SOME NEW IMPROVEMENTS OF YOUNG’S INEQUALITIES

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Abstract. In this paper, we obtain some improvements and generalizations of Young’s inequalities as the following:

(1) If $b \geq a$, we can get

$$\frac{(a\nabla b)^m - (a^{\tau}b)^m}{(a\nabla^\tau b)^m - (a^{\tau}b)^m} \leq \frac{v(1-v)}{\tau(1-\tau)};$$

(2) If $b \leq a$, we can get

$$\frac{(a\nabla b)^m - (a^{\tau}b)^m}{(a\nabla^\tau b)^m - (a^{\tau}b)^m} \geq \frac{v(1-v)}{\tau(1-\tau)}$$

for $m \in \mathbb{N}_+$. In addition, we obtain new result of Young’s inequality by using the expansions of the functions $(1-v) + vx - x^v$ with $0 < x < 2$.

1. Introduction

The Young’s inequality [8] is well known as the following: If $a, b > 0$ and $0 \leq v \leq 1$, then

$$a\nabla b = a^{1-v}b^v \leq (1-v)a + vb = a\nabla^\prime b$$

where equality holds if and only if $a = b$. Let $\frac{b}{a} = x$ in inequality (1.1), then we can obtain the equivalent inequality

$$0 \leq (1-v) + vx - x^v.$$

Liao, Wu and Zhao [7] showed the reverse inequality of the above Young’s inequality with Kantorovich constant

$$(1-v)a + vb \leq K(h, 2)R a^{1-v}b^v$$

where $a, b \geq 0$, $R = \max\{v, 1-v\}$ and $K(h, 2) = \frac{(h+1)^2}{4h}$ with $h = \frac{b}{a}$. He [2] and Hirzallah [3] refined Young’s inequality so that

$$r^2(a-b)^2 \leq [(1-v)a + vb]^2 - (a^{1-v}b^v)^2 \leq R^2(a-b)^2$$


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where \( a, b \geq 0, r = \min\{v, 1 - v\} \) and \( R = \max\{v, 1 - v\} \).

Alzer, Fonseca and Kovačec [1] presented the following Young’s inequalities

\[
\frac{v^m}{\tau^m} \leq \frac{(a \nabla_v b)^m - (a \nabla_\tau b)^m}{(a \nabla_\tau b)^m - (a \nabla_\tau_b)^m} \leq \frac{(1 - v)^m}{(1 - \tau)^m}
\] (1.5)

for \( 0 < v \leq \tau < 1 \) and \( m \in N_+ \).

Liao and Wu [5] replicated the above result as follows:

\[
\frac{v^m}{\tau^m} \leq \frac{(a \nabla_v b)^m - (a \nabla_\tau b)^m}{(a \nabla_\tau b)^m - (a \nabla_\tau_b)^m} \leq \frac{(1 - v)^m}{(1 - \tau)^m}
\] (1.6)

for \( 0 < v \leq \tau < 1 \) and \( m \in N_+ \).

Sababheh [10] obtained by convexity of function \( f \)

\[
\frac{v^m}{\tau^m} \leq \frac{(1 - v)f(\tau) + \tau f(0))^m - f^m(v)}{(1 - \tau)f(\tau) + \tau f(0))^m - f^m(\tau)} \leq \frac{(1 - v)^m}{(1 - \tau)^m}
\] (1.7)

for \( 0 < v \leq \tau < 1 \) and \( m \in N_+ \).

Ren [9] obtained the following inequalities:

\[
\begin{align*}
\frac{a \nabla_v b - a \nabla_\tau b}{a \nabla_\tau b - a \nabla_\tau b} &\leq \frac{v(1 - v)}{\tau(1 - \tau)}, & b - a &\geq 0 \\
\frac{a \nabla_v b - a \nabla_\tau b}{a \nabla_\tau b - a \nabla_\tau b} &\geq \frac{v(1 - v)}{\tau(1 - \tau)}, & b - a &\leq 0
\end{align*}
\] (1.8)

and

\[
\begin{align*}
\frac{(a \nabla_v b)^2 - (a \nabla_\tau b)^2}{(a \nabla_\tau b)^2 - (a \nabla_\tau b)^2} &\leq \frac{v(1 - v)}{\tau(1 - \tau)}, & b - a &\geq 0 \\
\frac{(a \nabla_v b)^2 - (a \nabla_\tau b)^2}{(a \nabla_\tau b)^2 - (a \nabla_\tau b)^2} &\geq \frac{v(1 - v)}{\tau(1 - \tau)}, & b - a &\leq 0
\end{align*}
\] (1.9)

for \( 0 < v \leq \tau < 1 \) and \( a, b > 0 \).

In addition, Zhu [11] obtained new Young’s inequalities by using the expansions of the functions \( \frac{(1 - v) + vx - x^v}{x^v} \).

