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ANOTHER PROOF OF HÖLDER'S INEQUALITY

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Abstract. In this note, we introduce an idea of deriving some inequalities, using the discriminants of polynomials. This gives another proof of the famous Hölder's inequality.

1. Introduction

Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial of degree n and $r_1, r_2, \cdots, r_n$ be the zeros of $p(x)$. Then the discriminant $D(p)$ is defined as

$$D(p) := a_n^{2n-2}\prod_{i<j}(r_i - r_j)^2 = (-1)^{n(n-1)/2}a_n^{2n-2}\prod_{i\neq j}(r_i - r_j).$$

The discriminant $D(p)$ can tell us some information about the roots of a given polynomial $p(x)$ of degree n with all the coefficients being real. For $n \geqslant 4$ one can say for instance that

1. the polynomial $p(x)$ has a multiple root if, and only if, $D(p) = 0$;
2. if $D(p) > 0$, then $p(x)$ has $2k$ pairs of complex conjugate roots and $n-4k$ real roots for some $k \geqslant 0$;
3. if $D(p) < 0$, then $p(x)$ has $2k+1$ pairs of complex conjugate roots and $n-4k-2$ real roots for some $k \geqslant 0$.

In this note, we introduce an idea of deriving an inequality, using the notion of discriminant of a polynomial. This can be seen as an extension of the Cauchy-Schwarz inequality. The idea is based on one standard proof of the Cauchy-Schwarz inequaility in linear algebra. Using this idea, we can give another proof of the famous Hölder's inequality.

Of course there are already huge number of other generalizations of the Cauchy-Schwarz inequality, for instance [1, 2, 3, 4, 5, 6, 7, 9, 10], to name a few some very recent publications. We hope that this note might be of interest of researchers in this field.

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2. Our result

In this section, we state and prove a result that gives a framework of deriving some inequalities for a set of positive real numbers. First we recall the following fact.

LEMMA 2.1. *Let $p(x)$ be a polynomial of degree n such that $p(a) \geqslant 0$ for all $a \in \mathbb{R}$. Then we have*

$$(-1)^{(n^2-n)/2} D(p) \geqslant 0.$$

Proof. First, note that n must be even. Put $n = 2k$. First, suppose that $p(x)$ has no real roots. Then $p(x)$ has k pairs of complex conjugate roots. Then the roots of $p(x)$ can be written as

$$\gamma_1, \overline{\gamma_1}, \gamma_2, \overline{\gamma_2}, \ldots, \gamma_k, \overline{\gamma_k}$$

where γ_i and $\overline{\gamma_i}$ are conjugate. Then the discriminant $D(p)$ can be written as

$$D(p) := (-1)^{k(2k-1)} a_n^{2n-2} \prod_{i=1}^{k} (\gamma_i - \overline{\gamma_i})(\overline{\gamma_i} - \gamma_i) \cdot \prod_{i \neq j} (\gamma_i - \overline{\gamma_j})(\overline{\gamma_i} - \gamma_j) \cdot \prod_{i \neq j} (\gamma_i - \gamma_j)(\overline{\gamma_i} - \overline{\gamma_j}).$$

We can check that $(\gamma_i - \overline{\gamma_j})(\overline{\gamma_i} - \gamma_j) \geqslant 0$, since $\gamma_i - \overline{\gamma_j}$ and $\overline{\gamma_i} - \gamma_j$ are conjugate. Thus

$$\prod_{i \neq j} (\gamma_i - \overline{\gamma_j})(\overline{\gamma_i} - \gamma_j) > 0.$$

Similarly, we know that

$$\prod_{i=1}^{k} (\gamma_i - \overline{\gamma_i})(\overline{\gamma_i} - \gamma_i) > 0 \text{ and } \prod_{i \neq j} (\gamma_i - \gamma_j)(\overline{\gamma_i} - \overline{\gamma_j}) > 0.$$

In conclusion, $D(p) > 0$ if k is even and $D(p) < 0$ if k is odd. We also know that $(-1)^{n(n-1)/2} = 1$ if k is even and $(-1)^{n(n-1)/2} = -1$ if k is odd. Therefore we can get the inequality

$$(-1)^{(n^2-n)/2} D(p) > 0.$$

Next, suppose that $p(x)$ has some real roots. We know that $D(p) = 0$ if and only if $p(t) = 0$ for some real number t , since p is of even degree and is nonnegative. Hence in conclusion, for any nonnegative polynomial $p(x)$, we have

$$(-1)^{(n^2-n)/2} D(p) \geqslant 0. \quad \square$$

Using this lemma, we can see that the following result holds.

THEOREM 2.2. *Consider a set of positive real numbers* $\{\alpha_i\}_{i=1}^m$, $\{\beta_i\}_{i=1}^m$. *Then for each nonnegative polynomial* $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in \mathbb{R}[x]$, *where* n *is an even integer, there is an inequality between* α_i*'s and* β_i*'s. This inequality comes from the inequality*

$$(-1)^{(n^2-n)/2} D(p^*) \geqslant 0,$$

for polynomial $p^*(x) := \sum_{i=1}^m \beta_i^n p\left(\frac{\alpha_i}{\beta_i}x\right)$ *by the expansion of the discriminant* $D(p^*)$.

Equality holds in the induced inequality if and only if α_i *and* β_i *satisfy* $p\left(\frac{\alpha_i}{\beta_i}t\right) = 0$ *for all* i *and some real number* t.

