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SOME INEQUALITIES ON THE CONVERGENT ABSCISSAS OF LAPLACE–STIELTJES TRANSFORMS

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Abstract. By making use of the basic properties of Laplace-Stieltjes transform, we establish some inequalities concerning the abscissa of convergence, the abscissa of absolute convergence and the abscissa of uniform convergence of Laplace-Stieltjes transform $\int_0^\infty e^{st}d\alpha(t)$. Moreover, we point out that our formulas for the three abscissas of convergence are consistent with the previous results given by Yu [26] for Laplace-Stieltjes transform under some conditions.

1. Introduction and some basic lemmas

Let $\alpha(x)$ be a bounded variation on any finite interval $[0,Y]$ $(0<Y<+\infty)$ and $\sigma,t\in\mathbb{R}$, then we call the following integral

$$\int_0^{+\infty} e^{sx}d\alpha(x), \qquad s=\sigma+it, \tag{1}$$

as a Laplace-Stieltjes transform, which is first named for Pierre-Simon Laplace and Thomas Joannes Stieltjes. Clearly, we can rewrite (1) as the form

$$\int_0^{+\infty} e^{st}d\alpha(t)=\lim_{R\to+\infty}\int_0^R e^{st}d\alpha(t). \tag{2}$$

If this limit in (2) exists for a given s, we say that the integral (1) converges for the value s, denote

$$F(s)=\int_0^{+\infty} e^{st}d\alpha(t). \tag{3}$$

If the limit (2) does not exist, then we say that the integral (1) diverges. If $\alpha(t)$ satisies

$$\begin{cases} \alpha(t)=a_1+a_2+\cdots+a_n, & \lambda_n<t<\lambda_{n+1},\\ \alpha(t)=0, & t=0,\\ \alpha(t)=\dfrac{\alpha(t+)+\alpha(t-)}{2}, & t>0, \end{cases} \tag{4}$$

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where

$$0 \leqslant \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots, \quad \lambda_n \to +\infty (n \to +\infty). \tag{5}$$

then (1) can become the Dirichlet series

$$f(s) = \sum_{n=1}^{+\infty} a_n e^{\lambda_n s}. \tag{6}$$

As is known to all, the study of the Laplace-Stieltjes transforms can be tracked back to about ninety years ago or even earlier (see [4, 18]). Nowadays, we can find that Laplace-Stieltjes transforms are widely and closely related to functional equation and analysis, theoretical and applied probability, optimization and queuing theory, machinery and control theory and so on.

In 1963, Yu [26] considered the convergence of Laplace-Stieltjes transforms

$$\int_0^{+\infty} e^{-sx} d\alpha(x),$$

which is different from (1), and obtained the Valiron-Knopp-Bohr formula concerning the associated abscissas of bounded convergence, absolute convergence and uniform convergence of Laplace-Stieltjes transforms. After that, many scholars had paid lots of attention to exploring the growth and the value distribution of entire (analytic) functions defined by such Laplace-Stieltjes transforms converges in the whole (half) plane, and obtained a number of important and meaningful results (see [1, 2, 3, 11, 16, 5]). The results about the singular points (singular directions) of analytic functions (entire functions) represented by such Laplace-Stieltjes transforms can be found in [13, 14, 24, 25]; and a series of results of the growth of such Laplace-Stieltjes transforms can refer to Refs. [1, 6, 7, 8, 12, 15, 19]. Since 2012, M. M. Sheremeta,Y. Y. Kong, X. Luo, H. Y. Xu have studied the growth of Laplace-Stieltjes transform (1), by applying the maximum term and the centre indexes of maximum terms of (1) (see [10, 9, 22]); G. S. Srivastava, Y. Y. Kong, S. Y. Liu, etc. have discussed the approximation of Laplace-Stieltjes transform (1), and obtained a series of the condition of equivalence concerning the error, the coefficients and the growth indexes (see [17, 20, 21, 23]).

However, to the knowledge of authors, there were seldom references focusing on the convergence of Laplace-Stieltjes transform (1)*. Naturally, this paper is devoted to investigate this problem, which will give an supplement of those references mentioned in the above.*

The paper is organized as follows. In Section 2, we will introduce three definitions of the convergent abscissas of Laplace-Stieltjes transform (1) including the convergent abscissas, absolute convergent abscissas and uniform convergent abscissas. In Section 3 we analyze the relationship between three convergent abscissas, and point out that our formulas of the convergent abscissas are consistent with the Valiron-Knopp-Bohr formula given by Yu [26]. The calculation formulas and properties of the three convergent abscissas will be discussed in Section 4, Section 5 and Section 6, respectively.

Before the formal discussion, we first introduce some lemmas on the basic properties of Laplace-Stieltjes, which are used in this paper.

LEMMA 1.1. (see [18]). *If $f(x)$ is of bounded variation and α is continuous in (a,b), then the Stieltjes integral of $f(x)$ with respect to $\alpha(x)$ from a to b exists and*

$$\int_a^b f(x)d\alpha(x) = f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b \alpha(x)df(x).$$

LEMMA 1.2. (see [18]). *If $f(x)$ is continuous and α is of bounded variation in $[a,b]$, then*

$$G(x) = \int_a^x f(t)d\alpha(t), \qquad a \leqslant x \leqslant b$$

is also of bounded variation in $[a,b]$ and

$$V_G(x) \leqslant \int_a^x |f(t)||d\alpha(t)|,$$

where $V_G(x)$ is the total variation of $G(x)$ in $[0,x]$, and

$$G(x+) - G(x) = f(x)[\alpha(x+) - \alpha(x)], \quad (a \leqslant x < b),$$

$$G(x) - G(x-) = f(x)[\alpha(x) - \alpha(x-)], \quad (a < x \leqslant b).$$

LEMMA 1.3. (see [18]). *If $f(x)$ and $\varphi(x)$ are continuous and α is of bounded variation in $[a,b]$, and if*

$$\beta(x) = \int_0^x \varphi(t)d\alpha(t), \ \ (a \leqslant x \leqslant b),$$

then

$$\int_a^b f(x)d\beta(x) = \int_a^b f(x)\varphi(x)d\alpha(x).$$

LEMMA 1.4. (see [18]). *If $f(x)$ is continuous and α is of bounded variation in $[a,b]$, then for any c in (a,b),*

$$\int_a^b f(x)d\alpha(x) = \int_a^c f(x)d\alpha(x) + \int_c^b f(x)d\alpha(x),$$

$$\left|\int_a^b f(x)d\alpha(x)\right| \leqslant \int_a^b |f(x)|dV_\alpha(x) \leqslant \max_{a\leqslant x\leqslant b} |f(x)|V_\alpha(b),$$

where $V_\alpha(x)$ is the total variation of $\alpha(x)$ in $[a,x]$.

