Journal of
Mathematical
Inequalities
Volume 17, Number 1 (2023), 191–204 doi:10.7153/jmi-2023-17-14

# THE SHARP BOUND OF THE THIRD HANKEL DETERMINANT FOR CONVEX FUNCTIONS OF ORDER $-1/2$

BOGUMIŁA KOWALCZYK, ADAM LECKO* AND DEREK K. THOMAS

(*Communicated by V. Rao Allu*)

*Abstract.* We prove the sharp inequality $|H_{3,1}(f)| \leqslant 1/16$ for the third Hankel determinant $H_{3,1}(f)$ for convex functions of order $-1/2$ i.e., functions $f$ analytic in $z \in \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ with $a_n := f^{(n)}(0)/n!$, $n \in \mathbb{N}$, $a_1 := 1$, such that

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > -\frac{1}{2}, \quad z \in \mathbb{D},$$

thus proving a recent conjecture.

## 1. Introduction

Let $\mathscr{H}$ be the class of all analytic functions in $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathscr{A}$ denote the subclass of $\mathscr{H}$ with functions $f \in \mathscr{A}$ having Taylor series

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad a_1 := 1. \tag{1}$$

Let $\mathscr{S}$ be the subfamily of $\mathscr{A}$, consisting of univalent functions, and $\mathscr{S}^*(\alpha)$ and $\mathscr{C}(\alpha)$ for $0 \leqslant \alpha < 1$ denote respectively the classes of starlike and convex functions of order $\alpha$. Then it is well-known that a function $f \in \mathscr{A}$ belongs to $\mathscr{S}^*(\alpha)$ if, and only if,

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in \mathbb{D},$$

and that $f \in \mathscr{C}(\alpha)$ if, and only if,

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad z \in \mathbb{D}.$$

We write $\mathscr{S}^*(0) =: \mathscr{S}^*$, and $\mathscr{C}(0) =: \mathscr{C}$ to denote the classes of starlike and convex functions respectively.

*Mathematics subject classification* (2020): 30C45, 30C50.
*Keywords and phrases*: Convex functions, Carathéodory functions, Hankel determinant, coefficients.
* Corresponding author.

A function $f \in \mathcal{A}$ belongs to $\mathcal{K}$, the class of close-to-convex functions if, and only if, there exist $g \in \mathcal{S}^*$ and $\tau \in (-\pi/2, \pi/2)$ such that $\mathrm{Re}[\mathrm{e}^{\mathrm{i}\tau}(zf'(z)/g(z))] > 0$ for $z \in \mathbb{D}$. The class $\mathcal{K}$ was first formally introduced by Kaplan in 1952 [7], who showed that $\mathcal{K} \subset \mathcal{S}$, so that $\mathcal{C} \subset \mathcal{S}^* \subset \mathcal{K} \subset \mathcal{S}$.

Little attention has been given to $\mathcal{C}(\alpha)$ for $\alpha < 0$. However in 1941, Ozaki [18] showed that functions in $\mathcal{A}$ are univalent if they satisfy the condition

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > -\frac{1}{2}, \quad z \in \mathbb{D}, \tag{2}$$

and we denote by $\mathcal{C}(-1/2)$ the class of functions $f$ satisfying (2).

We also note that $\mathcal{C}(-1/2) \subset \mathcal{K}$ follows from the original definition of Kaplan [7], and that Umezawa [23] subsequently proved that functions in $\mathcal{C}(-1/2)$ are not necessarily starlike, but are convex in one direction.

For $q, n \in \mathbb{N}$, the Hankel determinant $H_{q,n}(f)$ of functions $f \in \mathcal{A}$ given by (1) is defined as

$$H_{q,n}(f) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix}.$$

General results for Hankel determinants with applications can be found in [2], [22], [19] and [20]. For subclasses of $\mathcal{A}$, finding bounds of $|H_{q,n}(f)|$ for $q, n \in \mathbb{N}$, is an interesting and significant area of study. Hayman [6] examined the second Hankel determinant $H_{2,2}(f) = a_2a_4 - a_3^2$ for areally mean univalent functions, and recently many other authors have also examined the second Hankel determinant for a variety of subclasses of $\mathcal{A}$, (see e.g., [3], [4] for further references), often obtaining sharp bounds for $|H_{2,2}(f)|$. The problem of finding sharp bounds for the third Hankel determinant

$$H_{3,1}(f) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2) \tag{3}$$

is technically much more difficult.

Finding sharp bounds for Hankel determinants when $f \in \mathcal{K}$ also presents a difficult problem, and even the sharp bound for $|H_{2,2}(f)|$ is not known. Thus finding the sharp bound for $|H_{3,1}(f)|$ when $f$ belongs to a subclass of $\mathcal{K}$ represents a significant advance.

Some sharp bounds for $|H_{3,1}(f)|$ have been found e.g. for convex functions [10], starlike functions [11], starlike functions of order $1/2$ [13], functions $f \in \mathcal{A}$ which satisfy the condition $\mathrm{Re}\, f(z)/z > \alpha$, $z \in \mathbb{D}$, when $\alpha = 0$ and $\alpha = 1/2$ [9], and functions $f \in \mathcal{A}$ such that $|(z/f(z))^2 - 1| < 1$ for $z \in \mathbb{D}$ [16]. The sharp bound for $|H_{3,1}(f)|$ has also been found when the associate starlike function $g(z) = z$ for $z \in \mathbb{D}$, in the definition of $\mathcal{K}$ which represents perhaps the simplest subclass of $\mathcal{K}$ [8].

We note now that using standard techniques, it is a relatively simple exercise to show that if $f \in \mathcal{C}(-1/2)$, then $|H_{2,2}(f)| \leqslant 21/64$ and that this inequality is sharp, and in a recent unpublished paper, Obradović and Tuneski [17] gave the non-sharp bound $|H_{3,1}(f)| < (13/8)^2/30$ for $f \in \mathscr{C}(-1/2)$, and conjectured that the sharp bound is $|H_{3,1}(f)| \leqslant 1/16$.

