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# SOME NEW IMPROVEMENTS OF YOUNG'S INEQUALITIES

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*Abstract.* In this paper, we obtain some improvements and generalizations of Young's inequalities as the following:

(1) If $b \geqslant a$, we can get

$$\frac{(a\nabla_v b)^m - (a\sharp_v b)^m}{(a\nabla_\tau b)^m - (a\sharp_\tau b)^m} \leqslant \frac{v(1-v)}{\tau(1-\tau)};$$

(2) If $b \leqslant a$, we can get

$$\frac{(a\nabla_v b)^m - (a\sharp_v b)^m}{(a\nabla_\tau b)^m - (a\sharp_\tau b)^m} \geqslant \frac{v(1-v)}{\tau(1-\tau)}$$

for $m \in N_+$ and $0 < v \leqslant \tau < 1$. In addition, we obtain new result of Young's inequality by using the expansions of the functions $(1-v)+vx-x^v$ with $0 < x < 2$.

## 1. Introduction

The Young's inequality [8] is well known as the following: If $a,b > 0$ and $0 \leqslant v \leqslant 1$, then

$$a\sharp_v b = a^{1-v}b^v \leqslant (1-v)a + vb = a\nabla_v b \tag{1.1}$$

where equality holds if and only if $a = b$. Let $\frac{b}{a} = x$ in inequality (1.1), then we can obtain the equivalent inequality

$$0 \leqslant (1-v) + vx - x^v. \tag{1.2}$$

Liao, Wu and Zhao [7] showed the reverse inequality of the above Young's inequality with Kantorovich constant

$$(1-v)a + vb \leqslant K(h,2)^R a^{1-v}b^v \tag{1.3}$$

where $a$, $b \geqslant 0$, $R = \max\{v, 1-v\}$ and $K(h,2) = \frac{(h+1)^2}{4h}$ with $h = \frac{b}{a}$.

He [2] and Hirzallah [3] refined Young's inequality so that

$$r^2(a-b)^2 \leqslant [(1-v)a+vb]^2 - (a^{1-v}b^v)^2 \leqslant R^2(a-b)^2 \tag{1.4}$$

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where $a,\ b \geqslant 0,\ r = \min\{v, 1-v\}$ and $R = \max\{v, 1-v\}$.

Alzer, Fonseca and Kovačec [1] presented the following Young's inequalities

$$\frac{v^m}{\tau^m} \leqslant \frac{(a\nabla_v b)^m - (a\sharp_v b)^m}{(a\nabla_\tau b)^m - (a\sharp_\tau b)^m} \leqslant \frac{(1-v)^m}{(1-\tau)^m} \tag{1.5}$$

for $0 < v \leqslant \tau < 1$ and $m \in N_+$.

Liao and Wu [5] replicated the above result as follows:

$$\frac{v^m}{\tau^m} \leqslant \frac{(a\nabla_v b)^m - (a!_v b)^m}{(a\nabla_\tau b)^m - (a!_\tau b)^m} \leqslant \frac{(1-v)^m}{(1-\tau)^m} \tag{1.6}$$

for $0 < v \leqslant \tau < 1$ and $m \in N_+$.

Sababheh [10] obtained by convexity of function $f$

$$\frac{v^m}{\tau^m} \leqslant \frac{[(1-v)f(0)+vf(1)]^m - f^m(v)}{[(1-\tau)f(0)+\tau f(1)]^m - f^m(\tau)} \leqslant \frac{(1-v)^m}{(1-\tau)^m} \tag{1.7}$$

for $0 < v \leqslant \tau < 1$ and $m \in N_+$.

Ren [9] obtained the following inequalities:

$$\begin{cases} \dfrac{a\nabla_v b - a\sharp_v b}{a\nabla_\tau b - a\sharp_\tau b} \leqslant \dfrac{v(1-v)}{\tau(1-\tau)}, & b-a \geqslant 0 \\ \dfrac{a\nabla_v b - a\sharp_v b}{a\nabla_\tau b - a\sharp_\tau b} \geqslant \dfrac{v(1-v)}{\tau(1-\tau)}, & b-a \leqslant 0 \end{cases} \tag{1.8}$$

and

$$\begin{cases} \dfrac{(a\nabla_v b)^2 - (a\sharp_v b)^2}{(a\nabla_\tau b)^2 - (a\sharp_\tau b)^2} \leqslant \dfrac{v(1-v)}{\tau(1-\tau)}, & b-a \geqslant 0 \\ \dfrac{(a\nabla_v b)^2 - (a\sharp_v b)^2}{(a\nabla_\tau b)^2 - (a\sharp_\tau b)^2} \geqslant \dfrac{v(1-v)}{\tau(1-\tau)}, & b-a \leqslant 0 \end{cases} \tag{1.9}$$

for $0 < v \leqslant \tau < 1$ and $a,\ b > 0$.

In addition, Zhu [11] obtained new Young's inequalities by using the expansions of the functions $\frac{(1-v)+vx}{x^v}$.