2. Some concepts of abscissas of convergence of (1)

Let σ_c^F, σ_a^F, σ_u^F be the abscissas of convergence, absolute convergence and uniform convergence of Laplace-Stieltjes transform (1), respectively, which are defined by

$$\sigma_c^F = \sup\{\sigma_0 : (1) \ \textit{converges in } \Re s < \sigma_0, \ \sigma_0 \in R\},$$

$$\sigma_a^F = \sup\{\sigma_1 : (1)\ \textit{converges absolutely in}\ \Re s < \sigma_1,\ \sigma_1 \in R\},$$
$$\sigma_u^F = \sup\{\sigma_2 : (1)\ \textit{converges uniformly in}\ \Re s < \sigma_2,\ \sigma_2 \in R\}.$$

For the abscissas σ_c^F of convergence of Laplace-Stieltjes transform (1), we have

THEOREM 2.1. *Let* $s_0 = \sigma_0 + i\tau_0$ *and*

$$\beta(u) = \int_0^u e^{s_0 t}\, d\alpha(t), \quad (u \geqslant 0). \tag{7}$$

If

$$\sup_{0\leqslant u<+\infty} |\beta(u)| = \sup_{0\leqslant u<+\infty} \left| \int_0^u e^{s_0 t}\, d\alpha(t) \right| = M < \infty, \tag{8}$$

then (1) *converges for every* $s = \sigma + i\tau$ *such that* $\sigma < \sigma_0$*, and*

$$\int_0^{+\infty} e^{st}\, d\alpha(t) = (s_0 - s) \int_0^{+\infty} e^{st} \beta(t)\, dt, \tag{9}$$

the integral on the right hand side of (9) *converges absolutely.*

Proof. By Lemma 1.3, we have

$$\int_0^R e^{st}\, d\alpha(t) = \int_0^R e^{(s-s_0)t}\, d\beta(t). \tag{10}$$

With the analysis of integration by part, it follows that

$$\int_0^R e^{st}\, d\alpha(t) = e^{(s-s_0)R}\beta(R) - (s - s_0) \int_0^R e^{(s-s_0)t}\beta(t)\, dt. \tag{11}$$

Since

$$\left| e^{(s-s_0)R}\beta(R) \right| = e^{(\sigma-\sigma_0)R} |\beta(R)|,$$

thus by combining with (8), we can get

$$\lim_{R\to+\infty} e^{(s-s_0)R}\beta(R) = 0, \quad \sigma < \sigma_0. \tag{12}$$

On the other hand, the condition (8) can lead to

$$\left| \int_0^{+\infty} e^{(s-s_0)t}\beta(t)\, dt \right| \leqslant \int_0^{+\infty} e^{(\sigma-\sigma_0)t} |\beta(t)||dt| \leqslant M \int_0^{+\infty} e^{(\sigma-\sigma_0)t}\, dt = \frac{M}{\sigma_0 - \sigma}.$$

This shows that the integral $\int_0^{+\infty} e^{(s-s_0)t}\beta(t)\,dt$ converges absolutely for $\sigma < \sigma_0$.

Therefore, this completes the proof of Theorem 2.1. □

From Theorem 2.1, we have

COROLLARY 2.1. *If* (1) *converges for* $s_0 = \sigma_0 + i\tau_0$*, then* (1) *converges for all* $s = \sigma + i\tau$ *satisfying* $\sigma < \sigma_0$*.*

REMARK 2.1. From Corollary 2.1, we can see that the convergence region of (1) must be a half plane or whole plane. For example, the abscissa of convergence $\int_0^{+\infty} e^{st} e^{e^t} dt$ is $-\infty$, the abscissa of convergence $\int_0^{+\infty} e^{st} e^{-e^t} dt$ is $+\infty$, the abscissa of convergence $\int_0^{+\infty} e^{st} dt$ is 0.

Let $V(x)$ denote the total variation of $\alpha(x)$ in $[0,x]$. If $s=\sigma+i\tau$ and

$$\int_0^{+\infty} e^{\sigma t}|d\alpha(t)| = \int_0^{+\infty} e^{\sigma t} dV(x)$$

converges, then we say that Laplace-Stieltjes transform (1) converges absolutely at $s = \sigma + i\tau$. Similarly, if $\alpha(t)$ is stated as in (4), and the series $\sum\limits_{n=1}^{+\infty} |a_n| e^{\lambda_n \sigma}$ converges, then we say that Dirichlet series $\sum\limits_{n=1}^{+\infty} a_n e^{\lambda_n s}$ converges absolutely for $s=\sigma+i\tau$. For any fixed σ_0 and $\sigma \leqslant \sigma_0$, we have

$$\int_0^{+\infty} e^{\sigma t} dV(t) \leqslant \int_0^{+\infty} e^{\sigma_0 t} dV(t). \tag{13}$$

For the abscissa of absolute convergence of Laplace-Stieltjes transform (1), we have

THEOREM 2.2. *If* (1) *converges absolutely at the point* $s_0=\sigma_0+i\tau_0$, *then* (1) *converges absolutely for all* $s=\sigma+i\tau$ *satisfying* $\sigma \leqslant \sigma_0$.

REMARK 2.2. We can get from (13) that (1) converges uniformly for all $s=\sigma+i\tau$ satisfying $\sigma\leqslant\sigma_0$.

The following example show that the abscissa of convergence of (1) can be different from the abscissa of absolute convergence of (1).

EXAMPLE 2.1. Let $F_1(s)=\int_0^{+\infty} e^{st} e^{\theta t} \sin e^{\theta t} dt \;\; (\theta<0)$, in view of

$$|e^{st} e^{\theta t} \sin e^{\theta t}| \leqslant e^{(\sigma+\theta)t}, \quad \int_0^t |\sin e^{\theta t}| dt = \frac{1}{\theta}\int_1^{+\infty} \frac{|\sin u|}{u} du = \infty,$$

thus it follows that $\sigma_a^{F_1} < -\theta$. And in view of

$$\int_0^{+\infty} e^{st} e^{\theta t} \sin e^{\theta t} dt = \frac{1}{\theta}\int_1^{+\infty} \frac{\sin u}{u^{-s/k}} du,$$

then we have $\sigma_c^{F_1}=0$.

EXAMPLE 2.2. Let $F_2(s)=\int_0^{+\infty} e^{(s-1)t} e^{e^t} \sin e^{e^t} dt$. Set $u=e^{e^t}$, in view of

$$F_2(s)=\int_0^{+\infty} e^{(s-1)t} e^{e^t} \sin e^{e^t} dt = \int_e^{+\infty} \frac{\sin u}{(\log u)^{-s}} du,$$

thus we have $\sigma_c^{F_2}=0$ and $\sigma_a^{F_2}=-\infty$.