In this paper, we prove that $|H_{3,1}(f)| \leqslant 1/16$ for $f \in \mathscr{C}(-1/2)$, thus confirming this conjecture, and so giving the sharp bound for $|H_{3,1}(f)|$ for a significant subclass of $\mathscr{K}$.

Since functions in $\mathscr{C}(-1/2)$ can be represented using the Carathéodory class $\mathscr{P}$, [1] i.e., the class of functions $p \in \mathscr{H}$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in \mathbb{D}, \tag{4}$$

having a positive real part in $\mathbb{D}$, the coefficients of functions in $\mathscr{C}(-1/2)$ can be expressed in terms of coefficients of functions in $\mathscr{P}$. We therefore base our analysis on the well-known formula for $c_2$ (e.g., [21, p. 166]), the formula $c_3$ due to Libera and Zlotkiewicz [14, 15] and the formula for $c_4$ recently found in [12], all of which can be conveniently be expressed in the following lemma [12].

LEMMA 1.1. *If $p \in \mathscr{P}$ and is given by (4) with $c_1 \geqslant 0$, then*

$$c_1 = 2\zeta_1, \tag{5}$$

$$c_2 = 2\zeta_1^2 + 2(1-\zeta_1^2)\zeta_2, \tag{6}$$

$$c_3 = 2\zeta_1^3 + 4(1-\zeta_1^2)\zeta_1\zeta_2 - 2(1-\zeta_1^2)\zeta_1\zeta_2^2 + 2(1-\zeta_1^2)(1-|\zeta_2|^2)\zeta_3 \tag{7}$$

*and*

$$\begin{aligned} c_4 =& 2\zeta_1^4 + 2(1-\zeta_1^2)\zeta_2\left(\zeta_1^2\zeta_2^2 - 3\zeta_1^2\zeta_2 + 3\zeta_1^2 + \zeta_2\right) \\ &+ 2(1-\zeta_1^2)(1-|\zeta_2|^2)\zeta_3\left(2\zeta_1 - 2\zeta_1\zeta_2 - \overline{\zeta_2}\zeta_3\right) \\ &+ 2(1-\zeta_1^2)(1-|\zeta_2|^2)(1-|\zeta_3|^2)\zeta_4, \end{aligned} \tag{8}$$

*for some $\zeta_1 \in [0,1]$ and $\zeta_2, \zeta_3, \zeta_4 \in \overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leqslant 1\}$.*

## 2. Main result

THEOREM 2.1. *If $f \in \mathscr{C}(-1/2)$, then*

$$|H_{3,1}(f)| \leqslant \frac{1}{16}. \tag{9}$$

*The inequality is sharp.*

*Proof.* Let $f \in \mathscr{C}(-1/2)$ and be given by (1). Then by (2),

$$1 + \frac{zf''(z)}{f'(z)} = \frac{3}{2}p(z) - \frac{1}{2}, \quad z \in \mathbb{D}, \tag{10}$$

for some function $p \in \mathscr{P}$ given by (4). Substituting (1) and (4) into (10) and equating coefficients we obtain

$$\begin{aligned} a_2 &= \frac{3}{4}c_1, \quad a_3 = \frac{1}{8}\left(3c_1^2+2c_2\right), \quad a_4 = \frac{1}{64}\left(9c_1^3+18c_1c_2+8c_3\right), \\ a_5 &= \frac{3}{640}\left(9c_1^4+36c_1^2c_2+32c_1c_3+12c_2^2+16c_4\right). \end{aligned} \tag{11}$$

Hence from (3) we have

$$\begin{aligned} H_{3,1}(f) =&\frac{1}{20480}\left[96c_4(4c_2-3c_1^2)-320c_3^2+288c_1c_2c_3+144c_1^3c_3\right. \\ &\left.-252c_1^2c_2^2+108c_1^4c_2-32c_2^3-27c_1^6\right]. \end{aligned} \tag{12}$$

Since the class $\mathscr{C}(-1/2)$ and the functional $H_{3,1}(f)$ are rotationally invariant, we may assume that $a_2 \geqslant 0$, i.e., by (11) that $c_1 \in [0,2]$ ([1], see also [5, Vol. I, p. 80, Theorem 3]). Thus in view of (5) we assume that $\zeta_1 \in [0,1]$. Using (5)–(8) by straightforward algebraic computation we obtain

$$\begin{aligned} 96c_4(4c_2-3c_1^2) =&768\left[-\zeta_1^6-(1-\zeta_1^2)\zeta_1^4\zeta_2-(1-\zeta_1^2)\zeta_1^4\zeta_2^3+3(1-\zeta_1^2)\zeta_1^4\zeta_2^2\right. \\ &-(1-\zeta_1^2)\zeta_1^2\zeta_2^2+2(1-\zeta_1^2)^2\left(3\zeta_1^2\zeta_2^2+\zeta_1^2\zeta_2^4-3\zeta_1^2\zeta_2^3+\zeta_2^3\right) \\ &-(1-\zeta_1^2)(1-|\zeta_2|^2)\left(2\zeta_1^3\zeta_3-2\zeta_1^3\zeta_2\zeta_3-\zeta_1^2\overline{\zeta_2}\zeta_3^2\right) \\ &+2(1-\zeta_1^2)^2(1-|\zeta_2|^2)\left(2\zeta_1\zeta_2\zeta_3-2\zeta_1\zeta_2^2\zeta_3-|\zeta_2|^2\zeta_3^2\right) \\ &-(1-\zeta_1^2)(1-|\zeta_2|^2)(1-|\zeta_3|^2)\zeta_1^2\zeta_4 \\ &\left.+2(1-\zeta_1^2)(1-|\zeta_2|^2)(1-|\zeta_3|^2)\zeta_2\zeta_4\right], \end{aligned}$$