In this paper, we generalize a part of above results in section 2. In section 3 we obtain following results through using the expansions of the functions \( (1 - v) + vx - x^v \)

\[
\begin{align*}
(1 - v) + vx - x^v &\geq \sum_{k=2}^{2m} \alpha_k(v)(x-1)^k, & x \in (0, 1] \\
(1 - v) + vx - x^v &\leq \sum_{k=2}^{2m} \alpha_k(v)(x-1)^k, & x \in [1, \infty)
\end{align*}
\]

and

\[
(1 - v) + vx - x^v \geq \sum_{k=2}^{2m+1} \alpha_k(v)(x-1)^k
\]
for $0 \leq v \leq 1$, $m \in N_+$ and $x > 0$ where $\alpha_\ell(v) = \frac{(-1)^\ell v(1-v)(2-v)\cdots(k-1-v)}{k!}$. And our result is the improvement of [11, Corollary 1] when $m = 1$. Finally, we present trace norm, Hilbert-Schmidt norm and determinant version of results in section 2.

2. Generalized improvements of Young’s inequalities

We firstly show the generalization of Young’s inequality [9] for scalars under some conditions.

**Theorem 2.1.** Let $0 < v \leq \tau < 1$, $m \in N_+$ and $a$, $b$ are real positive numbers. Then

1. If $b \geq a$, we can get

\[
\frac{(a\sqrt{v}, b)^m - (a^\tau v, b)^m}{(a\sqrt{v}, b)^m - (a^\tau v, b)^m} \leq \frac{v(1-v)}{\tau(1-\tau)};
\]

(2.1)

2. If $b \leq a$, we can get

\[
\frac{(a\sqrt{v}, b)^m - (a^\tau v, b)^m}{(a\sqrt{v}, b)^m - (a^\tau v, b)^m} \geq \frac{v(1-v)}{\tau(1-\tau)}.
\]

(2.2)

**Proof.** Firstly, we have

\[
(1 - v + vx)^m - x^{m\nu} = (1 - v + vx - x^\nu)[(1 - v + vx)^{m-1} + (1 - v + vx)^{m-2}x^\nu + \cdots + (1 - v + vx)x^{(m-2)v} + x^{(m-1)v}].
\]

Then let $f(v) = (1 - v + vx)^{m-1} + (1 - v + vx)^{m-2}x^\nu + \cdots + (1 - v + vx)x^{(m-2)v} + x^{(m-1)v}$, we can get

\[
f'(v) = (m-1)(x-1)(1-v+vx)^{m-2} + (m-2)(x-1)(1-v+vx)^{m-3}x^\nu
\]

\[
+ (1 - v + vx)^{m-2}x^\nu \ln x + \cdots + (x-1)x^{(m-2)v} + (1 - v + vx)(m-2)x^{(m-2)v} \ln x
\]

\[
+ (m-1)x^{(m-1)v} \ln x
\]

\[
= (x-1)[(m-1)(1-v+vx)^{m-2} + (m-2)(1-v+vx)^{m-3}x^\nu + \cdots + x^{(m-2)v}] + \ln x[(1 - v + vx)^{m-2}x^\nu + (1 - v + vx)^{m-3}2x^{2v} + \cdots + (1 - v + vx)^{(m-2)v} + (m-1)x^{(m-1)v}] + \ln x
\]

(1) If $x \geq 1$, we have $1 - v + vx - (x-1)v \geq 1$. So it’s obvious that $f'(v) \geq 0$, it means that $f(v)$ is increasing on $[1, \infty)$, that is to say $\frac{f(v)}{f(\tau)} \leq 1$. Therefore

\[
\frac{(1 - v + vx)^m - x^{m\nu}}{(1 - \tau + \tau x)^m - x^{m\tau}} = \frac{(1 - v + vx)^m f(v)}{(1 - \tau + \tau x - \tau^\tau)^m f(\tau)} \leq \frac{(1 - v + vx - x^\nu)}{(1 - \tau + \tau x - \tau^\tau - x^\tau)} \leq \frac{v(1-v)}{\tau(1-\tau)} \quad \text{(by 1.8)}.
\]
(2) If $0 < x \leq 1$, we have $1 - v + vx = 1 + (x - 1)v \geq 0$ and $\ln x \leq 0$. So it’s obvious that $f'(v) \leq 0$, it means that $f(v)$ is decreasing on $(0, 1]$, that is to say $\frac{f(v)}{\tau} \geq 1$. Therefore

$$
\frac{(1 - v + vx)^m - x^{m\tau}}{(1 - \tau + \tau x)^m - x^{m\tau}} = \frac{((1 - v + vx) - x^v)f(v)}{((1 - \tau + \tau x) - x^\tau)f(\tau)}
\geq \frac{(1 - v + vx) - x^v}{(1 - \tau + \tau x) - x^\tau} \geq \frac{v(1 - v)}{\tau(1 - \tau)} \quad \text{(by } 1.8).\n$$

Taking $x = \frac{b}{a}$, we can get our desired results directly. □

REMARK 2.1. (1) Let $m = 2$, we can get [9, Theorem 2.3].