For the uniform convergence of Laplace-Stieltjes transform (1), we have

THEOREM 2.3. *If* (1) *converges uniformly for* $s_0 = \sigma_0 + i\tau,\ -\infty < \tau < +\infty$, *then* (1) *converges uniformly for all* $s = \sigma + i\tau$ *satisfying* $\sigma \leqslant \sigma_0$.

Proof. In view of Theorem 2.3, we should prove that for any $\varepsilon > 0$, there exists $r_0 > 0$ independent of σ, τ such that $r > r_0$,

$$\left|\int_r^{+\infty} e^{(\sigma+i\tau)t} d\alpha(t)\right| < \varepsilon,$$

holds for $\sigma < \sigma_0,\ -\infty < \tau < +\infty$.

Since (1) converges uniformly for $s_0 = \sigma_0 + i\tau,\ -\infty < \tau < +\infty$, then for any $\varepsilon(> 0)$, there exists $r_0 > 0$ such that

$$\left|\int_r^{+\infty} e^{(\sigma_0+i\tau)t} d\alpha(t)\right| < \frac{\varepsilon}{2}, \quad r > r_0, \quad -\infty < \tau < +\infty. \tag{14}$$

Let

$$\beta(t,\tau) = \int_t^{+\infty} e^{(\sigma_0+i\tau)y} d\alpha(y), \qquad t \geqslant 0,\ -\infty < \tau < +\infty,$$

thus for any $t \geqslant r_0,\ -\infty < \tau < +\infty$, it follows that

$$|\beta(t,\tau)| < \frac{\varepsilon}{2}, \tag{15}$$

and

$$\int_r^{r'} e^{(\sigma+i\tau)t} d\alpha(t) = -\int_r^{r'} e^{(\sigma-\sigma_0)t} d_t\beta(t,\tau), \quad r' > r,\ -\infty < \tau < +\infty. \tag{16}$$

Since

$$\begin{aligned}\int_r^{r'} e^{(\sigma+i\tau)t} d\alpha(t) = &-\beta(r',\tau)e^{(\sigma-\sigma_0)r'} + \beta(r,\tau)e^{(\sigma-\sigma_0)r} \\ &+ (\sigma-\sigma_0)\int_r^{r'} e^{(\sigma-\sigma_0)t}\beta(t,\tau)dt,\end{aligned} \tag{17}$$

and let $r' \to +\infty$, by combining with $\sigma \leqslant \sigma_0$, $|\beta(r',\tau)| \leqslant \frac{\varepsilon}{2}$ and $e^{(\sigma-\sigma_0)r'} \to +\infty$, $r' \to +\infty$, we obtain

$$\int_r^{+\infty} e^{(\sigma+i\tau)t} d\alpha(t) = \beta(r,\tau)e^{(\sigma-\sigma_0)r} + (\sigma-\sigma_0)\int_r^{+\infty} e^{(\sigma-\sigma_0)t}\beta(t,\tau)dt. \tag{18}$$

In view of (15) and (18), we can deduce that for any $r_0 > 0$,

$$\left|\int_r^{+\infty} e^{(\sigma+i\tau)t} d\alpha(t)\right| \leqslant \frac{\varepsilon}{2} - \frac{\varepsilon}{2}(\sigma-\sigma_0)\int_r^{+\infty} e^{(\sigma-\sigma_0)t} dt \leqslant \varepsilon.$$

Therefore, this completes the proof of Theorem 2.3. $\square$

3. Results

From Theorems 4.1, 5.1–6.1, we can obtain

THEOREM 3.1. *For Laplace-Stieltjes transform* (1)*, if* $\alpha(t)$ *satisfies* (4)*, and* λ_n *satisfies* (5)*, then the abscissas* σ_c^F *,* σ_a^F *,* σ_u^F *of convergence of* (3) *satisfy*

$$-\limsup_{n\to+\infty}\frac{\log n}{\lambda_n}-\limsup_{t\to+\infty}\frac{\log|\alpha(t)|}{t}\leqslant\sigma_a^F\leqslant\sigma_u^F\leqslant\sigma_c^F\leqslant-\limsup_{t\to+\infty}\frac{\log|\alpha(t)|}{t}. \quad (19)$$

Here, we give a brief proof of this result. From the assumptions of Theorem 3.1, we have the fact that

$$|\alpha(x)-\alpha(0)|=\left|\int_0^x d\alpha(t)\right|\leqslant\left|\int_0^x |e^{it\tau}|d\alpha(t)\right|\leqslant A(x),$$

If $\alpha(0)$ exists, then it follows that $\sigma_u^F\leqslant\sigma_c^F$.

On the other hand, in view of Lemma 1.4, for any $\tau\in(-\infty,+\infty)$, we have

$$|\beta(x,\tau)|\leqslant\int_0^x\left|e^{i\tau t}\right|dV(x)\leqslant V(x),$$

that is, $A(x)\leqslant V(x)$. By combining with Theorem 5.1 and Theorem 6.1, we have $\sigma_a^F\leqslant\sigma_u^F$. Thus, we can get by Theorem 5.2 that the conclusions of Theorem 3.1.

We can see that the conclusion (19) is very similar to the Valiron-Knopp-Bohr formula given by Yu [26]. Let us first recall the Valiron-Knopp-Bohr formula. If the sequence $\{\lambda_n\}$ satisfies (5),

$$\limsup_{n\to+\infty}(\lambda_{n+1}-\lambda_n)<+\infty, \quad (20)$$

and

$$\limsup_{n\to+\infty}\frac{\log n}{\lambda_n}=D<+\infty. \quad (21)$$

then the abscissa σ_c^F of convergence, the abscissa σ_a^F of absolute convergence, the abscissa σ_u^F of uniform convergence of (1) satisfy

$$-\limsup_{n\to+\infty}\frac{\log n}{\lambda_n}-\limsup_{n\to+\infty}\frac{\log A_n}{\lambda_n}\leqslant\sigma_c^F\leqslant-\limsup_{n\to+\infty}\frac{\log A_n}{\lambda_n}, \quad (22)$$

$$-\limsup_{n\to+\infty}\frac{\log n}{\lambda_n}-\limsup_{n\to+\infty}\frac{\log \widetilde{A}_n}{\lambda_n}\leqslant\sigma_a^F\leqslant-\limsup_{n\to+\infty}\frac{\log \widetilde{A}_n}{\lambda_n}, \quad (23)$$

$$-\limsup_{n\to+\infty}\frac{\log n}{\lambda_n}-\limsup_{n\to+\infty}\frac{\log A_n^*}{\lambda_n}\leqslant\sigma_u^F\leqslant-\limsup_{n\to+\infty}\frac{\log A_n^*}{\lambda_n}, \quad (24)$$

where

$$A_n=\sup_{\lambda_n<x\leqslant\lambda_{n+1}}|\alpha(x)-\alpha(\lambda_n)|,$$

$$\widetilde{A}_n = \int_{\lambda_n}^{\lambda_{n+1}} |d\alpha(x)| = V(\lambda_{n+1}) - V(\lambda_n),$$

$$A_n^* = \sup_{\lambda_n < x \leqslant \lambda_{n+1}, -\infty < t < \infty} \left| \int_{\lambda_n}^{x} e^{ity} d\alpha(y) \right|.$$

In view of the proof of the conclusion (22) given by Yu in [26], we know that the hypothesis of $\alpha(\lambda_n)$ being finite is the premise. For any $x > 0$, we have

$$|\alpha(t)| \leqslant |\alpha(t) - \alpha(\lambda_n)| + |\alpha(\lambda_n)|, \quad |\alpha(t) - \alpha(\lambda_n)| \leqslant |\alpha(t)| + |\alpha(\lambda_n)|,$$

thus we can prove that the conclusions of the abscissa of convergence σ_c^F of Laplace-Stieltjes transform (1) in (19) and (22) are consistent.