$$\begin{aligned} 320c_3^2 =&1280\left[\zeta_1^6+4(1-\zeta_1^2)^2\zeta_1^2\zeta_2^2-4(1-\zeta_1^2)^2\zeta_1^2\zeta_2^3+(1-\zeta_1^2)^2\zeta_1^2\zeta_2^4\right. \\ &+4(1-\zeta_1^2)\zeta_1^4\zeta_2-2(1-\zeta_1^2)\zeta_1^4\zeta_2^2+2(1-\zeta_1^2)(1-|\zeta_2|^2)\zeta_1^3\zeta_3 \\ &+(1-\zeta_1^2)^2(1-|\zeta_2|^2)^2\zeta_3^2+4(1-\zeta_1^2)^2(1-|\zeta_2|^2)\zeta_1\zeta_2\zeta_3 \\ &\left.-2(1-\zeta_1^2)^2(1-|\zeta_2|^2)\zeta_1\zeta_2^2\zeta_3\right], \end{aligned}$$

$$\begin{aligned} 288c_1c_2c_3 =&2304\left[\zeta_1^6+3(1-\zeta_1^2)\zeta_1^4\zeta_2-(1-\zeta_1^2)\zeta_1^4\zeta_2^2\right. \\ &+2(1-\zeta_1^2)^2\zeta_1^2\zeta_2^2-(1-\zeta_1^2)^2\zeta_1^2\zeta_2^3 \\ &\left.+(1-\zeta_1^2)(1-|\zeta_2|^2)\zeta_1^3\zeta_3+(1-\zeta_1^2)^2(1-|\zeta_2|^2)^2\zeta_1\zeta_2\zeta_3\right], \end{aligned}$$

$$\begin{aligned} 144c_1^3c_3 =&2304\left[\zeta_1^6+2(1-\zeta_1^2)\zeta_1^4\zeta_2-(1-\zeta_1^2)\zeta_1^4\zeta_2^2\right. \\ &\left.+(1-\zeta_1^2)(1-|\zeta_2|^2)\zeta_1^3\zeta_3\right], \end{aligned}$$

$$252c_1^2c_2^2 =4032\left[\zeta_1^6+2(1-\zeta_1^2)\zeta_1^4\zeta_2+(1-\zeta_1^2)^2\zeta_1^2\zeta_2^2\right],$$

$$108c_1^4c^2 =3456\left[\zeta_1^6+(1-\zeta_1^2)\zeta_1^4\zeta_2\right],$$

$$32c_2^2 = 256\left[\zeta_1^6 + 3(1-\zeta_1^2)\zeta_1^4\zeta_2 + 3(1-\zeta_1^2)^2\zeta_1^2\zeta_2^2 + (1-\zeta_1^2)^3\zeta_2^3\right],$$
$$27c_1^6 = 1728\zeta_1^6.$$

Substituting the above expression into (12) we obtain

$$\begin{aligned}H_{3,1}(f) =& \frac{1}{320}(1-\zeta_1^2)\left\{4\zeta_1^4\zeta_2 - (1+5\zeta_1^2)\zeta_1^2\zeta_2^2 + 4(3\zeta_1^4 - 11\zeta_1^2 + 5)\zeta_2^3\right.\\ &+4(1-\zeta_1^2)\zeta_1^2\zeta_2^4 + 4(1-|\zeta_2|^2)\left[2\zeta_1^2 + (1+5\zeta_1^2)\zeta_2 - 2(1-\zeta_1^2)\zeta_2^2\right]\zeta_1\zeta_3\\ &+4(1-|\zeta_2|^2)\left[3\zeta_1^2\overline{\zeta_2} - (1-\zeta_1^2)(5+|\zeta_2|^2)\right]\zeta_3^2\\ &\left.-12(1-|\zeta_2|^2)(1-|\zeta_3|^2)\left[\zeta_1^2 - 2(1-\zeta_1^2)\zeta_2\right]\zeta_4\right\},\end{aligned} \tag{13}$$

for some $\zeta_1 \in [0,1]$ and $\zeta_2, \zeta_3, \zeta_4 \in \overline{\mathbb{D}}$.

Since $|\zeta_4| \leqslant 1$, from (13) we obtain

$$\begin{aligned}|H_{3,1}(f)| \leqslant& \frac{1-\zeta_1^2}{320}\left\{\left|4\zeta_1^4\zeta_2 - (1+5\zeta_1^2)\zeta_1^2\zeta_2^2\right.\right.\\ &\left.+4(3\zeta_1^4 - 11\zeta_1^2 + 5)\zeta_2^3 + 4(1-\zeta_1^2)\zeta_1^2\zeta_2^4\right|\\ &+4(1-|\zeta_2|^2)\left|2\zeta_1^3 + (1+5\zeta_1^2)\zeta_1\zeta_2 - 2(1-\zeta_1^2)\zeta_1\zeta_2^2\right||\zeta_3|\\ &+4(1-|\zeta_2|^2)\left[\left|3\zeta_1^2\overline{\zeta_2} - (1-\zeta_1^2)(5+|\zeta_2|^2)\right|\right.\\ &\left.\left.-3\left|\zeta_1^2 - 2(1-\zeta_1^2)\zeta_2\right|\right]|\zeta_3|^2 + 12(1-|\zeta_2|^2)\left|\zeta_1^2 - 2(1-\zeta_1^2)\zeta_2\right|\right\}.\end{aligned}$$

**A.** Suppose that

$$\left|3\zeta_1^2\overline{\zeta_2} - (1-\zeta_1^2)(5+|\zeta_2|^2)\right| - 3\left|\zeta_1^2 - 2(1-\zeta_1^2)\zeta_2\right| \geqslant 0.$$

Then

$$|H_{3,1}(f)| \leqslant \frac{1}{320}h(\zeta_1, |\zeta_2|),$$

where $h : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$\begin{aligned}h(x,y) :=& (1-x^2)\left\{4(2x^3 - 5x^2 + 5) + 4(x^4 + 5x^3 + 3x^2 + x)y\right.\\ &+(5x^4 - 16x^3 + 17x^2 + 8x - 16)y^2 + 4[|3x^4 - 11x^2 + 5|\\ &\left.-5x^3 - 3x^2 - x]y^3 - 4(x^4 - 2x^3 - 2x^2 + 2x + 1)y^4\right\}.\end{aligned}$$

We show that $|h(x,y)| \leqslant 20$ for $(x,y) \in [0,1] \times [0,1]$.