(2) Let $a = b$, $b = a, v = 1 - \tau, \tau = 1 - v$ in inequality (2.1), we can also get inequality (2.2) directly.

(3) Let $0 < v \leq \tau < 1$, so $\frac{1 - v}{\tau} \geq 1$, therefore

(i) If $b \geq a$, we can get

$$
\frac{(a\nabla_v b)^m - (a^a_v b)^m}{(a\nabla_{\tau} b)^m - (a^a_{\tau} b)^m} \leq \frac{v(1 - v)}{\tau(1 - \tau)} \leq \frac{v(1 - v)^m}{\tau(1 - \tau)^m} \leq \frac{(1 - v)^m}{(1 - \tau)^m}.
$$

(ii) If $b \leq a$, we can get

$$
\frac{(a\nabla_v b)^m - (a^a_v b)^m}{(a\nabla_{\tau} b)^m - (a^a_{\tau} b)^m} \geq \frac{v(1 - v)}{\tau(1 - \tau)} \geq \frac{v^m(1 - v)}{\tau^m(1 - \tau)} \geq \frac{v^m}{\tau^m}.
$$

It is not difficult to see that Theorem 2.1 is the improvements of [1].

THEOREM 2.2. Let $\frac{1}{2} < v \leq \tau \leq 1$ and $a, b$ are real positive numbers. Then

$$
\frac{K(h, 2)^v a^a_v b - a\nabla_v b}{K(h, 2)^\tau a^a_{\tau} b - a\nabla_{\tau} b} \leq \frac{v}{\tau}
$$

(2.3)

where $K(h, 2) = \frac{(h+1)^2}{4h}$ and $h = \frac{b}{a}$.

Proof. Firstly, we let

$$
f(v) = \frac{K^v(x, 2)(x^v) - (1 - v + vx)}{v} = \frac{(\frac{x+1}{2})^{2v} - (1 - v + vx)}{v}.
$$

Then we can get

$$
f'(v) = \frac{v \left[2 \left(\frac{x+1}{2}\right)^{2v} \ln \left(\frac{x+1}{2}\right) - (x - 1)\right] - \left[(\frac{x+1}{2})^{2v} - (1 - v + vx)\right]}{v^2}
= \frac{(\frac{x+1}{2})^{2v} \left[2v \ln \left(\frac{x+1}{2}\right) - 1\right] + 1}{v^2}
= \frac{h(x)}{v^2}
$$
It means that \( x \in (0,1], \ h'(x) \leq 0; \ x \in [1,\infty), \ h'(x) \geq 0. \) So \( h(x) \geq h(1) = 0 \) and \( f'(v) \geq 0. \) Therefore \( f(v) \) is increasing on \((0, +\infty)\).

Taking \( x = \frac{b}{a}, \) we can get our desired results directly. \( \Box \)

**Theorem 2.3.** Let \( 0 < v \leq \tau \leq \frac{1}{2} \) and \( a, \ b \) are real positive numbers. Then

\[
\frac{(a\nabla_v b)^2 - (a\nabla_v b)^2 - v^2(a-b)^2}{(a\nabla_v b)^2 - (a\nabla_v b)^2 - \tau^2(a-b)^2} \geq \frac{v}{\tau}.
\] (2.4)

**Proof.** Firstly, we let \( f(v) = \frac{(1-v+vx)^2-x^2v^2(x-1)^2}{v^2}. \) Then

\[
f'(v) = v \left[ 2(x-1)(1-v+vx) - 2x^2v \ln x - 2v(x-1)^2 \right] - \left[ (1-v+vx)^2 - x^2v - v^2(x-1)^2 \right] \]

\[
= \frac{(1-v+vx)(vx - v - 1) + x^2v - 2vx^2v \ln x - v^2(x-1)^2}{v^2}
\]

and

\[
h'(x) = v(vx - v - 1) + v(1 - v + vx) + 2vx^2v - 4v^2x^{2v-1} \ln x - 2vx^{2v-1} - 2v^2(x-1)
\]

\[
= -4v^2x^{2v-1} \ln x
\]

It means that \( x \in (0,1], \ h'(x) \geq 0; \ x \in [1,\infty), \ h'(x) \leq 0. \) So \( h(x) \leq h(1) = 0 \) and \( f'(v) \leq 0. \) Therefore \( f(v) \) is decreasing on \((0, +\infty)\).