From the definitions of $A(x), \widetilde{A}_n$, A_n^* and $V(x)$, we have

$$\widetilde{A}_n \leqslant V(x) \leqslant n \max_{1 \leqslant k \leqslant n} \widetilde{A}_k, \quad A_n^* \leqslant A(x) \leqslant n \max_{1 \leqslant k \leqslant n} A_n^*.$$

If

$$\limsup_{n \to +\infty} \frac{\log \widetilde{A}_n}{\lambda_n} = -\infty, \quad \left(\limsup_{n \to \infty} \frac{\log A_n^*}{\lambda_n} = -\infty, \right) \quad \limsup_{n \to +\infty} \frac{\log n}{\lambda_n} = D < +\infty, \tag{25}$$

thus we can deduce from (19) and (23) that $\sigma_a^F = +\infty$ ($\sigma_u^F = +\infty$).

If

$$\limsup_{n \to +\infty} \frac{\log \widetilde{A}_n}{\lambda_n} = 0, \quad \left(\limsup_{n \to +\infty} \frac{\log A_n^*}{\lambda_n} = 0, \right) \quad \limsup_{n \to +\infty} \frac{\log n}{\lambda_n} = 0, \tag{26}$$

thus we can deduce from (19) and (23) that $\sigma_a^F = 0$ ($\sigma_u^F = 0$).

4. The abscissa of convergence of Laplace-Stieltjes transform

In this section, we will give the formula of the abscissa of convergence of Laplace-Stieltjes transform (1) as follows.

THEOREM 4.1. *If Laplace-Stieltjes transform* (1) *satisfies*

$$\limsup_{t \to +\infty} \frac{\log |\alpha(t)|}{t} = A \neq 0, \tag{27}$$

then $\sigma_c^F = -A$.

To prove Theorem 4.1, we require three lemmas below.

LEMMA 4.1. *For Laplace-Stieltjes transform* (1)*, if* $\alpha(x)$ *satisfies*

$$\alpha(t) = O(e^{\sigma_0 t}), \quad t \to +\infty, \tag{28}$$

for some real number σ_0*, then* (1) *converges for all* $s = \sigma + i\tau$ *satisfying* $\sigma < -\sigma_0$.

Proof. By Lemma 1.1, for any $0 < R < +\infty$, we have

$$\int_0^R e^{st} d\alpha(t) = e^{sR}\alpha(R) - \alpha(0) - s\int_0^R e^{st}\alpha(t)dt. \tag{29}$$

Since $\alpha(t)$ is a bounded variation function at any finite interval, and in view of (28), there exists a positive constant K such that $0 \leqslant t < +\infty$,

$$|\alpha(t)| \leqslant Ke^{\sigma_0 t}. \tag{30}$$

Let $R \to +\infty$, for $s = \sigma + i\tau$ with $\sigma < -\sigma_0$, it follows from (29) and (30) that

$$|\alpha(R)e^{sR}| \leqslant e^{(\sigma+\sigma_0)R} \to 0, \tag{31}$$

and

$$\left|\int_0^{+\infty} e^{st}\alpha(t)dt\right| \leqslant K\int_0^{+\infty} e^{(\sigma+\sigma_0)t}dt = -\frac{K}{\sigma+\sigma_0}. \tag{32}$$

Let $R \to +\infty$ in (29), and by combining with (31) and (32), one can prove that (1) converges for all $s = \sigma + i\tau$ with $\sigma < -\sigma_0$.

Therefore, this completes the proof of this lemma. □

COROLLARY 4.1. *If $\alpha(+\infty)$ exists, and*

$$\alpha(t) - \alpha(+\infty) = O(e^{\sigma_0 t}),$$

for some real number σ_0, then (1) converges for all $s = \sigma + i\tau$ with $\sigma < -\sigma_0$.

REMARK 4.1. In view of (29), we have

$$\int_0^{+\infty} e^{st} d\alpha(t) = -\alpha(0) - s\int_0^{+\infty} e^{st}\alpha(t)dt, \quad \sigma < -\sigma_0. \tag{33}$$

REMARK 4.2. It should be pointed out that the inverse proposition of Lemma 4.1 does not hold. For example, $\int_0^\infty e^{st}dt$ converges for all $s = \sigma + it$ with $\sigma < 0$, but $\alpha(t) = t$ does not satisfy (28).

LEMMA 4.2. *If (1) converges at the point $s_0 = \sigma_0 + i\tau$ with $\sigma_0 < 0$, then we have*

$$\alpha(t) = o(e^{-\sigma_0 t}), \quad t \to +\infty. \tag{34}$$

REMARK 4.3. The conclusions of Lemma 4.2 is not true for $\sigma_0 = 0$. For example, Let $\alpha(0) = 0$, $\alpha(t) = 1, (t > 0)$ in (1), then (1) converges for all s, but $\alpha(t) \neq o(1)$ as $t \to +\infty$.