**A1.** Suppose that $3x^4 - 11x^2 + 5 \geqslant 0$, which holds if, and only if, $x \in [0, x_0]$, where $x_0 := \sqrt{(11-\sqrt{61})/6} \approx 0.729$. Then

$$\begin{aligned}h(x,y) =& (1-x^2)\left[4(2x^3 - 5x^2 + 5) + 4(x^4 + 5x^3 + 3x^2 + x)y\right.\\ &+(5x^4 - 16x^3 + 17x^2 + 8x - 16)y^2\\ &+4(3x^4 - 5x^3 - 14x^2 - x + 5)y^3\\ &\left.-4(x^4 - 2x^3 - 2x^2 + 2x + 1)y^4\right], \quad x \in [0, x_0],\ y \in [0,1].\end{aligned}$$

I. On the vertices of $[0,x_0]\times[0,1]$ we have

$$h(0,0)=20,\quad h(0,1)=16,$$

$$h(x_0,0)=\frac{\sqrt{61}-5}{6}\left[8\left(\frac{11-\sqrt{61}}{6}\right)^{\frac{3}{2}}+\frac{10\sqrt{61}}{3}-\frac{50}{3}\right]\approx 5.839920<20,$$

$$h(x_0,1)=\frac{485\sqrt{61}-3685}{54}\approx 1.906872<20.$$

II. On the sides of $[0,x_0]\times[0,1]$ we have

(a) If $x=0,\ y\in(0,1),$ then

$$h(0,y)=-4y^4+20y^3-16y^2+20\leqslant 20,\quad y\in(0,1),$$

since the above inequality is equivalent to the obviously true inequality

$$-y^2(4-y)(1-y)\leqslant 0,\quad y\in(0,1).$$

(b) If $y=0,\ x\in(0,x_0),$ then for $x\in(0,x_0),$

$$h(x,0)=4(1-x^2)(2x^3-5x^2+5)=4(-2x^5+5x^4+2x^3-10x^2+5)\leqslant 20$$

since the last inequality is equivalent to the obviously true inequality

$$-x^2(5-x)(2-x^2)\leqslant 0,\quad x\in(0,x_0).$$

(c) If $y=1,\ x\in(0,x_0),$ then

$$h(x,1)=-17x^6+56x^4-59x^2+20\leqslant 20,\quad x\in(0,x_0),$$

since the last inequality is equivalent to the obviously true inequality

$$-x^2(17x^4-56x^2+59)\leqslant 0,\quad x\in(0,x_0).$$

(d) If $x=x_0,\ y\in(0,1),$ then

$$\begin{aligned}h(x_0,y)=&\frac{\sqrt{61}-5}{6}\left\{\left[8\left(\frac{11-\sqrt{61}}{6}\right)^{\frac{3}{2}}-8\sqrt{\frac{11-\sqrt{61}}{6}}+\frac{10\sqrt{61}}{9}-\frac{86}{9}\right]y^4\right.\\&-\left[20\left(\frac{11-\sqrt{61}}{6}\right)^{\frac{3}{2}}+4\sqrt{\frac{11-\sqrt{61}}{6}}-2\sqrt{61}+22\right]y^3\\&-\left[16\left(\frac{11-\sqrt{61}}{6}\right)^{\frac{3}{2}}-8\sqrt{\frac{11-\sqrt{61}}{6}}+\frac{53\sqrt{61}}{9}-\frac{364}{9}\right]y^2\\&+\left[20\left(\frac{11-\sqrt{61}}{6}\right)^{\frac{3}{2}}+4\sqrt{\frac{11-\sqrt{61}}{6}}-\frac{40\sqrt{61}}{9}+\frac{380}{9}\right]y\\&\left.+8\left(\frac{11-\sqrt{61}}{6}\right)^{\frac{3}{2}}+\frac{10\sqrt{61}}{3}-\frac{50}{3}\right\}=:h_1(y),\quad y\in(0,1).\end{aligned}$$

Since $h_1'(y) = 0$ only for $y = y_0 \approx 0.467385$, and at $y_0$ the function $h_1$ attains its maximum value $h_1(y_0) \approx 8.318$, we have

$$h(x_0, y) = h_1(y) \leqslant h_1(y_0) < 20, \quad y \in (0,1).$$

III. It remains to consider the set $(0, x_0) \times (0,1)$. Then all the real solutions of the system of equations

$$\begin{aligned}\frac{\partial h}{\partial x} =& -40x^4 + 80x^3 + 24x^2 - 80x \\ & - (24x^5 + 100x^4 + 32x^3 - 48x^2 - 24x - 4)y \\ & - (30x^5 - 80x^4 + 48x^3 + 72x^2 - 66x - 8)y^2 \\ & - (72x^5 - 100x^4 - 272x^3 + 48x^2 + 152x + 4)y^3 \\ & + (24x^5 - 40x^4 - 48x^3 + 48x^2 + 24x - 8)y^4 = 0,\end{aligned}$$

$$\begin{aligned}\frac{\partial h}{\partial y} =&(1 - x^2)\big[4(x^4 + 5x^3 + 3x^2 + x) \\ & + 2(5x^4 - 16x^3 + 17x^2 + 8x - 16)y \\ & + 12(3x^4 - 5x^3 - 14x^2 - x + 5)y^2 \\ & -16(x^4 - 2x^3 - 2x^2 + 2x + 1)y^3\big] = 0\end{aligned}$$

by a numerical computation are the following

$$\begin{cases} x_1 \approx -1.355145 \\ y_1 \approx -1.340102 \end{cases} \quad \begin{cases} x_2 \approx -1.009411 \\ y_2 \approx 6.825166 \end{cases} \quad \begin{cases} x_3 \approx -0.919818 \\ y_3 \approx 0,405793 \end{cases}$$

$$\begin{cases} x_4 \approx -0.794482 \\ y_4 \approx -0.849773 \end{cases} \quad \begin{cases} x_5 \approx 0.047594 \\ y_5 \approx 0.645015 \end{cases} \quad \begin{cases} x_6 \approx 0.735958 \\ y_6 \approx -0.823670 \end{cases}$$

$$\begin{cases} x_7 \approx 0.920936 \\ y_7 \approx -0.566691 \end{cases} \quad \begin{cases} x_8 \approx 1.315340 \\ y_8 \approx 12.353621 \end{cases} \quad \begin{cases} x_9 \approx 0 \\ y_9 \approx 0 \end{cases}$$

$$\begin{cases} x_{10} = 1 \\ y_{10} \approx -0.813927 \end{cases} \quad \begin{cases} x_{11} = 1 \\ y_{11} \approx -0.208735 \end{cases} \quad \begin{cases} x_{12} = 1 \\ y_{12} \approx 0.980996 \end{cases}$$

$$\begin{cases} x_{13} = -1 \\ y_{13} = (1 - 2\sqrt{2})/7 \end{cases} \quad \begin{cases} x_{14} = -1 \\ y_{14} = (1 + 2\sqrt{2})/7 \end{cases}.$$