In this paper, we generalize a part of above results in section 2. In section 3 we obtain following results through using the expansions of the functions $(1-v)+vx-x^v$

$$\begin{cases} (1-v)+vx-x^v \geqslant \displaystyle\sum_{k=2}^{2m} \alpha_k(v)(x-1)^k, & x \in (0,1] \\ (1-v)+vx-x^v \leqslant \displaystyle\sum_{k=2}^{2m} \alpha_k(v)(x-1)^k, & x \in [1,\infty) \end{cases}$$

and

$$(1-v)+vx-x^v \geqslant \sum_{k=2}^{2m+1} \alpha_k(v)(x-1)^k$$

for $0 \leqslant v \leqslant 1$, $m \in N_+$ and $x > 0$ where $\alpha_k(v) = \frac{(-1)^k v(1-v)(2-v)\cdots((k-1)-v)}{k!}$. And our result is the improvement of [11, Corollary 1] when $m = 1$. Finally, we present trace norm, Hilbert-Schmidt norm and determinant version of results in section 2.

## 2. Generalized impromențs of Young's inequalities

We firstly show the generalization of Young's inequality [9] for scalars under some conditions.

THEOREM 2.1. *Let $0 < v \leqslant \tau < 1$, $m \in N_+$ and $a$, $b$ are real positive numbers. Then*

*(1) If $b \geqslant a$, we can get*

$$\frac{(a\nabla_v b)^m - (a\sharp_v b)^m}{(a\nabla_\tau b)^m - (a\sharp_\tau b)^m} \leqslant \frac{v(1-v)}{\tau(1-\tau)}; \tag{2.1}$$

*(2) If $b \leqslant a$, we can get*

$$\frac{(a\nabla_v b)^m - (a\sharp_v b)^m}{(a\nabla_\tau b)^m - (a\sharp_\tau b)^m} \geqslant \frac{v(1-v)}{\tau(1-\tau)}. \tag{2.2}$$

*Proof.* Firstly, we have

$$\begin{aligned}
&(1-v+vx)^m - x^{mv} \\
&= (1-v+vx-x^v)[(1-v+vx)^{m-1} + (1-v+vx)^{m-2}x^v + \cdots \\
&\quad + (1-v+vx)x^{(m-2)v} + x^{(m-1)v}].
\end{aligned}$$

Then let $f(v) = (1-v+vx)^{m-1} + (1-v+vx)^{m-2}x^v + \cdots + (1-v+vx)x^{(m-2)v} + x^{(m-1)v}$, we can get

$$\begin{aligned}
f'(v) &= (m-1)(x-1)(1-v+vx)^{m-2} + (m-2)(x-1)(1-v+vx)^{m-3}x^v \\
&\quad + (1-v+vx)^{m-2}x^v \ln x + \cdots + (x-1)x^{(m-2)v} + (1-v+vx)(m-2)x^{(m-2)v}\ln x \\
&\quad + (m-1)x^{(m-1)v}\ln x \\
&= (x-1)[(m-1)(1-v+vx)^{m-2} + (m-2)(1-v+vx)^{m-3}x^v + \cdots + x^{(m-2)v}] \\
&\quad + \ln x[(1-v+vx)^{m-2}x^v + (1-v+vx)^{m-3}2x^{2v} + \cdots \\
&\quad + (1-v+vx)(m-2)x^{(m-2)v} + (m-1)x^{(m-1)v}].
\end{aligned}$$

(1) If $x \geqslant 1$, we have $1-v+vx = 1+(x-1)v \geqslant 1$. So it's obvious that $f'(v) \geqslant 0$, it means that $f(v)$ is increasing on $[1,\infty)$, that is to say $\frac{f(v)}{f(\tau)} \leqslant 1$. Therefore

$$\begin{aligned}
\frac{(1-v+vx)^m - x^{mv}}{(1-\tau+\tau x)^m - x^{m\tau}} &= \frac{((1-v+vx)-x^v)f(v)}{((1-\tau+\tau x)-x^\tau)f(\tau)} \\
&\leqslant \frac{(1-v+vx)-x^v}{(1-\tau+\tau x)-x^\tau} \\
&\leqslant \frac{v(1-v)}{\tau(1-\tau)} \quad \text{(by 1.8)}.
\end{aligned}$$

(2) If $0 < x \leqslant 1$, we have $1-v+vx = 1+(x-1)v \geqslant 0$ and $\ln x \leqslant 0$. So it's obvious that $f'(v) \leqslant 0$, it means that $f(v)$ is decreasing on $(0,1]$, that is to say $\frac{f(v)}{f(\tau)} \geqslant 1$. Therefore

$$\begin{aligned}\frac{(1-v+vx)^m - x^{mv}}{(1-\tau+\tau x)^m - x^{m\tau}} &= \frac{((1-v+vx)-x^v)f(v)}{((1-\tau+\tau x)-x^\tau)f(\tau)} \\ &\geqslant \frac{(1-v+vx)-x^v}{(1-\tau+\tau x)-x^\tau} \\ &\geqslant \frac{v(1-v)}{\tau(1-\tau)} \quad \text{(by 1.8).}\end{aligned}$$

Taking $x = \frac{b}{a}$, we can get our desired results directly. □

REMARK 2.1. (1) Let $m = 2$, we can get [9, Theorem 2.3].

(2) Let $a = b$, $b = a$, $v = 1-\tau$, $\tau = 1-v$ in inequality (2.1), we can also get inequality (2.2) directly.