Proof. Let

$$\beta(t) = \int_0^t e^{s_0 u} d\alpha(u), \qquad 0 < t < +\infty,$$

then
$$\alpha(t)-\alpha(0)=\int_0^t e^{-s_0u}d\beta(u). \tag{35}$$

In view of Lemma 1.1, it follows

$$\alpha(t)-\alpha(0)=\beta(t)e^{-s_0t}+s_0\int_0^t e^{-s_0u}\beta(u)du. \tag{36}$$

Since (1) converges for $s_0=\sigma_0+i\tau$, it follows that $\beta(+\infty)$ exists. By combining with (36), we have

$$\begin{aligned}\lim_{t\to+\infty}[\alpha(t)-\alpha(0)]e^{s_0t} &= \lim_{t\to+\infty}\left(\beta(+\infty)+s_0e^{s_0t}\int_0^t e^{-s_0u}\beta(u)du\right)\\ &= \lim_{t\to+\infty} s_0e^{s_0t}\int_0^t e^{-s_0u}[\beta(u)-\beta(+\infty)]du. \end{aligned}\tag{37}$$

In view of $\sigma_0<0$, it follows that the limit of the above equality is 0, that is,

$$\alpha(t)-\alpha(0)=o(e^{-\sigma_0t}),\quad \alpha(t)=o(e^{-\sigma_0t}),\quad t\to+\infty.$$

Therefore, this completes the proof of this lemma. □

LEMMA 4.3. *If* (1) *converges at the point* $s_0=\sigma_0+i\tau$ *with* $\sigma_0>0$*, then* $\alpha(+\infty)$ *exists and*
$$\alpha(t)-\alpha(+\infty)=o(e^{-\sigma_0t}),\quad t\to+\infty. \tag{38}$$

REMARK 4.4. The following example shows that the conclusion of Lemma 4.3 can not hold if $\sigma_0<0$ is replaced by $\sigma_0\leqslant 0$. Let $\alpha(t)=c$, $(t\leqslant 1)$, $\alpha(t)=3t^{\frac{1}{3}}$, $(t>1)$, $s=i$ in (1), then (1) becomes

$$\int_1^{+\infty}\frac{\cos t+i\sin t}{t^{\frac{2}{3}}}dt.$$

By the Dirichlet criterion in the anomalous integral, we obtain that $\alpha(+\infty)$ does not exist.

Proof. From the assumptions of Lemma (4.3), we have that (1) converges at the point $s=0$. By combining with $\int_0^\infty d\alpha(t)=\alpha(+\infty)-\alpha(0)$, we have that $\alpha(+\infty)$ exists. Since
$$\alpha(+\infty)-\alpha(t)=\int_t^{+\infty}e^{-s_0u}d\beta(u),$$

where $\beta(t)$ is stated as in Lemma 4.2, this leads to

$$\alpha(+\infty)-\alpha(t)=\lim_{R\to+\infty}e^{-s_0R}\beta(R)-e^{-s_0t}\beta(t)+s_0\int_t^{+\infty}e^{-s_0u}\beta(u)du. \tag{39}$$

Since (1) converges at the point $s_0=\sigma_0+i\tau$, it follows that $\beta(+\infty)$ exists. In view of $\sigma_0>0$, then we have $\lim_{R\to+\infty} e^{-s_0R}\beta(R)=0$. Thus, we can deduce from (39) that

$$\begin{aligned}\lim_{t\to+\infty}[\alpha(+\infty)-\alpha(t)]e^{s_0t} &= -\beta(+\infty)+\lim_{t\to+\infty} s_0e^{s_0t}\int_t^{+\infty} e^{-s_0u}\beta(u)du\\ &= -\lim_{t\to+\infty} s_0e^{s_0t}\int_t^{+\infty} e^{-s_0u}[\beta(+\infty)-\beta(u)]du\\ &=0,\end{aligned}$$

Therefore, this completes the proof of Lemma 4.3. □

The proof of Theorem 4.1. Two cases will be discussed below.

Case 1. Suppose that $A>0$. For $\varepsilon>0$, then $-(A+\varepsilon)<0$. In view of (27), it follows that

$$\alpha(t)=O(e^{(A+\varepsilon)t}),\quad t\to+\infty. \tag{40}$$

And by Lemma 4.1 and (40), we have that (1) converges for all $s=\sigma+i\tau$ with $\sigma<-(A+\varepsilon)$. Due to the arbitrariness of ε, thus it follows that (1) converges for all $s=\sigma+i\tau$ with $\sigma<-A$.

Next, we prove that (1) diverges for all $s=\sigma+i\tau$ with $\sigma>-A$. By means of reduction to absurdity, suppose that (1) converges for $s_1=\gamma+i\tau$ with $-A<\gamma<0$. In view of Lemma 4.2, it follows that

$$\alpha(t)=o(e^{-\gamma t}),\quad t\to+\infty. \tag{41}$$

This means that there exists a positive number K and $t_0\in\mathbb{R}$ such that for $t>t_0$,

$$|\alpha(t)|<Ke^{-\gamma t},$$

that is,

$$\log|\alpha(t)|<\log K-\gamma t,\quad (t>t_0).$$

Thus, it follows that

$$A=\limsup_{r\to+\infty}\frac{\log|\alpha(t)|}{t}\leqslant-\gamma,$$

which is contradiction with the hypothesis of $-A<\gamma$. Hence, we obtain that $\sigma_c^F=-A$ for $A>0$.

Case 2. Suppose that $A<0$. Similar to the argument as in Case 1, we can prove that (1) converges for all $s=\sigma+i\tau$ with $\sigma<-A$. Thus, it yields that (1) converges at the point $s=0$. This leads to $\alpha(+\infty)=0$. On the other hand, assume that (1) converges at the point $s=\gamma+i\tau$ with $\gamma>-A$. In view of Lemma 4.3, and combining with $\alpha(+\infty)=0$, it follows that

$$\alpha(t)=o(e^{-\gamma t}),\quad t\to+\infty.$$

Thus, we have

$$A=\limsup_{t\to+\infty}\frac{\log|\alpha(t)|}{t}\leqslant-\gamma,$$

which is a contradiction with $\gamma > -A$. Hence, we obtain that (1) converges for all $s = \gamma + i\tau$ with $\gamma < -A$.

From Case 1 and Case 2, we obtain $\sigma_c^F = -A$. This completes the proof of Theorem 4.1. □

From Theorem 4.1, we can get the following corollaries.

COROLLARY 4.2. *If Laplace-Stieltjes transform* (1) *satisfies*

$$\limsup_{r\to+\infty} \frac{\log|\alpha(t)|}{t} = +\infty(-\infty),$$

then $\sigma_c^F = -\infty(+\infty)$.

COROLLARY 4.3. *If the abscissa* σ_c^F *of convergence of Laplace-Stieltjes transform* (1) *satisfies* $\sigma_c^F \leqslant 0$*, then*

$$\sigma_c^F = -\limsup_{t\to+\infty} \frac{\log|\alpha(t)|}{t}.$$

Proof. From the assumption of Corollary 4.3, it follows that $\sigma_c^F \leqslant 0$. Set

$$A = \limsup_{t\to+\infty} \frac{\log|\alpha(t)|}{t}.$$

For $\sigma_c^F = 0$. If $A \neq 0$, thus it follows in view of Theorem 4.1 that $\sigma_c^F = -A \neq 0$, a contradiction.

For $\sigma_c^F < 0$. If $-A \neq \sigma_c^F$ and $A \neq 0$, thus it follows from Theorem 4.1 that $\sigma_c^F = -A$, this is also a contradiction. If $A = 0$, thus it follows from Theorem 4.1 that $\sigma_c^F \geqslant 0$, a contradiction. Hence, this completes the proof of Corollary 4.3. □

COROLLARY 4.4. *If Laplace-Stieltjes transform* (1) *satisfies that* $\alpha(t)$ *has no limit as* $t \to +\infty$*, and*

$$\limsup_{t\to+\infty} \frac{\log|\alpha(t)|}{t} = 0,$$

then $\sigma_c^F = 0$.