Thus $(x_5, y_5)$ is the unique critical point of $h$ in $(0, x_0) \times (0,1)$ with

$$h(x_5, y_5) \approx 18.099624 < 20.$$

**A2.** Suppose now that $3x^4 - 11x^2 + 5 < 0$, which holds if, and only if, $x \in (x_0, 1]$. Then

$$\begin{aligned}h(x,y) =&(1 - x^2)\big[4(2x^3 - 5x^2 + 5) + 4(x^4 + 5x^3 + 3x^2 + x)y \\ & + (5x^4 - 16x^3 + 17x^2 + 8x - 16)y^2 \\ & - 4(3x^4 + 5x^3 - 8x^2 + x + 5)y^3 \\ & -4(x^4 - 2x^3 - 2x^2 + 2x + 1)y^4\big], \quad x \in (x_0, 1],\ y \in [0,1].\end{aligned}$$

I. On the vertices of $(x_0,1]\times[0,1]$, we have

$$h(1,0)=h(1,1)=0.$$

II. On the sides of $(x_0,1]\times[0,1]$ we have
(a) If $x=1,\ y\in(0,1)$ then

$$h(1,y)=0,\quad y\in(0,1).$$

(b) If $y=0,\ x\in(x_0,1),$ then for $x\in(0,x_0)$,

$$h(x,0)=4(1-x^2)(2x^3-5x^2+5)=4(-2x^5+5x^4+2x^3-10x^2+5)\leqslant 20$$

since the last inequality is equivalent to the obviously true inequality

$$-x^2(5-x)(2-x^2)\leqslant 0,\quad x\in(x_0,1].$$

(c) If $y=1,\ x\in(x_0,1),$ then

$$h(x,1)=(1-x^2)(-7x^4+49x^2-20)=7x^6-56x^4+69x^2-20=:h_2(x),\quad x\in[0,1].$$

Since $h_2'(x)=42x^5-224x^3+138x<0$ for $x\in(0,1)$, the function $h_2$ decreases and

$$h(x,1)=h_2(x)\leqslant h_2(x_0)<h_2(0)=16,\quad x\in(x_0,1).$$

III. It remains to consider the set $(x_0,1)\times(0,1)$. Then all real solutions of the system of equations

$$\begin{aligned}\frac{\partial h}{\partial x}=&-40x^4+80x^3+24x^2-80x\\&-(24x^5+100x^4+32x^3-48x^2-24x-4)y\\&-(30x^5-80x^4+48x^3+72x^2-66x-8)y^2\\&+(72x^5+100x^4-176x^3-48x^2+104x-4)y^3\\&+(24x^5-40x^4-48x^3+48x^2+24x-8)y^4=0,\end{aligned}$$

$$\begin{aligned}\frac{\partial h}{\partial y}=&(1-x^2)\left[4(x^4+5x^3+3x^2+x)\right.\\&+2(5x^4-16x^3+17x^2+8x-16)y\\&-12(3x^4+5x^3-8x^2+x+5)y^2\\&\left.-16(x^4-2x^3-2x^2+2x+1)y^3\right]=0\end{aligned}$$

by a numerical computation are the following

$$\begin{cases} x_1 \approx -7.031128 \\ y_1 \approx 0.504740 \end{cases} \quad \begin{cases} x_2 \approx -1.669260 \\ y_2 \approx -0.859642 \end{cases} \quad \begin{cases} x_3 \approx -0.954646 \\ y_3 \approx -0.545208 \end{cases}$$
$$\begin{cases} x_4 \approx -0.926571 \\ y_4 \approx 0.228259 \end{cases} \quad \begin{cases} x_5 \approx -0.728005 \\ y_5 \approx 0.784228 \end{cases} \quad \begin{cases} x_6 \approx 0.004713 \\ y_6 \approx -0.643402 \end{cases}$$
$$\begin{cases} x_7 \approx 0.564827 \\ y_7 \approx -2.016450 \end{cases} \quad \begin{cases} x_8 \approx 0.886349 \\ y_8 \approx -0.809809 \end{cases} \quad \begin{cases} x_9 \approx 1.216558 \\ y_9 \approx 7.870078 \end{cases}$$
$$\begin{cases} x_{10} \approx 3.691002 \\ y_{10} \approx -8.082677 \end{cases} \quad \begin{cases} x_{11} = 0 \\ y_{11} = 0 \end{cases} \quad \begin{cases} x_{12} = -1 \\ y_{12} = 2/3 \end{cases}$$
$$\begin{cases} x_{13} = 1 \\ y_{13} \approx -1.223140 \end{cases} \quad \begin{cases} x_{14} = 1 \\ y_{14} = -0.202956 \end{cases} \quad \begin{cases} x_{15} = 1 \\ y_{15} = 1.342763 \end{cases}.$$

Thus the function $h$ has no critical point in $(x_0,1)\times(0,1)$.