Taking \( x = \frac{b}{a}, \) we can get our desired results directly. \( \Box \)

**Theorem 2.4.** Let \( \frac{1}{2} < v \leq \tau \leq 1 \) and \( a, \ b \) are real positive numbers. Then

\[
\frac{(a\nabla_v b)^2 + v^2(a-b)^2 - (a\nabla_v b)^2}{(a\nabla_v b)^2 + \tau^2(a-b)^2 - (a\nabla_v b)^2} \leq \frac{v}{\tau}.
\] (2.5)
Proof. Firstly, we let \( f(v) = \frac{x^{2v} + v^2(x-1)^2 - (1-v+vx)^2}{v^2} \). Then

\[
f'(v) = \frac{v \left[ 2x^{2v} \ln x + 2v(x-1)^2 - 2(x-1)(1-v+vx) \right]}{v^2} - \frac{\left[ x^{2v} + v^2(x-1)^2 - (1-v+vx)^2 \right]}{v^2}
= \frac{2x^{2v} \ln x + v^2(x-1)^2 - x^{2v} - (1-v+vx)(vx-v-1)}{v^2}
= \frac{h(x)}{v^2}
\]

and

\[
h'(x) = 4v^2x^{2v-1} \ln x + 2v^2(x-1) - [v(vx-v-1) + v(1-v+vx)]
= 4v^2x^{2v-1} \ln x.
\]

It means that \( x \in (0,1], \ h'(x) \leq 0; \ x \in [1,\infty), \ h'(x) \geq 0. \) So \( h(x) \geq h(1) = 0 \) and \( f''(v) \geq 0. \) Therefore \( f(v) \) is increasing on \((0,\infty)\).

Taking \( x = \frac{b}{a} \), we can get our desired results directly. \( \)

3. Some new results of Young-type inequalities

According to Newton’s binomial expansion for \( x \in (-1,1) \),

\[
(1+x)^v = 1 + vx + \frac{v(v-1)}{2!}x^2 + \frac{v(v-1)(v-2)}{3!}x^3 + \ldots + \frac{v(v-1)(v-2)\ldots[v-(k-1)]}{k!}x^k + \ldots.
\]

We can have if \( 0 \leq v \leq 1 \) and \( 0 < x < 2 \),

\[
(1-v) + vx - x^v = \sum_{k=2}^{\infty} \alpha_k(v)(x-1)^k \tag{3.1}
\]

where \( \alpha_k(v) = \frac{(-1)^k v(1-v)(2-v)\ldots[(k-1)-v]}{k!} \). And then we can get some new results of inequality \((1-v) + vx - x^v\) based on (3.1).

**Theorem 3.1.** Let \( 0 \leq v \leq 1, \ m \in \mathbb{N}_+ \) and \( x > 0. \) Then

\[
\begin{cases}
(1-v) + vx - x^v \geq \sum_{k=2}^{2m} \alpha_k(v)(x-1)^k, & x \in (0,1], \\
(1-v) + vx - x^v \leq \sum_{k=2}^{2m} \alpha_k(v)(x-1)^k, & x \in [1,\infty).
\end{cases} \tag{3.2}
\]
**Proof.** Suppose 

$$f(x) = (1 - v) + vx - x^v - \sum_{k=2}^{2m} \alpha_k(v)(x - 1)^k.$$ 

Then 

$$f'(x) = v - vx^{v-1} - \sum_{k=2}^{2m} k\alpha_k(v)(x - 1)^{k-1},$$ 

$$f''(x) = v(1 - v)x^{v-2} - v(1 - v) - \sum_{k=3}^{2m} k(k - 1)\alpha_k(v)(x - 1)^{k-2},$$ 

and so on.

Finally, we can get 

$$f(2m)(x) = (-1)^{2m}v(1 - v)(2 - v)\cdots(2m - 1 - v)x^{v-2m} - (2m)!\alpha_{2m}(v).$$

It means that $f(2m)(x) \geq 0$ on $(0, 1]$ and $f(2m)(x) \leq 0$ on $[1, +\infty)$, so that $f(2m-1)(x) \leq f(2m-2)(1) = 0$. Therefore $f(2m-2)(x)$ is decreasing on $(0, +\infty)$, $f(2m-2)(x) \geq 0$ on $(0, 1]$ and $f(2m-2)(x) \leq 0$ on $[1, +\infty)$ obviously. By that analogy, $f''(x)$ is decreasing on $(0, +\infty)$. It means that $f''(x) \geq 0$ on $(0, 1]$ and $f''(x) \leq 0$ on $[1, +\infty)$. So $f'(x) \leq f'(1) = 0$, that is, $f(x)$ is decreasing on $(0, +\infty)$. According to $f(1) = 0$, we can get desired results. $\square$