(3) Let $0 < v \leqslant \tau < 1$, so $\frac{1-v}{1-\tau} \geqslant 1$, therefore

(i) If $b \geqslant a$, we can get

$$\frac{(a\nabla_v b)^m - (a\sharp_v b)^m}{(a\nabla_\tau b)^m - (a\sharp_\tau b)^m} \leqslant \frac{v(1-v)}{\tau(1-\tau)} \leqslant \frac{v(1-v)^m}{\tau(1-\tau)^m} \leqslant \frac{(1-v)^m}{(1-\tau)^m};$$

(ii) If $b \leqslant a$, we can get

$$\frac{(a\nabla_v b)^m - (a\sharp_v b)^m}{(a\nabla_\tau b)^m - (a\sharp_\tau b)^m} \geqslant \frac{v(1-v)}{\tau(1-\tau)} \geqslant \frac{v^m(1-v)}{\tau^m(1-\tau)} \geqslant \frac{v^m}{\tau^m}.$$

It is not difficult to see that Theorem 2.1 is the improvements of [1].

THEOREM 2.2. *Let $\frac{1}{2} < v \leqslant \tau \leqslant 1$ and $a$, $b$ are real positive numbers. Then*

$$\frac{K(h,2)^v a\sharp_v b - a\nabla_v b}{K(h,2)^\tau a\sharp_\tau b - a\nabla_\tau b} \leqslant \frac{v}{\tau} \tag{2.3}$$

*where $K(h,2) = \frac{(h+1)^2}{4h}$ and $h = \frac{b}{a}$.*

*Proof.* Firstly, we let

$$\begin{aligned}f(v) &= \frac{K^v(x,2)(x^v) - (1-v+vx)}{v} \\ &= \frac{\left(\frac{x+1}{2}\right)^{2v} - (1-v+vx)}{v}.\end{aligned}$$

Then we can get

$$\begin{aligned}f'(v) &= \frac{v\left[2\left(\frac{x+1}{2}\right)^{2v}\ln\left(\frac{x+1}{2}\right) - (x-1)\right] - \left[\left(\frac{x+1}{2}\right)^{2v} - (1-v+vx)\right]}{v^2} \\ &= \frac{\left(\frac{x+1}{2}\right)^{2v}\left[2v\ln\left(\frac{x+1}{2}\right) - 1\right] + 1}{v^2} \\ &= \frac{h(x)}{v^2}\end{aligned}$$

and

$$\begin{aligned} h'(x) &= v\left(\frac{x+1}{2}\right)^{2v-1}\left[2v\ln\left(\frac{x+1}{2}\right)-1\right]+v\left(\frac{x+1}{2}\right)^{2v-1} \\ &= 2v^2\left(\frac{x+1}{2}\right)^{2v-1}\ln\left(\frac{x+1}{2}\right). \end{aligned}$$

It means that $x\in(0,1]$, $h'(x)\leqslant 0$; $x\in[1,\infty)$, $h'(x)\geqslant 0$. So $h(x)\geqslant h(1)=0$ and $f'(v)\geqslant 0$. Therefore $f(v)$ is increasing on $(0,+\infty)$.

Taking $x=\frac{b}{a}$, we can get our desired results directly. □

THEOREM 2.3. *Let* $0<v\leqslant\tau\leqslant\frac{1}{2}$ *and* $a$*,* $b$ *are real positive numbers. Then*

$$\frac{(a\nabla_v b)^2-(a\sharp_v b)^2-v^2(a-b)^2}{(a\nabla_\tau b)^2-(a\sharp_\tau b)^2-\tau^2(a-b)^2}\geqslant\frac{v}{\tau}. \tag{2.4}$$

*Proof.* Firstly, we let $f(v)=\frac{(1-v+vx)^2-x^{2v}-v^2(x-1)^2}{v}$. Then

$$\begin{aligned} f'(v) &= \frac{v\left[2(x-1)(1-v+vx)-2x^{2v}\ln x-2v(x-1)^2\right]-\left[(1-v+vx)^2-x^{2v}-v^2(x-1)^2\right]}{v^2} \\ &= \frac{(1-v+vx)(vx-v-1)+x^{2v}-2vx^{2v}\ln x-v^2(x-1)^2}{v^2} \\ &= \frac{h(x)}{v^2} \end{aligned}$$

and

$$\begin{aligned} h'(x) &= v(vx-v-1)+v(1-v+vx)+2vx^{2v-1}-4v^2x^{2v-1}\ln x-2vx^{2v-1}-2v^2(x-1) \\ &= -4v^2x^{2v-1}\ln x. \end{aligned}$$

It means that $x\in(0,1]$, $h'(x)\geqslant 0$; $x\in[1,\infty)$, $h'(x)\leqslant 0$. So $h(x)\leqslant h(1)=0$ and $f'(v)\leqslant 0$. Therefore $f(v)$ is decreasing on $(0,+\infty)$.