**B.** Suppose that

$$\left|3\zeta_1^2\overline{\zeta_2} - (1-\zeta_1^2)(5+|\zeta_2|^2)\right| - 3\left|\zeta_1^2 - 2(1-\zeta_1^2)\zeta_2\right| < 0.$$

Then

$$|H_{3,1}(f)| \leqslant \frac{1}{320} g(\zeta_1, |\zeta_2|),$$

where $g:\mathbb{R}^2\to\mathbb{R}$ is defined by

$$\begin{aligned} g(x,y) :=& (1-x^2)\left[4(2x^3+3x^2)+4(x^4+5x^3-6x^2+x+6)y\right. \\ &+(5x^4-16x^3-11x^2+8x)y^2 \\ &+4(|3x^4-11x^2+5|-5x^3+6x^2-x-6)y^3 \\ &\left.-4(x^4-2x^3-x^2+2x)y^4\right]. \end{aligned}$$

We show now that $|g(x,y)| \leqslant 20$ for $(x,y)\in[0,1]\times[0,1]$.

**B1.** Suppose that $3x^4-11x^2+5 \geqslant 0$, which holds if, and only if, $x\in[0,x_0]$, where $x_0 := \sqrt{(11-\sqrt{61})/6} \approx 0.729$. Then

$$\begin{aligned} g(x,y) =& (1-x^2)\left[4(2x^3+3x^2)+4(x^4+5x^3-6x^2+x+6)y\right. \\ &+(5x^4-16x^3-11x^2+8x)y^2+4(3x^4-5x^3-5x^2-x-1)y^3 \\ &\left.-4(x^4-2x^3-x^2+2x)y^4\right], \quad x\in[0,x_0],\ y\in[0,1]. \end{aligned}$$

I. On the vertices of $[0,x_0]\times[0,1]$ we have

$$g(0,0)=0, \quad g(0,1)=20,$$

$$g(x_0,0) = \frac{2\sqrt{61}-10}{3}\left[2\left(\frac{11-\sqrt{61}}{6}\right)^{\frac{3}{2}} - \frac{\sqrt{61}}{6} + \frac{11}{2}\right] \approx 4.440417 < 20,$$

$$g(x_0,1) = \frac{485\sqrt{61}-3685}{54} \approx 1.906872 < 20.$$

II. On the sides of $[0,x_0]\times[0,1]$ we have
(a) If $x=0,\ y\in(0,1)$, then

$$g(0,y)=24y-4y^3\leqslant g(0,1)=20,\quad y\in(0,1).$$

(b) If $y=0,\ x\in(0,x_0)$, then

$$g(x,0)=-8x^5-12x^4+8x^3+12x^2=:g_1(x),\quad x\in[0,x_0].$$

Since

$$g_1'(x)=-40x^4-48x^3+24x^2+24x>0,\quad x\in(0,x_0),$$

$g$ increases and so

$$g(x,y)=g_1(x)\leqslant g_1(x_0)=\frac{485\sqrt{61}-3685}{54}\approx 1.906872<20,\quad x\in(0,x_0).$$

Note that $g_1'$ has in $(0,1)$ a unique zero at $x\approx 0.7334\notin(0,x_0)$.
(c) If $y=1,\ x\in(0,x_0)$, then

$$h(x,1)=-17x^6+56x^4-59x^2+20\leqslant 20,\quad x\in(0,x_0),$$

since the last inequality is equivalent to the obviously true inequality

$$-x^2(17x^4-56x^2+59)\leqslant 0,\quad x\in(0,x_0).$$

(d) If $x=x_0,\ y\in(0,1)$, then

$$\begin{aligned}g(x_0,y)=&\frac{\sqrt{61}-5}{6}\left\{\left[8\left(\frac{11-\sqrt{61}}{6}\right)^{\frac{3}{2}}-8\sqrt{\frac{11-\sqrt{61}}{6}}+\frac{16\sqrt{61}}{9}-\frac{116}{9}\right]y^4\right.\\&-\left[15\left(\frac{11-\sqrt{61}}{6}\right)^{\frac{3}{2}}+4\sqrt{\frac{11-\sqrt{61}}{6}}+4\sqrt{61}-20\right]y^3\\&-\left[16\left(\frac{11-\sqrt{61}}{6}\right)^{\frac{3}{2}}-8\sqrt{\frac{11-\sqrt{61}}{6}}+\frac{11\sqrt{61}}{9}-\frac{46}{9}\right]y^2\\&+\left[20\left(\frac{11-\sqrt{61}}{6}\right)^{\frac{3}{2}}+4\sqrt{\frac{11-\sqrt{61}}{6}}+\frac{14\sqrt{61}}{9}+\frac{2}{9}\right]y\\&\left.+8\left(\frac{11-\sqrt{61}}{6}\right)^{\frac{3}{2}}-2\sqrt{61}+22\right\}=:g_2(y),\quad y\in(0,1).\end{aligned}$$

Observe that $g_2'(y)=0$ only for $y=y_0\approx 0.320862$ with $g_2(y_0)\approx 7.353760$. Thus by Part I we deduce that

$$g(x_0,y)=g_2(x)\leqslant\max\{g(x_0,0),g(x_0,1),g_2(y_0)\}\leqslant 7.36<20.$$

III. It remains to consider the set $(0,x_0)\times(0,1)$. Then all real solutions of the system of equations

$$\begin{aligned}\frac{\partial g}{\partial x}=&-40x^4-48x^3+24x^2+24x\\&-(24x^5+100x^4-112x^3-48x^2+96x-4)y\\&-(30x^5-80x^4-64x^3+72x^2+22x-8)y^2\\&-(72x^5-100x^4-128x^3+48x^2+32x+4)y^3\\&+(24x^5-40x^4-32x^3+48x^2+8x-8)y^4=0,\end{aligned}$$

$$\begin{aligned}\frac{\partial g}{\partial y}=&(1-x^2)\left[4(x^4+5x^3-6x^2+x+6)\right.\\&+2(5x^4-16x^3-11x^2+8x)y\\&+12(3x^4-5x^3-5x^2-x-1)y^2\\&\left.-16(x^4-2x^3-x^2+2x)y^3\right]=0\end{aligned}$$

by a numerical computation are the following

$$\begin{cases}x_1\approx-3.155561\\y_1\approx-0.458540\end{cases}\quad\begin{cases}x_2\approx-1.158451\\y_2\approx 4.101246\end{cases}\quad\begin{cases}x_3\approx-0.919613\\y_3\approx 0.729786\end{cases}$$