**Theorem 3.2.** Let $0 \leq v \leq 1$, $m \in \mathbb{N}_+$ and $x > 0$. Then 

$$(1 - v) + vx - x^v \geq \sum_{k=2}^{2m+1} \alpha_k(v)(x - 1)^k. \quad (3.3)$$

**Proof.** Suppose 

$$f(x) = (1 - v) + vx - x^v - \sum_{k=2}^{2m+1} \alpha_k(v)(x - 1)^k.$$ 

Then 

$$f'(x) = v - vx^{v-1} - \sum_{k=2}^{2m+1} k\alpha_k(v)(x - 1)^{k-1},$$ 

$$f''(x) = v(1 - v)x^{v-2} - v(1 - v) - \sum_{k=3}^{2m+1} k(k - 1)\alpha_k(v)(x - 1)^{k-2},$$ 

and so on.
It's obvious that \( f(0) = f_0 \), and \( f(\infty) = f_\infty \). Therefore \( f(x) \) is increasing on \((0, \infty)\). By that analogy, \( f'(x) \) is increasing on \((0, \infty)\). It means that \( f'(x) \leq 0 \) on \((0, 1]\) and \( f'(x) \geq 0 \) on \([1, \infty)\). So \( f(x) \geq f(1) = 0 \). By simple shift, we can get final result. \( \square \)

**COROLLARY 3.1.** Let \( 0 \leq v \leq 1 \) and \( x > 0 \). Then

\[
\begin{cases}
(1 - v) + vx - x^v \geq \frac{v(1 - v)}{2} (x - 1)^2, & x \in (0, 1], \\
(1 - v) + vx - x^v \leq \frac{v(1 - v)}{2} (x - 1)^2, & x \in [1, \infty).
\end{cases}
\] (3.4)

**Proof.** Let \( m = 1 \) in Theorem 3.2, we can get desired results. \( \square \)

**REMARK 3.1.** Because \( x^v \geq 1 \) on \([1, \infty)\) and \( 0 < x^v \leq 1 \) on \((0, 1]\), so

\[
(1 - v) + vx - x^v \geq \frac{v(1 - v)}{2} (x - 1)^2 \geq x^v \frac{v(1 - v)}{2} (x - 1)^2, \quad x \in (0, 1],
\]

\[
(1 - v) + vx - x^v \leq \frac{v(1 - v)}{2} (x - 1)^2 \leq x^v \frac{v(1 - v)}{2} (x - 1)^2, \quad x \in [1, \infty).
\]

It’s not hard to see that the inequality 3.4 is the improvement of [11, Corollary 1].

**REMARK 3.2.** Let \( x = \frac{b}{a} \) in Theorem 3.1 and Theorem 3.2, we can get

\[
\begin{cases}
(1 - v)a + vb - a^{-v}b^v \geq \sum_{k=2}^{2m} \alpha_k(v)a^{1-k}(b-a)^k, & a \geq b > 0 \\
(1 - v)a + vb - a^{-v}b^v \leq \sum_{k=2}^{2m} \alpha_k(v)a^{1-k}(b-a)^k, & a \geq b > 0
\end{cases}
\]

and

\[
(1 - v)a + vb - a^{-v}b^v \geq \sum_{k=2}^{2m+1} \alpha_k(v)a^{1-k}(b-a)^k.
\]

It’s obvious that \( \sum_{k=2}^{n} \alpha_k(v)a^{1-k}(b-a)^k \) is greater than 0 where \( a \geq b > 0 \).
Corollary 3.2. Let $0 \leq v \leq \tau \leq 1$ and $a \geq b > 0$. Then
\[
\frac{(a\nabla,b) - (a\nabla^*b) - \sum_{k=2}^{n} \alpha_k(v)a_1^{-k}(b-a)^k}{(a\nabla_\tau b) - (a\nabla^*_\tau b) - \sum_{k=2}^{n} \alpha_k(\tau)a_1^{-k}(b-a)^k} \geq \frac{v(1-v)(2-v)\cdots(n-v)}{\tau(1-\tau)(2-\tau)\cdots(n-\tau)}.
\] (3.5)

