.5).

Proof. Replacing x by $2^n x$ and dividing the both sides of (4.5) by $|2|^n$, we obtain

$$\left\|\frac{f(2^{n+1}x)}{2^{n+1}}-\frac{f(2^n x)}{2^n}\right\| \leqslant \frac{1}{|2|}\frac{1}{|2|^n}\Phi(2^n x) \tag{5.7}$$

for all $x \in X$. It follows from (5.1) and (5.7) that the sequence $\{\frac{f(2^n x)}{2^n}\}$ is Cauchy for all $x \in X$. Since Y is the non-Archimedean Banach space, we conclude that $\{\frac{f(2^n x)}{2^n}\}$ is convergent. Hence, we can define a mapping $A : X \to Y$ as

$$A(x) = \lim_{n\to\infty}\frac{f(2^n x)}{2^n}$$

for all $x \in X$.

Using induction one can show that

$$\left\|f(x)-\frac{f(2^n x)}{2^n}\right\| \leqslant \frac{1}{|2|}\max\left\{\frac{1}{|2|^k}\Phi(2^k x) : 0 \leqslant k < n\right\} \tag{5.8}$$

for all $n \in \mathbb{N}$ and all $x \in X$. By taking n to approach infinity in (5.8) and using (5.2) one obtain (5.5). By (5.1) and (5.4), we get

$$\begin{aligned}&\|2A(x)+2A(y)+2A(z)-f(x+y)-f(y+z)\|\\ &= \lim_{n\to\infty}\frac{1}{|2|^n}\|2f(2^n x)+2f(2^n y)+2f(2^n z)\\ &\quad -f(2^n(x+y))-f(2^n(y+z))\|\\ &\leqslant \lim_{n\to\infty}\frac{1}{|2|^n}\|f(2^n(x+z))\|+\lim_{n\to\infty}\frac{1}{|2|^n}\varphi(2^n x,2^n y,2^n z)\\ &= \|A(x+z)\|\end{aligned} \tag{5.9}$$

for all $x,y,z \in X$. Thus, by Lemma 2.2, the mapping $A : X \to Y$ is additive. To prove the uniqueness property of A, let A' be another additive mapping satisfying (5.5). Then

$$\begin{aligned}\|A(x) - A'(x)\| &= \lim_{i\to\infty} |2|^{-i}\|A(2^i x) - A'(2^i x)\| \\ &\leqslant \lim_{i\to\infty} |2|^{-i} \max\{\|A(2^i x) - f(2^i x)\|, \|f(2^i x) - A'(2^i x)\|\} \\ &\leqslant \frac{1}{|2|} \lim_{i\to\infty}\lim_{n\to\infty} \max\left\{\frac{\Phi(2^k x)}{|2|^k} : i \leqslant k < n+i\right\} \\ &= 0\end{aligned}$$

for all $x \in X$. Therefore, we have $A = A'$, and the proof is complete. □

COROLLARY 5.1. *Let $\rho : [0,\infty) \to [0,\infty)$ be function satisfying*
(i) $\rho(|2|t) \leqslant \rho(|2|)\rho(t)$ *for all* $t \geqslant 0$,
(ii) $\rho(|2|) \leqslant |2|^{\lambda}$, *where λ is a fixed real number in $\lambda \in [1,\infty)$.*
Let $\delta > 0$, and let $f : X \to Y$ be a mapping with $f(0) = 0$ and satisfying the functional inequality

$$\begin{aligned}&\|2f(x) + 2f(y) + 2f(z) - f(x+y) - f(y+z)\| \\ &\qquad \leqslant \|f(x+z)\| + \delta[\rho(\|x\|) + \rho(\|y\|) + \rho(\|z\|)]\end{aligned} \tag{5.10}$$

for all $x,y,z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leqslant \frac{1}{|2|}\max\{3, 2|2|\}\delta\rho(\|x\|) \tag{5.11}$$

for all $x \in X$.

Proof. Defining $\varphi : X^3 \to [0,\infty)$ by

$$\varphi(x,y) := \delta[\rho(\|x\|) + \rho(\|y\|) + \rho(\|z\|)].$$

Since $|2|^{-1}\rho(|2|) < |2|^{\lambda-1} \leqslant 1$, we have

$$\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y, 2^n z)}{|2|^n} \leqslant \lim_{n\to\infty} \left(\frac{\rho(|2|)}{|z|}\right)^n \varphi(x,y,z) = 0$$

for all $x,y,z \in X$. It follows from (5.3) that

$$\begin{aligned}\Phi(x) :&= \max\{\varphi(x,x,-x), |2|\varphi(x,0,-x)\} \\ &= \max\{3, 2|2|\}\delta\rho(\|x\|)\end{aligned} \tag{5.12}$$