Proof. Note that

$$\begin{aligned} p^*(x) &= \sum_{i=1}^m \beta_i^n p\left(\frac{\alpha_i}{\beta_i}x\right) \\ &= a_n\left(\sum_{i=1}^m \alpha_i^n\right)x^n + a_{n-1}\left(\sum_{i=1}^m \alpha_i^{n-1}\beta_i\right)x^{n-1} + \cdots + a_0\left(\sum_{i=1}^m \beta_i^n\right). \end{aligned}$$

The point is that for a nonnegative polynomial $p(x)$, the polynomial $p^*(x)$ is also a nonnegative polynomial , since β_i's are positive. Then by Lemma 2.1, we get the inequality

$$(-1)^{(n^2-n)/2} D(p^*) \geqslant 0.$$

Finally, we know that $(-1)^{(n^2-n)/2} D(p^*) = 0$ if and only if $p^*(x)$ has a multiple real root t, and this is equivalent to $\sum_{i=1}^m \beta_i^n p\left(\frac{\alpha_i}{\beta_i}t\right) = 0$ for some real number t. By our assumption of $p(x)$, we conclude that $(-1)^{(n^2-n)/2} D(p^*) = 0$ if and only if $p\left(\frac{\alpha_i}{\beta_i}t\right) = 0$ for all i and some real number t. $\square$

REMARK 2.3. It is not difficult to generate nonnegative polynomials. For example, for any real polynomial $q(x)$ of degree $< 2n$, there exists a constant c such that $x^{2n} - q(x) + c \geqslant 0$ for all $x \in \mathbb{R}$ and one can apply Theorem 2.2 for an infinite number of polynomials of the desired property. The main problem is that many of these inequalities may not be so practical, due to many terms usually appearing in the inequalities.

EXAMPLE 2.4. Let us give an example about the case when $n = 2$ and $p(x) = x^2 - 2x + 1 = (x-1)^2$, which recovers the classical Cauchy-Schwarz inequality. Take any positive real numbers $\{\alpha_i\}_{i=1}^m$ and $\{\beta_i\}_{i=1}^m$. Then we see that $\alpha_i^2 x^2 - 2\alpha_i\beta_i x + \beta_i^2 = (\alpha_i x - \beta_i)^2 \geqslant 0$ on $\mathbb{R}$, for any i. Adding up for all i, we have the polynomial $p^*(x)$ given by

$$p^*(x) = \left(\sum_{i=1}^m \alpha_i^2\right)x^2 - 2\left(\sum_{i=1}^m \alpha_i\beta_i\right)x + \left(\sum_{i=1}^m \beta_i^2\right),$$

which is also nonnegative on $\mathbb{R}$. Therefore, we see that the discriminant of $p^*(x)$ is

$$4\left[\left(\sum_{i=1}^{m}\alpha_i\beta_i\right)^2-\left(\sum_{i=1}^{m}\alpha_i^2\right)\left(\sum_{i=1}^{m}\beta_i^2\right)\right]\leqslant 0,$$

which gives the inequality

$$\left(\sum_{i=1}^{m}\alpha_i\beta_i\right)^2\leqslant\left(\sum_{i=1}^{m}\alpha_i^2\right)\left(\sum_{i=1}^{m}\beta_i^2\right).$$

3. Another proof of Hölder's inequality

In this section, we give another proof of Hölder's inequality using Theorem 2.2. To do this, we first recall the following Swan's discriminant formula for trinomials:

THEOREM 3.1. ([8, Theorem 2]) *The trinomial* $f(x)=x^n-ax^k+b$ *with* $0<k<n$ *has discriminant*

$$D(f)=(-1)^{\frac{n(n-1)}{2}}b^{k-1}\left[n^Nb^{N-K}-(n-k)^{N-K}k^Ka^N\right]^d,$$

where $d=(n,k), N=n/d, K=k/d$.

Now we consider the polynomial $p(x)=kx^{2n}-2nx^k+(2n-k)$ such that $0<k<2n$ and $(k,2n)=1$. Then it is easy to see that $p(x)$ is nonnegative. Applying Theorem 2.2, we see that for any positive real numbers $\{\alpha_i\}_{i=1}^m$ and $\{\beta_i\}_{i=1}^m$, the polynomial $p^*(x)=kax^{2n}-2nbx^k+(2n-k)c$ with $a=\sum_{i=1}^m\alpha_i^{2n}$, $b=\sum_{i=1}^m\alpha_i^k\beta_i^{2n-k}$ and $c=\sum_{i=1}^m\beta_i^{2n}$, has the property

$$(-1)^{(2n^2-n)}D(p^*)\geqslant 0.$$

Using the Swan's discriminant formula for trinomials, we get the following inequality:

$$\left(\sum_{i=1}^{m}\alpha_i^k\beta_i^{2n-k}\right)^{2n}\leqslant\left(\sum_{i=1}^{m}\alpha_i^{2n}\right)^{k}\left(\sum_{i=1}^{m}\beta_i^{2n}\right)^{2n-k},\tag{3.1}$$

or equivalently with $a_i=\alpha_i^k$ and $b_i=\beta_i^{2n-k}$,

$$\sum_{i=1}^{m}a_ib_i\leqslant\left(\sum_{i=1}^{m}a_i^{\frac{2n}{k}}\right)^{\frac{k}{2n}}\left(\sum_{i=1}^{m}b_i^{\frac{2n}{2n-k}}\right)^{\frac{2n-k}{2n}},$$

for $(p,q)=\left(\frac{2n}{k},\frac{2n}{2n-k}\right)$.

The continuity argument gives the general Hölder's inequality, for $p,q\in(1,\infty)$ with $\frac{1}{p}+\frac{1}{q}=1$,

$$\sum_{i=1}^{m}a_ib_i\leqslant\left(\sum_{i=1}^{m}a_i^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{m}b_i^q\right)^{\frac{1}{q}}.$$

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