Proof. Firstly, we let $x \in (0,2)$ and
\[
f(v) = (1-v) + vx - x^v - \sum_{k=2}^{n} \alpha_k(v)(x-1)^k.
\]
According to (3.1), we have
\[
f(v) = \sum_{k=n+1}^{\infty} \alpha_k(v)(x-1)^k,
\]
so
\[
\frac{(1-v) + vx - x^v - \sum_{k=2}^{n} \alpha_k(v)(x-1)^k}{(1-\tau) + \tau x - x^\tau - \sum_{k=2}^{n} \alpha_k(\tau)(x-1)^k} = \frac{\sum_{k=n+1}^{\infty}(-1)^k \alpha_k(v)(1-x)^k}{\sum_{k=n+1}^{\infty}(-1)^k \alpha_k(\tau)(1-x)^k}.
\]
Let
\[
\beta_k(v) = (-1)^k \alpha_k(v) = \frac{v(1-v)(2-v)\cdots(k-1-v)}{k}.
\]
When $k \geq n+1$, we can get
\[
\frac{\beta_k(v)}{\beta_k(\tau)} = \frac{v(1-v)(2-v)\cdots(k-1-v)}{\tau(1-\tau)(2-\tau)\cdots(k-1-\tau)} \geq \frac{v(1-v)(2-v)\cdots(n-v)}{\tau(1-\tau)(2-\tau)\cdots(n-\tau)}.
\]
Therefore
\[
\frac{(1-v) + vx - x^v - \sum_{k=2}^{n} \alpha_k(v)(x-1)^k}{(1-\tau) + \tau x - x^\tau - \sum_{k=2}^{n} \alpha_k(\tau)(x-1)^k} = \frac{\sum_{k=n+1}^{\infty} \beta_k(v)(1-x)^k}{\sum_{k=n+1}^{\infty} \beta_k(\tau)(1-x)^k} \geq \frac{v(1-v)(2-v)\cdots(n-v)}{\tau(1-\tau)(2-\tau)\cdots(n-\tau)}.
\]
Taking $x = \frac{k}{a}$, we can get our desired results directly. \qed

4. Applications

Let $M_n(\mathbb{C})$ denotes the space of all $n \times n$ complex matrices and $M_n^+(\mathbb{C})$ denotes the space of all $n \times n$ positive semidefinite matrices in $M_n(\mathbb{C})$. A norm $|||\cdot|||$ is called unitarily invariant norm if $|||UAV||| = |||A|||$ for all $A \in M_n(\mathbb{C})$ and for all unitary matrices $U, V \in M_n(\mathbb{C})$. For $A = [a_{ij}] \in M_n(\mathbb{C})$, the trace norm and Hilbert-Schmidt norm of $A$ are defined by
\[
||A||_1 = tr|A| = \sum_{i=1}^{n} s_i(A), \quad ||A||_2 = \sqrt{\sum_{i=1}^{n} s_i^2(A)} = \sqrt{\sum_{i,j=1}^{n} |a_{ij}|^2}
\]
where $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A)$ are the singular values of $A$, that is, the eigenvalues of the positive matrix $|A| = (A^*A)^{1/2}$, arranged in decreasing order and repeated according to multiplicity and $\text{tr}$ is the usual trace function. As is known to all, the Hilbert-Schmidt norm is unitarily invariant.

**Lemma 4.1.** Let $A, B \in M_n^+(\mathbb{C})$, then

$$\det(A + B)^{1/2} \geq \det A^{1/2} + \det B^{1/2}.$$  

**Lemma 4.2.** ([5]) Let $A, B, X \in M_n(\mathbb{C})$ and $A, B \in M_n^+(\mathbb{C})$. If $0 \leq v \leq 1$, then for any unitarily invariant norm $||| \cdot |||$  

$$|||A'XB^{1-v}||| \leq |||AX|||v|||XB|||1-v.$$  

**Theorem 4.1.** Let $A, B \in M_n^+(\mathbb{C})$, $m \in N_+$ and $0 < v \leq \tau < 1$. Then

1. If $B \geq A \geq 0$, we can get

$$\frac{||(1 - v)A + vB||^m_1 - (||A||_1^{1-v}||B||_1^v)^m}{v(1 - v)} \leq \frac{||(1 - \tau)A + \tau B||^m_1 - (||A||_1^{1-\tau}||B||_1^\tau)^m}{\tau(1 - \tau)};$$

2. If $A \geq B \geq 0$, we can get

$$\frac{||(1 - v)A + vB||^m_1 - (||A||_1^{1-v}||B||_1^v)^m}{v(1 - v)} \geq \frac{||(1 - \tau)A + \tau B||^m_1 - (||A||_1^{1-\tau}||B||_1^\tau)^m}{\tau(1 - \tau)}.$$

**Proof.** Suppose $B \geq A$ and by Theorem 2.1, we have

$$\frac{||(1 - v)A + vB||^m_1}{v(1 - v)} = \frac{(\text{tr}((1 - v)A + vB))^m}{v(1 - v)}$$

$$\leq \frac{(\text{tr}(A)^{1-v} \text{tr}(B)^v)^m + \frac{v(1 - v)}{\tau(1 - \tau)}[(\text{tr}(A)^{1-v} \text{tr}(B)^v)^m - (\text{tr}(A)^{1-\tau} \text{tr}(B)^\tau)^m]}{\tau(1 - \tau)}$$

$$\leq \frac{||(1 - v)A + vB||^m_1}{v(1 - v)} \geq \frac{(||A||_1^{1-v}||B||_1^v)^m + \frac{v(1 - v)}{\tau(1 - \tau)}[(||A||_1^{1-\tau}||B||_1^\tau)^m]}{\tau(1 - \tau)}.$$

Using the same method we can get (2) similarly, so we omit it.  