$$\begin{cases}x_4\approx-0.768375\\y_4\approx-0.127076\end{cases}\quad\begin{cases}x_5\approx 0.052913\\y_5\approx -1.489266\end{cases}\quad\begin{cases}x_6\approx 0.190502\\y_6\approx -2.680476\end{cases}$$

$$\begin{cases}x_7\approx 0.898787\\y_7\approx-0.656863\end{cases}\quad\begin{cases}x_8\approx 1.367317\\y_8\approx 14.375248\end{cases}\quad\begin{cases}x_9\approx 1.480316\\y_9\approx -0.804262\end{cases}$$

$$\begin{cases}x_{10}=1\\y_{10}\approx 0.981079\end{cases}\quad\begin{cases}x_{11}=-1\\y_{11}\approx -1.462304\end{cases}\quad\begin{cases}x_{12}=-1\\y_{12}\approx 0.209961\end{cases}$$

$$\begin{cases}x_{13}=-1\\y_{13}\approx 1.085675\end{cases}.$$

Thus the function $g$ has no critical point in $(x_0,1)\times(0,1)$.

**B2.** Suppose that $3x^4-11x^2+5<0$, which holds if, and only if, $x\in(x_0,1]$. Then

$$\begin{aligned}g(x,y)=&(1-x^2)\left[4(2x^3+3x^2)+4(x^4+5x^3-6x^2+x+6)y\right.\\&+(5x^4-16x^3-11x^2+8x)y^2-4(3x^4+5x^3-17x^2+x+11)y^3\\&\left.-4(x^4-2x^3-x^2+2x)y^4\right],\quad x\in(x_0,1],\ y\in[0,1].\end{aligned}$$

I. On the vertices of $(x_0,1]\times[0,1]$ we have

$$g(1,0)=g(1,1)=0.$$

II. On the sides of $(x_0,1]\times[0,1]$ we have

(a) If $x=1,\ y\in(0,1)$, then

$$g(1,y)=0,\quad y\in(0,1).$$

(b) If $y=0$, $x\in(x_0,1)$, then

$$g(x,0)=g_1(x), \quad x\in(x_0,1),$$

where $g_1$ is defined in Part B1.II(b).

(c) If $y=1$, $x\in(x_0,1)$, then

$$g(x,1)=7x^6-56x^4+69x^2-20=: g_3(x), \quad x\in(x_0,1).$$

Since $g_3'(x)=42x^5-224x^3+138x=0$ only for $x=\sqrt{8/3-\sqrt{1687}/21}\approx 0.843092$, where $g_3$ attains its maximum, we deduce that for $x\in(x_0,1)$,

$$g(x,1)=g_3(x)\leqslant g\left(\sqrt{\frac{8}{3}-\frac{\sqrt{1687}}{21}}\right)=\frac{482\sqrt{1687}}{189}-\frac{2740}{27}\approx 3.265804<20.$$

III. It remains to consider the set $(x_0,1)\times(0,1)$. Then all real solutions of the system of equations

$$\begin{aligned}\frac{\partial g}{\partial x}=&-40x^4-48x^3+24x^2+24x\\&-(24x^5+100x^4-112x^3-48x^2+96x-4)y\\&-(30x^5-80x^4-64x^3+72x^2+22x-8)y^2\\&+(72x^5+100x^4-320x^3-48x^2+224x-4)y^3\\&+(24x^5-40x^4-32x^3+48x^2+8x-8)y^4=0,\end{aligned}$$

$$\begin{aligned}\frac{\partial g}{\partial y}=&(1-x^2)\big[4(x^4+5x^3-6x^2+x+6)\\&+2(5x^4-16x^3-11x^2+8x)y\\&-12(3x^4+5x^3-17x^2+x+11)y^2\\&-16(x^4-2x^3-x^2+2x)y^3\big]=0\end{aligned}$$

by a numerical computation are the following

$$\begin{cases}x_1\approx -6.224355\\ y_1\approx 0.479164\end{cases}\quad \begin{cases}x_2\approx -0.968969\\ y_2\approx 0.414411\end{cases}\quad \begin{cases}x_3\approx -0.831610\\ y_3\approx -0.338257\end{cases}$$

$$\begin{cases}x_4\approx -0.772427\\ y_4\approx -0.136537\end{cases}\quad \begin{cases}x_5\approx -0.729807\\ y_5\approx -0.781026\end{cases}\quad \begin{cases}x_6\approx -0.721994\\ y_6\approx 0.753954\end{cases}$$

$$\begin{cases}x_7\approx 0.004694\\ y_7\approx -0.426300\end{cases}\quad \begin{cases}x_8\approx 0.590399\\ y_8\approx 0.484376\end{cases}\quad \begin{cases}x_9\approx 0.763704\\ y_9\approx -0.752592\end{cases}$$

$$\begin{cases}x_{10}\approx 0.830666\\ y_{10}\approx -10.538708\end{cases}\quad \begin{cases}x_{11}\approx 0.898772\\ y_{11}\approx -1.054646\end{cases}\quad \begin{cases}x_{12}\approx 1.233102\\ y_{12}\approx 5.163931\end{cases}$$

$$\begin{cases}x_{13}\approx 1.547826\\ y_{13}\approx -1.289887\end{cases}\quad \begin{cases}x_{14}\approx 2.756334\\ y_{14}\approx -8.717327\end{cases}\quad \begin{cases}x_{15}\approx 5.779609\\ y_{15}\approx -0.380734\end{cases}$$

$$\begin{cases} x_{16} = 1 \\ y_{16} \approx -1.928459 \end{cases} \begin{cases} x_{17} = 1 \\ y_{17} \approx -0.623757 \end{cases} \begin{cases} x_{18} = 1 \\ y_{18} \approx 1.38555 \end{cases}$$
$$\begin{cases} x_{19} = -1 \\ y_{19} \approx -0.856098 \end{cases} \begin{cases} x_{20} = -1 \\ y_{20} \approx 0.225831 \end{cases} \begin{cases} x_{21} = -1 \\ y_{21} \approx 0.574711 \end{cases}.$$

Thus the function $g$ has no critical point in $(x_0, 1) \times (0, 1)$.