for all $x \in X$. By direct calculation,

$$\tilde{\varphi}(x) = \lim_{n\to\infty} \max\left\{\frac{1}{|2|^k}\Phi(2^k x) : 0 \leqslant k < n\right\} = \Phi(x), \tag{5.13}$$

exists and

$$\lim_{i\to\infty}\lim_{n\to\infty}\max\left\{\frac{1}{|2|^k}\Phi(2^k x): i\leqslant k<n+i\right\}=\lim_{i\to\infty}\frac{1}{|2|^i}\Phi(2^i x)=0 \tag{5.14}$$

holds for all $x\in X$. Applying Theorem 5.1, we infer that

$$\begin{aligned}\|f(x)-A(x)\| &\leqslant \frac{1}{|2|}\tilde{\varphi}(x)=\frac{1}{|2|}\Phi(x)\\ &=\frac{1}{|2|}\max\{3,2|2|\}\delta\rho(\|x\|)\end{aligned} \tag{5.15}$$

for all $x\in X$, and the proof is complete. □

COROLLARY 5.2. *Let $\omega:[0,\infty)\to[0,\infty)$ be function satisfying*
(i) $\omega(|2|^{-1}t)\leqslant\omega(|2|^{-1})\omega(t)$ *for all* $t\geqslant 0$,
(ii) $\omega(|2|^{-1})\leqslant|2|^{-\mu}$, *where μ is a fixed real number in $\mu\in(-\infty,1]$.*
Let $\delta>0$, and let $f:X\to Y$ be a mapping with $f(0)=0$ and satisfying the functional inequality

$$\begin{aligned}\|2f(x)+2f(y)+2f(z)&-f(x+y)-f(y+z)\|\\ &\leqslant\|f(x+z)\|+\delta[\omega(\|x\|)+\omega(\|y\|)+\omega(\|z\|)]\end{aligned} \tag{5.16}$$

for all $x,y,z\in X$. Then there exists a unique additive mapping $A:X\to Y$ such that

$$\|f(x)-A(x)\|\leqslant\frac{1}{|2|^{\mu}}\max\{3,2|2|\}\delta\omega(\|x\|) \tag{5.17}$$

for all $x\in X$.

Proof. Let $\varphi:X^3\to[0,\infty)$ defined by

$$\varphi(x,y,z):=\delta[\omega(\|x\|)+\omega(\|y\|)+\omega(\|z\|)].$$

Since $|2|\omega(|2|^{-1})<|2|^{1-\mu}\leqslant 1$, we have

$$\lim_{n\to\infty}|2|^n\varphi\left(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\right)\leqslant\lim_{n\to\infty}|2|^n\omega(|2|^{-1})^n\varphi(x,y,z)=0$$

for all $x,y,z\in X$. Also

$$\tilde{\varphi}(x)=\lim_{n\to\infty}\max\left\{|2|^k\Phi(\frac{x}{2^k}):1\leqslant k<n+1\right\}=|2|\Phi\left(\frac{x}{2}\right)$$

and

$$\lim_{i\to\infty}\lim_{n\to\infty}\max\left\{|2|^k\Phi(\frac{x}{2^k}):i+1\leqslant k<n+i+1\right\}=\lim_{i\to\infty}|2|^{i+1}\Phi\left(\frac{x}{2^{i+1}}\right)=0$$

for all $x\in X$. Hence the result follows by Theorem 5.1. □

REFERENCES

[1] M. R. ABDOLLAHPOUR, R. AGHAYARIA AND M. TH. RASSIAS, *Hyers-Ulam stability of associated Laguerre differential equations in a subclass of analytic functions*, J. Math. Anal. Appl. **437** (2016), 605–612.

[2] M. R. ABDOLLAHPOUR AND M. TH. RASSIAS, *Hyers-Ulam stability of hypergeometric differential equations*, Aequationes Math. **93** (2019), 691–698.

[3] T. AOKI, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan, **2** (1950), 64–66.

[4] D. G. BOURGIN, *Classes of transformations and bordering transformations*, Bull. Amer. Math. Soc. **57** (1951), 223–237.

[5] L. CĂDARIU AND V. RADU, *On the stability of the Cauchy functional equation: A fixed point approach*, Grazer Math. Ber. **346** (2004), 43–52.

[6] S. CZERWIK, *Functional Equations and Inequalities in Several Variables*, World Scientific, 2002.

[7] J. B. DIAZ AND B. MARGOLIS, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc. **74** (1968), 305–309.

[8] W. FECHNER, *Stability of a functional inequality associated with the Jordan-von Neumann functional equation*, Aequationes Math. **71** (2006), 149–161.