**Theorem 4.2.** Let $A, B \in M_n^+(\mathbb{C})$, $m \in N_+$ and $0 < v \leq \tau < 1$. Then

1. If $B \geq A \geq 0$, we can get

$$\det((1 - \tau)A + \tau B)^m$$

$$\geq \frac{\tau(1 - \tau)}{v(1 - v)} [((1 - v)\det A^{1/2} + v\det B^{1/2})^{mn} - \det(A^{1-v}B^v)^m] + \det(A^{1-\tau}B^\tau)^m;$$
(2) If \( A \geq B \geq 0 \), we can get

\[
\det((1 - \nu)A + \nu B)^m \geq \det(A^{1-\nu}B^\nu)^m + \frac{\nu(1 - \nu)}{\tau(1 - \tau)} \left[ (1 - \tau)\det A^{\frac{1}{\tau}} + \tau \det B^{\frac{1}{\tau}} \right]^{mn} - \det(A^{1-\tau}B^\tau)^m.
\]

Proof. Suppose \( B \geq A \) and by Theorem 2.1 and Lemma 4.1, we have

\[
\det((1 - \tau)A + \tau B)^m = \left[ \det((1 - \tau)A + \tau B)^{\frac{1}{\tau}} \right]^{mn} \geq \left[ (1 - \tau)\det A^{\frac{1}{\tau}} + \tau \det B^{\frac{1}{\tau}} \right]^{mn} \geq \frac{\tau(1 - \tau)}{\nu(1 - \nu)} \left[ (1 - \nu)\det A^{\frac{1}{\nu}} + \nu \det B^{\frac{1}{\nu}} \right]^{mn} - \det(A^{1-\tau}B^\tau)^m.
\]

Using the same method we can get (2) similarly, so we omit it. \( \square \)

**Theorem 4.3.** Let \( A, B, X \in M_n(\mathbb{C}) \) with \( A, B \in M_n^+(\mathbb{C}) \), \( m \in N_+ \) and \( 0 < \nu \leq \tau < 1 \), then for any unitarily invariant norm \( ||| \cdot ||| \), we have

(1) If \( B \geq A \geq 0 \), we can get

\[
[(1 - \tau)|||AX||| + \tau|||XB|||]^{mn} \geq \frac{\tau(1 - \tau)}{\nu(1 - \nu)} \left[ (1 - \nu)|||AX||| + \nu|||XB||| \right]^{mn} - (|||AX|||^{1-\nu}|||XB|||^{\nu})^m + |||A^{1-\tau}XB^\tau|||^{mn};
\]

(2) If \( A \geq B \geq 0 \), we can get

\[
[(1 - \nu)|||AX||| + \nu|||XB|||]^{mn} \geq \frac{\nu(1 - \nu)}{\tau(1 - \tau)} \left[ (1 - \tau)|||AX||| + \tau|||XB||| \right]^{mn} - (|||AX|||^{1-\tau}|||XB|||^{\tau})^m + |||A^{1-\nu}XB^\nu|||^{mn}.
\]

Proof. Suppose \( B \geq A \) and by Theorem 2.1 and Lemma 4.2, we have

\[
[(1 - \tau)|||AX||| + \tau|||XB|||]^{mn} - |||A^{1-\tau}XB^\tau|||^{mn} \geq \left[ (1 - \tau)|||AX||| + \tau|||XB||| \right]^{mn} - (|||AX|||^{1-\tau}|||XB|||^{\tau})^m \geq \frac{\tau(1 - \tau)}{\nu(1 - \nu)} \left[ (1 - \nu)|||AX||| + \nu|||XB||| \right]^{mn} - (|||AX|||^{1-\nu}|||XB|||^{\nu})^m.
\]

Using the same method we can get (2) similarly, so we omit it. \( \square \)
\textbf{Theorem 4.4.} Let $A, B \in M_n^{+}(\mathbb{C})$ and $\frac{1}{2} < v \leq \tau \leq 1$. Then

$$K(h, 2)^v ||A||^{1-v}_1 ||B||^{1-v}_1 - ||(1-v)A + vB||_1 \leq \frac{K(h, 2)^\tau ||A||^{1-\tau}_1 ||B||^{1-\tau}_1 - ||(1-\tau)A + \tau B||_1}{\tau}$$

where $K(h, 2) = \frac{(h+1)^2}{4h}$ and $h = \frac{tr(B)}{tr(A)}$.