Taking $x=\frac{b}{a}$, we can get our desired results directly. □

THEOREM 2.4. *Let* $\frac{1}{2}<v\leqslant\tau\leqslant 1$ *and* $a$*,* $b$ *are real positive numbers. Then*

$$\frac{(a\sharp_v b)^2+v^2(a-b)^2-(a\nabla_v b)^2}{(a\sharp_\tau b)^2+\tau^2(a-b)^2-(a\nabla_\tau b)^2}\leqslant\frac{v}{\tau}. \tag{2.5}$$

*Proof.* Firstly, we let $f(v)=\frac{x^{2v}+v^2(x-1)^2-(1-v+vx)^2}{v}$. Then

$$\begin{aligned} f'(v) &= \frac{v\left[2x^{2v}\ln x+2v(x-1)^2-2(x-1)(1-v+vx)\right]}{v^2} \\ &\quad -\frac{\left[x^{2v}+v^2(x-1)^2-(1-v+vx)^2\right]}{v^2} \\ &= \frac{2vx^{2v}\ln x+v^2(x-1)^2-x^{2v}-(1-v+vx)(vx-v-1)}{v^2} \\ &= \frac{h(x)}{v^2} \end{aligned}$$

and

$$\begin{aligned} h'(x) &= 4v^2x^{2v-1}\ln x+2v^2(x-1)-\left[v(vx-v-1)+v(1-v+vx)\right] \\ &= 4v^2x^{2v-1}\ln x. \end{aligned}$$

It means that $x\in(0,1]$, $h'(x)\leqslant 0$; $x\in[1,\infty)$, $h'(x)\geqslant 0$. So $h(x)\geqslant h(1)=0$ and $f'(v)\geqslant 0$. Therefore $f(v)$ is increasing on $(0,+\infty)$.

Taking $x=\frac{b}{a}$, we can get our desired results directly. □

## 3. Some new results of Young-type inequalities

According to Newton's binomial expansion for $x\in(-1,1)$,

$$\begin{aligned} (1+x)^v &= 1+vx+\frac{v(v-1)}{2!}x^2+\frac{v(v-1)(v-2)}{3!}x^3+\cdots \\ &\quad +\frac{v(v-1)(v-2)\cdots[v-(k-1)]}{k!}x^k+\cdots. \end{aligned}$$

We can have if $0\leqslant v\leqslant 1$ and $0<x<2$,

$$(1-v)+vx-x^v=\sum_{k=2}^{\infty}\alpha_k(v)(x-1)^k \tag{3.1}$$

where $\alpha_k(v)=\frac{(-1)^k v(1-v)(2-v)\cdots((k-1)-v)}{k!}$. And then we can get some new results of inequality $(1-v)+vx-x^v$ based on (3.1).

THEOREM 3.1. *Let* $0\leqslant v\leqslant 1$, $m\in N_+$ *and* $x>0$. *Then*

$$\begin{cases} (1-v)+vx-x^v\geqslant \sum_{k=2}^{2m}\alpha_k(v)(x-1)^k, & x\in(0,1], \\ (1-v)+vx-x^v\leqslant \sum_{k=2}^{2m}\alpha_k(v)(x-1)^k, & x\in[1,\infty). \end{cases} \tag{3.2}$$

*Proof.* Suppose

$$f(x) = (1-v)+vx-x^v - \sum_{k=2}^{2m} \alpha_k(v)(x-1)^k.$$

Then

$$\begin{aligned}
f'(x) &= v - vx^{v-1} - \sum_{k=2}^{2m} k\alpha_k(v)(x-1)^{k-1},\\
f''(x) &= v(1-v)x^{v-2} - v(1-v) - \sum_{k=3}^{2m} k(k-1)\alpha_k(v)(x-1)^{k-2},\\
&\ \ \vdots\\
f^{(2m-1)}(x) &= (-1)^{2m-1}v(1-v)(2-v)\cdots((2m-2)-v)x^{v-(2m-1)}\\
&\quad -(2m-1)!\alpha_{2m-1}(v)-(2m)!\alpha_{2m}(v)(x-1),\\
f^{(2m)}(x) &= (-1)^{2m}v(1-v)(2-v)\cdots[(2m-1)-v]x^{v-2m}-(2m)!\alpha_{2m}(v).
\end{aligned}$$

Finally, we can get

$$\begin{aligned}
f^{(2m)}(x) &= (-1)^{2m}v(1-v)(2-v)\cdots[(2m-1)-v])x^{v-2m}-(2m)!\alpha_{2m}(v)\\
&= v(1-v)(2-v)\cdots[(2m-1)-v]x^{v-2m} - v(1-v)(2-v)\cdots[(2m-1)-v]\\
&= v(1-v)(2-v)\cdots[(2m-1)-v](x^{v-2m}-1).
\end{aligned}$$

It means that $f^{(2m)}(x) \geqslant 0$ on $(0,1]$ and $f^{(2m)}(x) \leqslant 0$ on $[1,+\infty)$, so that $f^{(2m-1)}(x) \leqslant f^{(2m-1)}(1) = 0$. Therefore $f^{(2m-2)}(x)$ is decreasing on $(0,+\infty)$, $f^{(2m-2)}(x) \geqslant 0$ on $(0,1]$ and $f^{(2m-2)}(x) \leqslant 0$ on $[1,+\infty)$ obviously. By that analogy, $f''(x)$ is decreasing on $(0,+\infty)$. It means that $f''(x) \geqslant 0$ on $(0,1]$ and $f''(x) \leqslant 0$ on $[1,+\infty)$. So $f'(x) \leqslant f'(1) = 0$, that is, $f(x)$ is decreasing on $(0,+\infty)$. According to $f(1) = 0$, we can get desired results. $\square$