**C.** Summarizing, we see that the bounds obtained in Parts A and B give

$$|H_{3,1}(f)| \leqslant \frac{1}{320} \cdot 20 = \frac{1}{16}.$$

We finally note that equality in (9) holds for the function $f \in \mathscr{C}(-1/2)$ satisfying (10) with

$$p(z) := \frac{1+z^3}{1-z^3}, \quad z \in \mathbb{D},$$

for which $a_2 = a_3 = a_5 = 0$, and $a_4 = 1/4$. This completes the proof of the theorem. $\square$

REMARK 2.2. We note that using Lemma 1.1 it is a relatively simple exercise to prove that $|H_{2,2}(f)| \leqslant 21/64$ when $f \in \mathscr{C}(-1/2)$, and that this inequality is sharp.

## REFERENCES


[1] C. CARATHÉODORY, *Über den Variabilitatsbereich der Koeffizienten von Potenzreihen, die gegebene werte nicht annehmen*, Math. Ann. **64** (1907), 95–115.

[2] L. CARLITZ, *Hankel determinants and Bernoulli numbers*, Tohoku Math. J. **5** (1954), 272–276.

[3] N. E. CHO, B. KOWALCZYK, O. S. KWON, A. LECKO, Y. J. SIM, *The bounds of some determinants for starlike functions of order alpha*, Bull. Malays. Math. Sci. Soc. **41** (2018), 523–535.

[4] N. E. CHO, B. KOWALCZYK, O. S. KWON, A. LECKO, Y. J. SIM, *The bound of the Hankel detrminant for strongly starlike functions of order alpha*, J. Math. Ineq. **11** (2017), no. 2, 429–439.

[5] A. W. GOODMAN, *Univalent Functions*, Mariner, Tampa, Florida, 1983.

[6] W. K. HAYMAN, *On the second Hankel determinant of mean univalent functions*, Proc. London Math. Soc. **18** (3) (1968), 77–94.

[7] W. KAPLAN, *Close to convex schlicht functions*, Michigan Math. J. **1** (1952), 169–186.

[8] B. KOWALCZYK, A. LECKO, *The sharp bound of the third Hankel determinant for functions of bounded turning*, Bol. Soc. Mat. Mex. **27** (2021), Art. 69, pp. 13.

[9] B. KOWALCZYK, A. LECKO, M. LECKO, Y. J. SIM, *The sharp bound of the third Hankel determinant for some classes of analytic functions*n Bull. Korean Math. Soc. **55** (2018), no. 6, 1859–1868.

[10] B. KOWALCZYK, A. LECKO, Y. J. SIM, *The sharp bound of the Hankel determinant of the third kind for convex functions*, Bull. Aust. Math. Soc. **97** (2018), 435–445.

[11] B. KOWALCZYK, A. LECKO, D. K. THOMAS, *The sharp bound of the third Hankel determinant for starlike functions*, Forum Math. **34** (2022), no. 5, 1249–1254.

[12] O. S. KWON, A. LECKO, Y. J. SIM, *On the fourth coefficient of functions in the Carathéodory class*, Comput. Methods Funct. Theory **18** (2018), 307–314.

[13] A. LECKO, Y. J. SIM, B. ŚMIAROWSKA, *The Sharp Bound of the Hankel Determinant of the Third Kind for Starlike Functions of Order 1/2*, Complex Anal. Oper. Theory **13** (2019), 2231–2238.

[14] R. J. LIBERA, E. J. ZLOTKIEWICZ, *Early coefficients of the inverse of a regular convex function*, Proc. Amer. Math. Soc. **85** (1982), no. 2, 225–230.

[15] R. J. LIBERA, E. J. ZLOTKIEWICZ, *Coefficient bounds for the inverse of a function with derivatives in $\mathscr{P}$*, Proc. Amer. Math. Soc. **87** (1983), no. 2, 251–257.

[16] M. OBRADOVIĆ, N. TUNESKI, *Some properties of the class $\mathscr{U}$*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **73** (2019), 49–56.

[17] M. OBRADOVIĆ, N. TUNESKI, *Improved upper bound of third order Hankel determinant for Ozaki close-to-convex functions*, Preprint, arXiv:2004.10680 [math.CV].

[18] S. OZAKI, *On the theory of multivalent functions*, Sci. Rep. Tokyo Bunrika Daigaku Sect. A **4** (1941), 45–87.

[19] C. POMMERENKE, *On the coefficients and Hankel determinants of univalent functions*, J. London Math. Soc. **41** (1966), 111–122.

[20] C. POMMERENKE, *On the Hankel determinants of Univalent functions*, Mathematika **14** (1967), 108–112.

[21] C. POMMERENKE, *Univalent Functions*, Vandenhoeck & Ruprecht, Göttingen, 1975.

[22] I. J. SCHOENBERG, *On the maxima of certain Hankel determinants and the zeros of the classical orthogonal polynomials*, Indag. Math. **21** (1959), 282–290.

[23] T. UMEZAWA, *Analytic functions convex in one direction*, J. Math. Soc. Japan **4** (1952), 194–20.

(Received June 19, 2022)

*Bogumiła Kowalczyk*
*Department of Complex Analysis, Faculty of Mathematics and Computer Science*
*University of Warmia and Mazury in Olsztyn*
*ul. Słoneczna 54, 10-710 Olsztyn, Poland*
*e-mail:* b.kowalczyk@matman.uwm.edu.pl

*Adam Lecko*
*Department of Complex Analysis, Faculty of Mathematics and Computer Science*
*University of Warmia and Mazury in Olsztyn*
*ul. Słoneczna 54, 10-710 Olsztyn, Poland*
*e-mail:* alecko@matman.uwm.edu.pl

*Derek K. Thomas*
*Department of Complex Analysis, Department of Mathematics*
*Swansea University Bay Campus*
*Swansea, SA1 8EN, United Kingdom*
*e-mail:* d.k.thomas@swansea.ac.uk