\textbf{Proof.} According to Theorem 2.4, we have

$$||(1-v)A + vB||_1 = (1-v)tr(A) + vtr(B)$$

$$\geq K(h, 2)^v tr(A)^{1-v} tr(B)^v - v\frac{K(h, 2)^\tau tr(A)^{1-\tau} tr(B)^\tau - ((1-\tau)tr(A) + \tau tr(B))}{\tau}$$

$$= K(h, 2)^v ||A||^{1-v}_1 ||B||^{1-v}_1 - v\frac{K(h, 2)^\tau ||A||^{1-\tau}_1 ||B||^{1-\tau}_1 - ||(1-\tau)A + \tau B||_1}{\tau}.$$

This completes the proof. \qed

\textbf{Theorem 4.5.} Suppose $A, B, X \in M_n(\mathbb{C})$ such that $A, B \in M_n^{+}(\mathbb{C})$. Then

(1) if $0 < v \leq \tau \leq \frac{1}{2}$, we have

$$\sqrt{v} \left( ||(1-v)AX + vXB||_2^2 - ||A^{1-v}XB^v||_2^2 - v^2 ||AX - BX||_2^2 \right)$$

$$\geq \frac{\sqrt{v} \left( ||(1-\tau)AX + \tau XB||_2^2 - ||A^{1-\tau}XB^\tau||_2^2 - \tau^2 ||AX - XB||_2^2 \right)}{\tau}.$$

(2) if $\frac{1}{2} < v \leq \tau \leq 1$, we have

$$\sqrt{v} \left( ||(1-v)AX + vXB||_2^2 - ||A^{1-v}XB^v||_2^2 - v^2 ||AX - XB||_2^2 \right)$$

$$\leq \frac{\sqrt{v} \left( ||(1-\tau)AX + \tau XB||_2^2 - ||A^{1-\tau}XB^\tau||_2^2 - \tau^2 ||AX - XB||_2^2 \right)}{\tau}.$$

\textbf{Proof.} Since $A$ and $B$ are positive semidefinite, it follows by spectral theorem that there exist unitary matrices $U, V \in M_n(\mathbb{C})$, such that $A = U\Lambda_1 U^*, \ B = V\Lambda_2 V^*$, where $\Lambda_1 = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, $\Lambda_2 = \text{diag}(\mu_1, \mu_2, \ldots, \mu_n)$ for $\lambda_i, \mu_i$ are eigenvalues of $A$ and $B$ respectively, $i, l = 1, 2, \ldots, n$.

For our computations, let $Y = U^* XV = [y_{il}]$. Then we have

$$(1-v)AX + vXB = U[(1-v)\Lambda_1 Y + v\Lambda_2 Y]V^* = U[(1-v)\lambda_i + v\mu_i]y_{il}]V^*,$$

$$A^{1-v}XB^v = U[\lambda_i^{1-v} \mu_i^v]y_{il}]V^*, \ AX - XB = U[(\lambda_i - \mu_i)y_{il}]V^*,$$

So

$$\sqrt{v} \left( ||(1-v)AX + vXB||_2^2 - ||A^{1-v}XB^v||_2^2 - v^2 ||AX - XB||_2^2 \right)$$

$$\leq \frac{\sqrt{v} \left( ||(1-\tau)AX + \tau XB||_2^2 - ||A^{1-\tau}XB^\tau||_2^2 - \tau^2 ||AX - XB||_2^2 \right)}{\tau}.$$
\[
= \sum_{i,l}^{n} [(1-v) \lambda_i + v \mu_l] |y_{il}|^2 - \sum_{i,l}^{n} (\lambda_i^{1-v} \mu_l^v) |y_{il}|^2 - v^2 \sum_{i,l}^{n} (\lambda_i - \mu_l)^2 |y_{il}|^2
\]

\[
= \sum_{i,l}^{n} [(1-v) \lambda_i + v \mu_l] |y_{il}|^2 - (\lambda_1^{1-v} \mu_1^v) |y_{il}|^2 - v^2 (\lambda_i - \mu_l)^2 |y_{il}|^2
\]

\[
\geq \frac{v}{\tau} \sum_{i,l}^{n} [(1-\tau) \lambda_i + \tau \mu_l] |y_{il}|^2 - (\lambda_1^{1-\tau} \mu_1^\tau)^2 |y_{il}|^2 - \tau^2 (\lambda_i - \mu_l)^2 |y_{il}|^2
\]

\[
= \frac{v}{\tau} \left( \sum_{i,l}^{n} [(1-\tau) \lambda_i + \tau \mu_l] |y_{il}|^2 - \sum_{i,l}^{n} (\lambda_i^{1-\tau} \mu_l^\tau) |y_{il}|^2 - \tau^2 \sum_{i,l}^{n} (\lambda_i - \mu_l)^2 |y_{il}|^2 \right)
\]

\[
= \frac{v}{\tau} \left[ ||(1-\tau)AX + \tau XB||_2^2 - ||A^{1-\tau}XB^\tau||_2^2 - \tau^2 ||AX - XB||_2^2 \right].
\]

Using the same method we can get (2) similarly, so we omit it. \[\square\]

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