REMARK 4.5. The following example shows that the condition "$\alpha(t)$ has no limit as $t \to +\infty$" in Corollary 4.4 can not be removed. For example, the abscissa of convergence of the integral $\int_0^{+\infty} e^{st} d(1-e^{-t})$ is $\sigma_c^F = -1$, but this shows that $\alpha(t) = 1 - e^{-t}$,

$$\limsup_{t\to+\infty} \frac{\log|\alpha(t)|}{t} = \limsup_{t\to+\infty} \frac{\log|1-e^{-t}|}{t} = 0.$$

Proof. Here we use reduction to absurdity. Suppose that $\sigma_c^F \neq 0$. If $\sigma_c^F > 0$, it follows from Theorem 4.1 that $\sigma_c^F = -\limsup_{t\to+\infty} \frac{\log|\alpha(t)|}{t}$, which is a contradiction with the assumption of Corollary 4.4. If $\sigma_c^F < 0$, in view of Corollary 4.3, we have $\sigma_c^F = -\limsup_{t\to+\infty} \frac{\log|\alpha(t)|}{t} = 0$, this is also a contradiction. Hence, we have $\sigma_c^F = 0$. $\square$

COROLLARY 4.5. *For Laplace-Stieltjes transform* (1)*, if* $\alpha(+\infty)$ *exists and*

$$\limsup_{t\to+\infty} \frac{\log|\alpha(t)-\alpha(+\infty)|}{t} = A \geqslant 0,$$

then $\sigma_c^F = -A$.

COROLLARY 4.6. *If the abscissa of convergence of Laplace-Stieltjes transform* (1) *satisfies* $\sigma_c^F > 0$*, then we have that* $\alpha(+\infty)$ *exists and*

$$\sigma_c^F = -\limsup_{t\to+\infty} \frac{\log|\alpha(t)-\alpha(+\infty)|}{t}.$$

Now, we give an application of Theorem 4.1 for Dirichlet series as follows.

THEOREM 4.2. *If Dirichlet series* (6) *satisfies*

$$\limsup_{n\to+\infty} \frac{\log\left|\sum_{k=1}^{n} a_k\right|}{\lambda_n} = A \neq 0, \tag{42}$$

then we have $\sigma_c^f = -A$.

Proof. In view of Theorem 4.1, we only need to prove that $\alpha(t)$ satisfies (27). Denote $s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$, For $\lambda_n < t < \lambda_{n+1}$, and combining with the definition of $\alpha(t)$, we have

$$\frac{\log|\alpha(t)|}{t} = \frac{\log|s_n|}{t} < \frac{\log|s_n|}{\lambda_n}. \tag{43}$$

Thus, it follows

$$\gamma := \limsup_{t\to+\infty} \frac{\log|\alpha(t)|}{t} \leqslant \limsup_{n\to+\infty} \frac{\log|s_n|}{\lambda_n} = A. \tag{44}$$

Next, we prove that $\gamma < A$ in (44) does not hold. We use reduction to absurdity. Suppose that $\gamma < A$, then there exists $\gamma < \gamma_1 < A$ such that for sufficient large t,

$$\frac{\log|\alpha(t)|}{t} < \gamma_1,$$

thus for sufficient large n, it follows

$$\frac{\log|s_n|}{t} < \gamma_1.$$

Let $t \to \lambda_n$, the above inequality leads to

$$A = \limsup_{n\to+\infty} \frac{\log\left|\sum_{k=1}^{n} a_k\right|}{\lambda_n} \leqslant \gamma_1,$$

this can yield a contradiction with $\gamma < A$. Hence, we have

$$\limsup_{t\to+\infty} \frac{\log|\alpha(t)|}{t} = A.$$

Thus, by combining with Theorem 4.1, one can prove Theorem 4.2 easily. □

5. The abscissa of absolute convergence of (1)

In this section, we will give the formula of the abscissa of absolute convergence of Laplace-Stieltjes transform (1) as follows.

THEOREM 5.1. *If Laplace-Stieltjes transform* (1) *satisies*

$$\limsup_{t\to+\infty} \frac{\log V(t)}{t} = A \neq 0, \tag{45}$$

where $V(x)$ *is the total variation of* $\alpha(x)$ *in* $[0,x]$. *Then* $\sigma_a^F = -A$.

Proof. By using the same argument as in the proof of Theorem 4.1, one can prove the conclusion of Theorem 5.1 easily. □

The following result reveals the relationship on the abscissa of between convergence and absolute convergence of Laplace-Stieltjes transform (1).

THEOREM 5.2. *If Laplace-Stieltjes transform* (1) *satisfies that* $\alpha(t)$ *is monotonic in* $(\lambda_n, \lambda_{n+1})$ $(n = 0, 1, \ldots,)$, *where the sequence* $\{\lambda_n\}_{n=1}^{+\infty}$ *satisfy* (5), *then*

$$\sigma_a^F - \sigma_c^F \geqslant -\limsup_{n\to+\infty} \frac{\log n}{\lambda_n}.$$

Proof. Let

$$\limsup_{n\to+\infty} \frac{\log n}{\lambda_n} = D < +\infty. \tag{46}$$

If $D = \infty$, the conclusion of Theorem 5.2 holds clearly.

Since D is a non-negative number, we consider the following auxiliary series

$$\sum_{n=1}^{\infty} e^{\lambda_n s}.$$

By observing this series, it follows that $a_n = 1$, $n = 1,2,\ldots$. If $D \neq 0$, in view of Theorem 4.2, the abscissa of convergence of the above series is $-D$, and since this series diverges at the point $s = 0$, so the abscissa of convergence of this series is $-D$.

Assume that (1) converges at the point $s_0 = \sigma_0 + i\tau_0$, then there exists $r_0 > 0$ such that for any $r'', r' > r_0$,

$$\left|\int_{r'}^{r''} e^{s_0 t}\, d\alpha(t)\right| \leqslant 1.$$

In particular, there exists a positive integer $N \in \mathbb{N}_+$ such that for $n \geqslant N$,

$$\left|\int_{\lambda_n+}^{t} e^{s_0 t}\, d\alpha(t)\right| \leqslant 1, \quad t > \lambda_n > r_0. \tag{47}$$

By applying Lemma 1.2 for (47), it leads to

$$\left|R''_n e^{s_0\lambda_n}\right| \leqslant 1, \qquad \left|R'_{n+1} e^{s_0\lambda_n}\right| \leqslant 1, \tag{48}$$

where

$$R''_n = \alpha(\lambda_n+) - \alpha(\lambda_n), \qquad R'_{n+1} = \alpha(\lambda_{n+1}) - \alpha(\lambda_{n+1}-).$$

Now, we only prove that $\int_0^{+\infty} e^{st}\, dV(t)$ converges for all $s = \sigma + i\tau$ with $\sigma < \sigma_0 - D$, or the following series

$$\sum_{n=0}^{+\infty} \int_{\lambda_n}^{\lambda_{n+1}} e^{\sigma t}\, dV(t)$$

converges.