[9] Z. GAJDA, *On stability of additive mappings*, Internat. J. Math. Math. Sci. **14** (1991), 431–434.

[10] P. GĂVRUŢĂ, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.

[11] A. GILÁNYI, *Eine zur Parallelogrammgleichung äquivalente Ungleichung*, Aequationes Math. **62** (2001), 303–309.

[12] A. GILÁNYI, *On a problem by K. Nikodem*, Math. Ineq. Appl. **5** (2002), 707–710.

[13] M. E. GORDJI AND M. B. SAVADKOUHI, *Stability of a mixed type cubic-quartic functional equation in non-Archimedean spaces*, Appl. Math. Lett. **23** (2010), 1198–1202.

[14] M. E. GORDJI AND M. B. SAVADKOUHI, *Stability of cubic and quartic functional equations in non-Archimedean spaces*, Acta Appl. Math. **110** (2010), 1321–1329.

[15] D. H. HYERS, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A. **27** (1941), 222–224.

[16] D. H. HYERS, G. ISAC AND TH. M. RASSIAS, *Stability of Functional Equations in Several variables*, Birkhäuser, Basel, 1998.

[17] S. M. JUNG, *Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis*, Springer Science, New York, 2011.

[18] S. M. JUNG, M. TH. RASSIAS AND C. MORTICI, *On a functional equation of trigonometric type*, Appl. Math. Comput. **252** (2015), 294–303.

[19] PL. KANNAPPAN, *Functional Equations and Inequalities with Applications*, Springer Science, New York, 2009.

[20] S. O. KIM, A. BODAGHI AND C. PARK, *Stability of functional inequalities associated with the Cauchy-Jensen additive functional equalities in non-Archimedean Banach spaces*, J. Nonlinear Sci. Appl. **8** (2015), 776–786.

[21] Y. H. LEE, S. M. JUNG AND M. TH. RASSIAS, *Uniqueness theorems on functional inequalities concerning cubic-quadratic-additive equation*, J. Math. Inequal. **12** (2018), 43–61.

[22] A. K. MIRMOSTAFAEE, *Approximately additive mappings in non-Archimedean normed spaces*, Bull. Korean Math. Soc. **46** (2009), 387–400.

[23] M. S. MOSLEHIAN AND TH. M. RASSIAS, *Stability of functional equations in non-Archimedean spaces*, Appl. Anal. Discrete Math. **1** (2007), 325–334.

[24] M. S. MOSLEHIAN AND GH. SADEGHI, *A Mazur-Ulam theorem in non-Archimedean normed spaces*, Nonlinear Anal. **69** (2008), 3405–3408.

[25] A. NAJATI AND F. MORADLOU, *Hyers-Ulam-Rassias stability of the Apollonius type quadratic mapping in non-Archimedean spaces*, Tamsui Oxf. J. Math. Sci. **24** (2008), 367–380.

[26] A. PANSUWAN, W. SINTUNAVARAT, C. Y. CHOI AND Y. J. CHO, *Ulam-Hyers stability, well-posedness and limit shadowing property of the fixed point problems in M-metric spaces*, J. Nonlinear Sci. Appl. **9** (2016), 4489–4499.

[27] C. PARK, Y. S. CHO AND M.-H. HAN, *Functional inequalities associated with Jordan-von Neumann-type additive functional equations*, J. Ineq. Appl. **2007** (2007), Article ID 41820, 13 pages.

[28] C. PARK, J. S. AN AND F. MORADLOU, *Additive functional inequalities in Banach modules*, J. Ineq. Appl. **2008** (2008), Article ID 592504, 10 pages.
[29] C. PARK, *Additive ρ-functional inequalities in non-Archimedean normed spaces*, J. Math. Inequal. **9** (2015), 397–407.
[30] C. PARK, *Cubic and quartic ρ-functional inequalities in fuzzy Banach spaces*, J. Math. Inequal. **10** (2016), 1123–1136.
[31] C. PARK AND M. TH. RASSIAS, *Additive functional equations and partial multipliers in C^*-algebras*, Revista de la Real Academia de Ciencias Exactas, Serie A. Matemáticas, **113** (2019), 2261–2275.
[32] TH. M. RASSIAS, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.
[33] TH. M. RASSIAS, *Functional Equations, Inequalities and Applications*, Kluwer Academic, Dordrecht, 2003.
[34] J. RÄTZ, *On inequalities associated with the Jordan-von Neumann functional equation*, Aequationes Math. **66** (2003), 191–200.
[35] P. K. SAHOO AND PL. KANNAPPAN, *Introduction to Functional Equations*, CRC Press, Boca Raton, 2011.
[36] S. M. ULAM, *Problems in Modern Mathematics, Chapter VI*, Science Editions, Wiley, New York, 1964.

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