THEOREM 3.2. *Let $0 \leqslant v \leqslant 1$, $m \in N_+$ and $x > 0$. Then*

$$(1-v)+vx-x^v \geqslant \sum_{k=2}^{2m+1} \alpha_k(v)(x-1)^k. \tag{3.3}$$

*Proof.* Suppose

$$f(x) = (1-v)+vx-x^v - \sum_{k=2}^{2m+1} \alpha_k(v)(x-1)^k.$$

Then

$$\begin{aligned}
f'(x) &= v - vx^{v-1} - \sum_{k=2}^{2m+1} k\alpha_k(v)(x-1)^{k-1},\\
f''(x) &= v(1-v)x^{v-2} - v(1-v) - \sum_{k=3}^{2m+1} k(k-1)\alpha_k(v)(x-1)^{k-2},\\
&\ \ \vdots
\end{aligned}$$

$$f^{(2m)}(x) = (-1)^{2m} v(1-v)(2-v)\cdots[(2m-1)-v]x^{v-2m} - (2m)!\alpha_{2m}(v) \\ - (2m+1)!\alpha_{2m+1}(v)(x-1),$$
$$f^{(2m+1)}(x) = (-1)^{2m+1} v(1-v)(2-v)\cdots(2m-v)x^{v-(2m+1)} - (2m+1)!\alpha_{2m+1}(v).$$

Finally, we can get

$$\begin{aligned} f^{(2m+1)}(x) &= (-1)^{2m+1} v(1-v)(2-v)\cdots(2m-v)x^{v-(2m+1)} - (2m+1)!\alpha_{2m+1}(v) \\ &= -v(1-v)(2-v)\cdots(2m-v)x^{v-(2m+1)} + v(1-v)(2-v)\cdots(2m-v) \\ &= v(1-v)(2-v)\cdots(2m-v)(1-x^{v-(2m+1)}). \end{aligned}$$

It's obvious that $f^{(2m+1)}(x) \leqslant 0$ on $(0,1]$ and $f^{(2m+1)}(x) \geqslant 0$ on $[1,+\infty)$, so that $f^{(2m)}(x) \geqslant f^{(2m)}(1) = 0$. Therefore $f^{(2m-1)}(x)$ is increasing on $(0,+\infty)$, so $f^{(2m-1)}(x) \leqslant 0$ on $(0,1]$ and $f^{(2m-1)}(x) \geqslant 0$ on $[1,+\infty)$. By that analogy, $f'(x)$ is increasing on $(0,+\infty)$. It means that $f'(x) \leqslant 0$ on $(0,1]$ and $f'(x) \geqslant 0$ on $[1,+\infty)$. So $f(x) \geqslant f(1) = 0$. By simple shift, we can get final result. □

COROLLARY 3.1. *Let* $0 \leqslant v \leqslant 1$ *and* $x > 0$. *Then*

$$\begin{cases} (1-v) + vx - x^v \geqslant \dfrac{v(1-v)}{2}(x-1)^2, & x \in (0,1], \\ (1-v) + vx - x^v \leqslant \dfrac{v(1-v)}{2}(x-1)^2, & x \in [1,\infty). \end{cases} \tag{3.4}$$

*Proof.* Let $m = 1$ in Theorem 3.2, we can get desired results. □

REMARK 3.1. Because $x^v \geqslant 1$ on $[1,\infty)$ and $0 < x^v \leqslant 1$ on $(0,1]$, so

$$(1-v) + vx - x^v \geqslant \frac{v(1-v)}{2}(x-1)^2 \geqslant x^v \frac{v(1-v)}{2}(x-1)^2, \qquad x \in (0,1],$$
$$(1-v) + vx - x^v \leqslant \frac{v(1-v)}{2}(x-1)^2 \leqslant x^v \frac{v(1-v)}{2}(x-1)^2. \qquad x \in [1,\infty).$$

It's not hard to see that the inequality 3.4 is the improvement of [11, Corollary 1].

REMARK 3.2. Let $x = \frac{b}{a}$ in Theorem 3.1 and Theorem 3.2, we can get

$$\begin{cases} (1-v)a + vb - a^{1-v}b^v \geqslant \displaystyle\sum_{k=2}^{2m} \alpha_k(v) a^{1-k}(b-a)^k, & a \geqslant b > 0 \\ (1-v)a + vb - a^{1-v}b^v \leqslant \displaystyle\sum_{k=2}^{2m} \alpha_k(v) a^{1-k}(b-a)^k, & b \geqslant a > 0 \end{cases}$$

and

$$(1-v)a + vb - a^{1-v}b^v \geqslant \sum_{k=2}^{2m+1} \alpha_k(v) a^{1-k}(b-a)^k.$$

It's obvious that $\sum_{k=2}^{n} \alpha_k(v) a^{1-k}(b-a)^k$ is greater than 0 where $a \geqslant b > 0$.