For $\lambda_n < t < \lambda_{n+1}$, in view of the monotonicity of $\alpha(t)$, we have

$$\int_{\lambda_n}^{\lambda_{n+1}} e^{\sigma t}\, dV(t) = \left|\int_{\lambda_{n+1}}^{\lambda_{n+1}-} e^{\sigma t}\, d\alpha(t)\right| + |R''_n| e^{\lambda_n\sigma} + |R'_{n+1}| e^{\lambda_{n+1}\sigma}, \tag{49}$$

where

$$\sum_{n=1}^{+\infty} |R''_n| e^{\lambda_n\sigma} = \sum_{n=1}^{+\infty} e^{(\sigma-\sigma_0)\lambda_n}, \qquad \sum_{n=1}^{+\infty} |R'_{n+1}| e^{\lambda_n\sigma} = \sum_{n=1}^{+\infty} e^{(\sigma-\sigma_0)\lambda_{n+1}}.$$

In view of $0 \leqslant \limsup\limits_{n\to+\infty} \frac{\log n}{\lambda_n} = D < \sigma_0 - \sigma$, for any ε $(0 < \varepsilon < \sigma_0 - \sigma - D)$, there exists a positive integer N_1 such that

$$\lambda_n > \frac{\log n}{D+\varepsilon}, \quad n > N_1. \tag{50}$$

and combining with (50), we have

$$e^{(\sigma-\sigma_0)\lambda_n} \leqslant e^{\frac{\sigma-\sigma_0}{D+\varepsilon}\log n} = \frac{1}{n^{\frac{\sigma_0-\sigma}{D+\varepsilon}}}, \quad n > N_1. \tag{51}$$

Due to $\varepsilon < \sigma_0 - \sigma - D$, that is, $\frac{\sigma_0-\sigma}{D+\varepsilon} > 1$, thus the series $\sum\limits_{n=1}^{+\infty} \frac{1}{n^{\frac{\sigma_0-\sigma}{D+\varepsilon}}}$ converges, this leads to $\sum\limits_{n=1}^{+\infty} |R_n''|e^{\lambda_n\sigma}$ converges. Similarly, we can get that the series $\sum\limits_{n=1}^{+\infty} |R_{n+1}'|e^{\lambda_n\sigma}$ converges.

Next, we will discuss the convergence of the series $\sum\limits_{n=1}^{+\infty} \left|\int_{\lambda_{n+1}}^{\lambda_{n+1}-} e^{\sigma t}d\alpha(t)\right|$. Set

$$\beta_n(t) = \int_{\lambda_n+}^{t} e^{s_0 u}d\alpha(u), \qquad r_0 < \lambda_n < t.$$

Due to

$$\begin{aligned}\int_{\lambda_n+}^{\lambda_{n+1}-} e^{\sigma t}d\alpha(t) &= \int_{\lambda_n+}^{\lambda_{n+1}-} e^{(\sigma-s_0)t}d\beta_n(t)\\ &= \beta_n(\lambda_{n+1}-)e^{(\sigma-s_0)\lambda_{n+1}-} - (\sigma-s_0)\int_{\lambda_n+}^{\lambda_{n+1}-} e^{(\sigma-s_0)t}dt,\end{aligned} \tag{52}$$

and in view of (48), it follows that $|\beta_n(\lambda_{n+1}-)| \leqslant 1$. Thus, for sufficient large n and any $t > \lambda_n$, we have

$$\begin{aligned}\sum_{n=1}^{+\infty}\left|\int_{\lambda_n+}^{\lambda_{n+1}-} e^{\sigma t}d\alpha(t)\right| &\leqslant \sum_{n=1}^{+\infty} e^{(\sigma-\sigma_0)\lambda_{n+1}} + K\sum_{n=1}^{+\infty}\int_{\lambda_n}^{\lambda_{n+1}} e^{(\sigma-\sigma_0)t}dt\\ &= \sum_{n=1}^{+\infty} e^{(\sigma-\sigma_0)\lambda_{n+1}} + K\int_0^{+\infty} e^{(\sigma-\sigma_0)t}dt,\end{aligned} \tag{53}$$

where K is a constant. By making use of $\sigma - \sigma_0 < -D < 0$, it follows that

$$\int_0^{+\infty} e^{(\sigma-\sigma_0)t}dt = \frac{1}{\sigma_0-\sigma}.$$

And by combining with (49), we obtain that $\int_0^{+\infty} e^{st}dV(t)$ converges for all $s = \sigma + i\tau$ with $\sigma < \sigma_0 - D$, which implies that (1) converges for all $s = \sigma + i\tau$ with $\sigma < \sigma_0 - D$.

Therefore, this completes the proof of Theorem 5.2. □

From Theorem 5.2, one can get the following result on the abscissa of between convergence and absolute convergence of Dirichlet series.

COROLLARY 5.1. *For Dirichlet series $f(s) = \sum\limits_{n=1}^{+\infty} a_n e^{\lambda_n s}$, if $\{\lambda_n\}_{n=1}^{+\infty}$ satisfies (5), then*

$$\sigma_a^f - \sigma_c^f \geqslant -\limsup_{n\to+\infty}\frac{\log n}{\lambda_n},$$

where σ_a^f, σ_c^f are the abscissas of convergence and absolute convergence of Dirichlet series, respectively.

6. The abscissa of uniform convergence of Laplace-Stieltjes transform

For $x \geqslant 0$, $-\infty < \tau < +\infty$, denote

$$\beta(x,\tau) = \int_0^x e^{i\tau t} d\alpha(t), \quad A(x) = \sup_{-\infty<\tau<+\infty} |\beta(x,\tau)|.$$

Now, the formula of the abscissa of uniform convergence of (1) will be given below.

THEOREM 6.1. *If Laplace-Stieltjes transform* (1) *satisfies*

$$\limsup_{x\to+\infty} \frac{\log A(x)}{x} = \gamma > 0, \tag{54}$$

then $\sigma_u^F = -\gamma$.

To prove Theorem 6.1, we first introduce some lemmas as follows.

LEMMA 6.1. *Let* $s = \sigma + i\tau$, *and for a fixed real number* γ,

$$A(x) = O(e^{\gamma x}), \quad x \to +\infty, \tag{55}$$

then for any $\varepsilon > 0$, (1) *converges uniformly on the left half* $\sigma \leqslant -\gamma - \varepsilon$, $-\infty < \tau < +\infty$.