COROLLARY 3.2. *Let* $0 \leqslant v \leqslant \tau \leqslant 1$ *and* $a \geqslant b > 0$. *Then*

$$\frac{(a\nabla_v b) - (a\sharp_v b) - \sum_{k=2}^n \alpha_k(v) a^{1-k}(b-a)^k}{(a\nabla_\tau b) - (a\sharp_\tau b) - \sum_{k=2}^n \alpha_k(\tau) a^{1-k}(b-a)^k} \geqslant \frac{v(1-v)(2-v)\cdots(n-v)}{\tau(1-\tau)(2-\tau)\cdots(n-\tau)}. \tag{3.5}$$

*Proof.* Firstly, we let $x \in (0,2)$ and

$$f(v) = (1-v) + vx - x^v - \sum_{k=2}^n \alpha_k(v)(x-1)^k.$$

According to (3.1), we have

$$f(v) = \sum_{k=n+1}^{\infty} \alpha_k(v)(x-1)^k,$$

so

$$\frac{(1-v) + vx - x^v - \sum_{k=2}^n \alpha_k(v)(x-1)^k}{(1-\tau) + \tau x - x^\tau - \sum_{k=2}^n \alpha_k(\tau)(x-1)^k} = \frac{\sum_{k=n+1}^{\infty} (-1)^k \alpha_k(v)(1-x)^k}{\sum_{k=n+1}^{\infty} (-1)^k \alpha_k(\tau)(1-x)^k}.$$

Let

$$\beta_k(v) = (-1)^k \alpha_k(v) = \frac{v(1-v)(2-v)\cdots(k-1-v)}{k}.$$

When $k \geqslant n+1$, we can get

$$\frac{\beta_k(v)}{\beta_k(\tau)} = \frac{v(1-v)(2-v)\cdots(k-1-v)}{\tau(1-\tau)(2-\tau)\cdots(k-1-\tau)} \geqslant \frac{v(1-v)(2-v)\cdots(n-v)}{\tau(1-\tau)(2-\tau)\cdots(n-\tau)}.$$

Therefore

$$\begin{aligned} \frac{(1-v) + vx - x^v - \sum_{k=2}^n \alpha_k(v)(x-1)^k}{(1-\tau) + \tau x - x^\tau - \sum_{k=2}^n \alpha_k(\tau)(x-1)^k} &= \frac{\sum_{k=n+1}^{\infty} \beta_k(v)(1-x)^k}{\sum_{k=n+1}^{\infty} \beta_k(\tau)(1-x)^k} \\ &\geqslant \frac{v(1-v)(2-v)\cdots(n-v)}{\tau(1-\tau)(2-\tau)\cdots(n-\tau)}. \end{aligned}$$

Taking $x = \frac{b}{a}$, we can get our desired results directly. □

## 4. Applications

Let $M_n(\mathbb{C})$ denotes the space of all $n \times n$ complex matrices and $M_n^+(\mathbb{C})$ denotes the space of all $n \times n$ positive semidefinite matrices in $M_n(\mathbb{C})$. A norm $|||.|||$ is called unitarily invariant norm if $|||UAV||| = |||A|||$ for all $A \in M_n(\mathbb{C})$ and for all unitary matrices $U, V \in M_n(\mathbb{C})$. For $A = [a_{ij}] \in M_n(\mathbb{C})$, the trace norm and Hilbert-Schmidt norm of $A$ are defined by

$$\|A\|_1 = tr|A| = \sum_{i=1}^n s_i(A), \quad \|A\|_2 = \sqrt{\sum_{i=1}^n s_i^2(A)} = \sqrt{\sum_{i,j=1}^n |a_{ij}|^2}$$

where $s_1(A) \geqslant s_2(A) \geqslant \cdots \geqslant s_n(A)$ are the singular values of $A$, that is, the eigenvalues of the positive matrix $|A| = (A^*A)^{\frac{1}{2}}$, arranged in decreasing order and repeated according to multiplicity and tr is the usual trace function. As is known to all, the Hilbert-Schmidt norm is unitarily invariant.

LEMMA 4.1. *Let $A,B \in M_n^+(\mathbb{C})$, then*

$$\det(A+B)^{\frac{1}{n}} \geqslant \det A^{\frac{1}{n}} + \det B^{\frac{1}{n}}.$$

LEMMA 4.2. ([5]) *Let $A,B,X \in M_n(\mathbb{C})$ and $A,B \in M_n^+(\mathbb{C})$. If $0 \leqslant v \leqslant 1$, then for any unitarily invariant norm $|||\cdot|||$*

$$|||A^v X B^{1-v}||| \leqslant |||AX|||^v |||XB|||^{1-v}.$$

THEOREM 4.1. *Let $A,B \in M_n^+(\mathbb{C})$, $m \in N_+$ and $0 < v \leqslant \tau < 1$. Then*
*(1) If $B \geqslant A \geqslant 0$, we can get*

$$\frac{\|(1-v)A+vB\|_1^m - (\|A\|_1^{1-v}\|B\|_1^v)^m}{v(1-v)} \leqslant \frac{\|(1-\tau)A+\tau B\|_1^m - (\|A\|_1^{1-\tau}\|B\|_1^\tau)^m}{\tau(1-\tau)};$$

*(2) If $A \geqslant B \geqslant 0$, we can get*

$$\frac{\|(1-v)A+vB\|_1^m - (\|A\|_1^{1-v}\|B\|_1^v)^m}{v(1-v)} \geqslant \frac{\|(1-\tau)A+\tau B\|_1^m - (\|A\|_1^{1-\tau}\|B\|_1^\tau)^m}{\tau(1-\tau)}.$$