Proof. Since for any $r > 0$, $-\infty < \tau < +\infty$, we have

$$\begin{aligned}\int_0^r e^{(\sigma+i\tau)t} d\alpha(t) &= \int_0^r e^{\sigma t} d_t\beta(t,\tau)\\ &= e^{\sigma r}\beta(r,\tau) - \sigma\int_0^r \beta(t,\tau)e^{\tau t}dt.\end{aligned} \tag{56}$$

thus, in view of (56), for any $\sigma \leqslant -\gamma-\varepsilon$, $-\infty < \tau < +\infty$, we obtain

$$|e^{\sigma r}\beta(r,\tau)| \leqslant e^{(-\gamma-\varepsilon)r}A(r) = O(e^{-\varepsilon r}), \tag{57}$$

and

$$\left|\sigma\int_r^{+\infty} e^{\sigma t}\beta(t,\tau)dt\right| \leqslant |\sigma|\int_r^{+\infty} e^{\sigma t}A(t)dt \leqslant K\frac{(\gamma+\varepsilon)}{\varepsilon}e^{-\varepsilon r}, \tag{58}$$

where K is a bounded quantity independent of σ, τ.

Let $r \to +\infty$, in view of $\varepsilon > 0$, then for any $-\infty < \tau < +\infty$, it follows

$$\lim_{r\to+\infty} e^{\sigma r}\beta(r,\tau) = 0, \qquad \lim_{r\to+\infty} \sigma\int_r^{+\infty} e^{\sigma t}\beta(t,\tau)dt = 0. \tag{59}$$

By combining with (56) and (59), we can complete the proof of this lemma. $\square$

LEMMA 6.2. *If the integral*

$$\int_0^{+\infty} e^{(\gamma+i\tau)t} d\alpha(t)$$

converges uniformly for any fixed negative number γ *and* $-\infty<\tau<+\infty$*, then*

$$A(x)=O(e^{-\gamma x}),\quad x\to+\infty.$$

Proof. From the assumptions of this lemma, there exists a real number r_0 independent of τ such that for any $r\geqslant r_0$,

$$\left|\int_{r_0}^{r} e^{(\gamma+i\tau)t} d\alpha(t)\right|\leqslant 1. \tag{60}$$

Let

$$\xi(x,\tau)=\int_0^x e^{(x+i\tau)t} d\alpha(t),\quad x\geqslant 0,\ -\infty<\tau<+\infty.$$

Thus, in view of (60), for any $x>r_0$, we have

$$\begin{aligned} |\zeta(x,\tau)| &\leqslant \left|\int_0^{r_0} e^{(\gamma+i\tau)t} d\alpha(t)\right|+1 \\ &\leqslant \int_0^{r_0} e^{\gamma t}|d\alpha(t)|+1=K_1, \end{aligned} \tag{61}$$

where K_1 is a bounded quantity. On the other hand, since

$$\beta(x,\tau)=\int_0^x e^{-\gamma t} d_t\zeta(t,\tau)=\zeta(x,\tau)e^{-\gamma x}+\gamma\int_0^x e^{-\gamma t}\zeta(t,\tau)dt, \tag{62}$$

in view of (61) and (62), we can deduce that

$$|\beta(x,\tau)|\leqslant K_1e^{-\gamma x}+K_1(e^{-\gamma x}-1)\leqslant 2K_1e^{-\gamma x},$$

that is,

$$A(x)=O(e^{-\gamma x}),\qquad x\to+\infty.$$

Therefore, this completes the proof of this lemma. □

Now, we start to prove Theorem 6.1.

The Proof of Theorem 6.1. In view of (54), for any $\varepsilon>0$, we have

$$A(x)=O\left(e^{(\gamma+\frac{\varepsilon}{2})x}\right).\qquad x\to+\infty.$$

By Lemma 6.1, we obtain that (1) converges uniformly for all $s=\sigma+i\tau$ such that $\sigma\leqslant-(\gamma+\frac{\varepsilon}{2})-\frac{\varepsilon}{2}=-\gamma-\varepsilon$, $-\infty<\tau<+\infty$.

Now, we only prove that (1) does not converge uniformly for $s=\sigma+i\tau$ such that $\sigma<-\gamma+\varepsilon$, $-\infty<\tau<+\infty$. We use reduction to absurdity. If there exists ϑ satisfying

$-\gamma < \vartheta < -\gamma + \varepsilon < 0$, (1) converges uniformly for $s = \sigma + i\tau$ such that $\sigma = \vartheta$ and $-\infty < \tau < \infty$. Thus, it follows from Lemma 6.2 that

$$A(x) = O(e^{-\vartheta x}), \qquad x \to +\infty,$$

that is,

$$\limsup_{x \to +\infty} \frac{\log A(x)}{x} \leqslant -\vartheta.$$

In view of the above inequality and (54), and combining with the assumption of ϑ, we obtain $\gamma \leqslant -\vartheta < \gamma$, this is a contradiction.

Therefore, we complete the proof of Theorem 6.1. □

From Theorem 6.1, we can get the following corollary.

COROLLARY 6.1. *If $\sigma_u^F < 0$, then*

$$\sigma_u^F = -\limsup_{x \to +\infty} \frac{\log A(x)}{x}.$$

Proof. If

$$\limsup_{x \to +\infty} \frac{\log A(x)}{x} > 0,$$

thus the conclusion holds clearly by Theorem 6.1.

If $\sigma_u^F < 0$ and

$$\limsup_{x \to +\infty} \frac{\log A(x)}{x} \leqslant 0,$$

we have

$$A(x) = O\left(e^{-\frac{\sigma_u^F}{2}x}\right), \qquad x \to +\infty.$$

Taking $\varepsilon = -\frac{\sigma_u^F}{8}$, we can deduce that (1) converges uniformly for

$$\sigma \leqslant \frac{\sigma_u^F}{2} - \left(-\frac{\sigma_u^F}{8}\right) = \frac{5}{8}\sigma_u^F < \sigma_u^F,$$

this is a contradiction.

Therefore, this completes the proof of this corollary. □

7. Conclusions

From the above argument, we can see that the convergence abscissa formulas of Laplace Stieltjes transform (1) given in this paper are consistent with the Valiron-Knopp-Bohr formulas given by Yu [26] for the cases of Laplace-Stieltjes transform (1) converging in the whole plane (the conditions (20), (21), (25)) or at the half plane (the conditions (20), (21), (26)).

From the foregoing, we discuss the convergence of Laplace-Stieltjes transform (1) by using a method different from that in [26], and establish some formulas involving three convergence abscissas of Laplace-Stieltjes transform (1) which is very similar to Valiron-Knopp-Bohr formula in Ref. [26].

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