*Proof.* Suppose $B \geqslant A$ and by Theorem 2.1, we have

$$\begin{aligned}
&\|(1-v)A+vB\|_1^m \\
&= (tr((1-v)A)+tr(vB))^m \\
&= ((1-v)tr(A)+vtr(B))^m \\
&\leqslant (tr(A)^{1-v}tr(B)^v)^m + \frac{v(1-v)}{\tau(1-\tau)}[((1-\tau)tr(A)+\tau tr(B))^m - (tr(A)^{1-\tau}tr(B)^\tau)^m] \\
&= (\|A\|_1^{1-v}\|B\|_1^v)^m + \frac{v(1-v)}{\tau(1-\tau)}[\|(1-\tau)A+\tau B\|_1^m - (\|A\|_1^{1-\tau}\|B\|_1^\tau)^m].
\end{aligned}$$

Using the same method we can get (2) similarly, so we omit it. □

THEOREM 4.2. *Let $A,B \in M_n^+(\mathbb{C})$, $m \in N_+$ and $0 < v \leqslant \tau < 1$. Then*
*(1) If $B \geqslant A \geqslant 0$, we can get*

$$\begin{aligned}
&\det((1-\tau)A+\tau B)^m \\
&\geqslant \frac{\tau(1-\tau)}{v(1-v)}\left[[(1-v)\det A^{\frac{1}{n}} + v\det B^{\frac{1}{n}}]^{mn} - \det(A^{1-v}B^v)^m\right] + \det\left(A^{1-\tau}B^\tau\right)^m;
\end{aligned}$$

*(2) If $A \geqslant B \geqslant 0$, we can get*

$$\begin{aligned}&\det((1-v)A+vB)^m\\ \geqslant\ &\det(A^{1-v}B^v)^m+\frac{v(1-v)}{\tau(1-\tau)}\left[[(1-\tau)\det A^{\frac{1}{n}}+\tau\det B^{\frac{1}{n}}]^{mn}-\det(A^{1-\tau}B^{\tau})^m\right].\end{aligned}$$

*Proof.* Suppose $B \geqslant A$ and by Theorem 2.1 and Lemma 4.1, we have

$$\begin{aligned}&\det((1-\tau)A+\tau B)^m\\ =\ &\left[\det((1-\tau)A+\tau B)^{\frac{1}{n}}\right]^{mn}\\ \geqslant\ &\left[(1-\tau)\det A^{\frac{1}{n}}+\tau\det B^{\frac{1}{n}}\right]^{mn}\\ \geqslant\ &\frac{\tau(1-\tau)}{v(1-v)}\left[[(1-v)\det A^{\frac{1}{n}}+v\det B^{\frac{1}{n}}]^{mn}-[\det A^{\frac{1-v}{n}}\det B^{\frac{v}{n}}]^{mn}\right]\\ &+\left[\det A^{\frac{1-\tau}{n}}\det B^{\frac{\tau}{n}}\right]^{mn}\\ =\ &\frac{\tau(1-\tau)}{v(1-v)}\left[[(1-v)\det A^{\frac{1}{n}}+v\det B^{\frac{1}{n}}]^{mn}-\det(A^{1-v}B^{v})^m\right]\\ &+\det\left(A^{1-\tau}B^{\tau}\right)^m.\end{aligned}$$

Using the same method we can get (2) similarly, so we omit it. □

THEOREM 4.3. *Let $A,B,X \in M_n(\mathbb{C})$ with $A,B \in M_n^+(\mathbb{C})$, $m \in N_+$ and $0 < v \leqslant \tau < 1$, then for any unitarily invariant norm $|||\cdot|||$, we have*

*(1) If $B \geqslant A \geqslant 0$, we can get*

$$\begin{aligned}&[(1-\tau)|||AX|||+\tau|||XB|||]^m\\ \geqslant\ &\frac{\tau(1-\tau)}{v(1-v)}\left[[(1-v)|||AX|||+v|||XB|||]^m-(|||AX|||^{1-v}|||XB|||^{v})^m\right]+|||A^{1-\tau}XB^{\tau}|||^m;\end{aligned}$$

*(2) If $A \geqslant B \geqslant 0$, we can get*

$$\begin{aligned}&[(1-v)|||AX|||+v|||XB|||]^m\\ \geqslant\ &\frac{v(1-v)}{\tau(1-\tau)}\left[[(1-\tau)|||AX|||+\tau|||XB|||]^m-(|||AX|||^{1-\tau}|||XB|||^{\tau})^m\right]+|||A^{1-v}XB^{v}|||^m.\end{aligned}$$

*Proof.* Suppose $B \geqslant A$ and by Theorem 2.1 and Lemma 4.2, we have

$$\begin{aligned}&[(1-\tau)|||AX|||+\tau|||XB|||]^m-|||A^{1-\tau}XB^{\tau}|||^m\\ \geqslant\ &[(1-\tau)|||AX|||+\tau|||XB|||]^m-(|||AX|||^{1-\tau}|||XB|||^{\tau})^m\\ \geqslant\ &\frac{\tau(1-\tau)}{v(1-v)}\left[[(1-v)|||AX|||+v|||XB|||]^m-(|||AX|||^{1-v}|||XB|||^{v})^m\right].\end{aligned}$$

Using the same method we can get (2) similarly, so we omit it. □

THEOREM 4.4. *Let $A,B \in M_n^+(\mathbb{C})$ and $\frac{1}{2} < v \leqslant \tau \leqslant 1$. Then*

$$\frac{K(h,2)^v\|A\|_1^{1-v}\|B\|_1^v - \|(1-v)A+vB\|_1}{v} \leqslant \frac{K(h,2)^\tau\|A\|_1^{1-\tau}\|B\|_1^\tau - \|(1-\tau)A+\tau B\|_1}{\tau}$$

*where $K(h,2) = \frac{(h+1)^2}{4h}$ and $h = \frac{tr(B)}{tr(A)}$.*

*Proof.* According to Theorem 2.4, we have

$$\begin{aligned}
&\|(1-v)A+vB\|_1\\
&= (1-v)tr(A)+vtr(B)\\
&\geqslant K(h,2)^v tr(A)^{1-v}tr(B)^v - \frac{v}{\tau}[K(h,2)^\tau tr(A)^{1-\tau}tr(B)^\tau - ((1-\tau)tr(A)+\tau tr(B))]\\
&= K(h,2)^v\|A\|_1^{1-v}\|B\|_1^v - \frac{v}{\tau}[K(h,2)^\tau\|A\|_1^{1-\tau}\|B\|_1^\tau - \|(1-\tau)A+\tau B\|_1].
\end{aligned}$$

This completes the proof. □

THEOREM 4.5. *Suppose $A,B,X \in M_n(\mathbb{C})$ such that $A,B \in M_n^+(\mathbb{C})$. Then*
*(1) if $0 < v \leqslant \tau \leqslant \frac{1}{2}$, we have*

$$\begin{aligned}
&\frac{\|(1-v)AX+vXB\|_2^2 - \|A^{1-v}XB^v\|_2^2 - v^2\|AX-BX\|_2^2}{v}\\
&\geqslant \frac{\|(1-\tau)AX+\tau XB\|_2^2 - \|A^{1-\tau}XB^\tau\|_2^2 - \tau^2\|AX-XB\|_2^2}{\tau};
\end{aligned}$$

*(2) if $\frac{1}{2} < v \leqslant \tau \leqslant 1$, we have*

$$\begin{aligned}
&\frac{\|(1-v)AX+vXB\|_2^2 - \|A^{1-v}XB^v\|_2^2 - v^2\|AX-XB\|_2^2}{v}\\
&\leqslant \frac{\|(1-\tau)AX+\tau XB\|_2^2 - \|A^{1-\tau}XB^\tau\|_2^2 - \tau^2\|AX-XB\|_2^2}{\tau}.
\end{aligned}$$

*Proof.* Since $A$ and $B$ are positive semidefinite, it follows by spectral theorem that there exist unitary matrices $U,V \in M_n(\mathbb{C})$, such that $A = U\Lambda_1 U^*$, $B = V\Lambda_2 V^*$, where $\Lambda_1 = diag(\lambda_1,\lambda_2,\cdots,\lambda_n)$, $\Lambda_2 = diag(\mu_1,\mu_2,\cdots,\mu_n)$ for $\lambda_i,\mu_l$ are eigenvalues of $A$ and $B$ respectively, $i,l = 1,2,\cdots,n$.

For our computations, let $Y = U^*XV = [y_{il}]$. Then we have

$$(1-v)AX+vXB = U[(1-v)\Lambda_1 Y + vY\Lambda_2]V^* = U[((1-v)\lambda_i+v\mu_l)y_{il}]V^*,$$

$$A^{1-v}XB^v = U[(\lambda_i^{1-v}\mu_l^v)y_{il}]V^*, \quad AX-XB = U[(\lambda_i-\mu_l)y_{il}]V^*,$$

So

$$\|(1-v)AX+vXB\|_2^2 - \|A^{1-v}XB^v\|_2^2 - v^2\|AX-XB\|_2^2$$

$$
\begin{aligned}
&= \sum_{i,l}^{n} [(1-v)\lambda_i + v\mu_l]^2 |y_{il}|^2 - \sum_{i,l}^{n} (\lambda_i^{1-v}\mu_l^v)^2 |y_{il}|^2 - v^2 \sum_{i,l}^{n} (\lambda_i - \mu_l)^2 |y_{il}|^2 \\
&= \sum_{i,l}^{n} \left( [(1-v)\lambda_i + v\mu_l]^2 - (\lambda_i^{1-v}\mu_l^v)^2 - v^2(\lambda_i - \mu_l)^2 \right) |y_{il}|^2 \\
&\geqslant \frac{v}{\tau} \sum_{i,l}^{n} \left( [(1-\tau)\lambda_i + \tau\mu_l]^2 - (\lambda_i^{1-\tau}\mu_l^\tau)^2 - \tau^2(\lambda_i - \mu_l)^2 \right] |y_{il}|^2 \\
&= \frac{v}{\tau} \left( \sum_{i,l}^{n} [(1-\tau)\lambda_i + \tau\mu_l]^2 |y_{il}|^2 - \sum_{i,l}^{n} (\lambda_i^{1-\tau}\mu_l^\tau)^2 |y_{il}|^2 - \tau^2 \sum_{i,l}^{n} (\lambda_i - \mu_l)^2 |y_{il}|^2 \right) \\
&= \frac{v}{\tau} \left[ \|(1-\tau)AX + \tau XB\|_2^2 - \|A^{1-\tau}XB^\tau\|_2^2 - \tau^2 \|AX - XB\|_2^2 \right].
\end{aligned}
$$

Using the same method we can get (2) similarly, so we omit it. □

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