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ON THE LOCAL RITT RESOLVENT CONDITION

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Abstract. Let T be a linear bounded operator on a complex Banach space $\mathscr{X}$. In this paper, we introduce a local version of the Ritt resolvent condition $[LR]$ for Banach space operators T. We start by showing that this concept is weaker than the classical Ritt condition $[R]$. We prove that, for operators with single-valued extension property (SVEP), estimate $[LR]$ extends, with a larger constant, to some sector K_δ. Moreover, by extending some Ritt's theorems to the local case for operators with the SVEP, several characterizations of the local sublinear decay of $T^n - T^{n+1}$ have been established.

1. Introduction and preliminaries

Let $\mathscr{X}$ be a complex Banach space and let $\|.\|$ be the operator norm induced by the vector norm in $\mathscr{X}$, and let $B(\mathscr{X})$ be the algebra of bounded linear operators on $\mathscr{X}$. We denote the spectrum of $T \in B(\mathscr{X})$ by $\sigma(T)$, the identity operator on $\mathscr{X}$ by I, and the resolvent of T by $R(T,\lambda) = (\lambda I - T)^{-1}$, $\lambda \notin \sigma(T)$. Let us recall (see, e.g., [9, 17]) that an operator T with spectrum in the unit disc is said to satisfy the Ritt resolvent condition with constant $M \geqslant 1$ if

$$\|R(T,\lambda)\| \leqslant \frac{M}{|\lambda - 1|} \text{ for all } |\lambda| > 1. \qquad [R]$$

An operator $T \in B(\mathscr{X})$ is called power bounded, if there exists a constant $M \geqslant 0$ such that

$$\|T^n\| \leqslant M, \quad \text{for all } n \in \mathbb{N}. \tag{1}$$

In the literature, the estimate of the powers of operators under various resolvent conditions has been largely studied [9, 10, 11, 13, 16, 17, 18, 19, 21, 22].

In [21], Ritt proved that for Banach space setting The condition $[R]$ yield $\|T^n\| = O(n)$ as $n \longrightarrow \infty$. In [16], it was shown that if Ritt resolvent condition holds for an operator T acting on a Banach space, then $\|T^n\| = O(\log n)$ as $n \longrightarrow \infty$, and $\left\|T^n - T^{n+1}\right\| \to 0$ as $n \longrightarrow \infty$. These results have been generalized by Pater for operators acting on locally convex spaces [19, Theorem 3]. Another important study was

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made by Moore when he extended the notions of states and of numerical ranges of operators to the case of locally convex spaces [15]. In this work, we introduce the local Ritt resolvent condition and relate it to the local power boundedness and the local decay of $T^n - T^{n+1}$. In fact, we prove local versions of some results of [19, 21]. For this we need to introduce some preliminaries on local spectral theory; for more details on this subject, we refer to [6, 14].

The local resolvent set $\rho_T(x)$ of T at $x \in \mathscr{X}$ is defined as the set of all complex $\lambda \in \mathbb{C}$ for which there exists an analytic $\mathscr{X}$-valued function w on some open neighborhood U of λ such that

$$(\mu I - T)w(\mu) = x \text{ for all } \mu \in U.$$

The local spectrum $\sigma_T(x)$ of T at x is the complement in $\mathbb{C}$ of $\rho_T(x)$. It is well known that the resolvent mapping is unbounded. On the other hand, as observed in [12], the behavior of local resolvent functions may be quite different.

An operator $T \in B(\mathscr{X})$ is said to have the single-valued extension property (hereafter referred to as SVEP) if, for every open set $U \subseteq \mathbb{C}$, the only analytic solution $w : U \to \mathscr{X}$ of the equation

$$(\lambda I - T)w(\lambda) = 0 \quad (\lambda \in U),$$

is the constant function $w \equiv 0$.

If T has SVEP, then, for every $x \in \mathscr{X}$, there exists a unique analytic function $\hat{x}_T(.) : \rho_T(x) \to \mathscr{X}$ such that

$$(\lambda I - T)\hat{x}_T(\lambda) = x \text{ for all } \lambda \in \rho_T(x).$$

This function is called the local resolvent function of T at x and satisfies

$$\hat{x}_T(\lambda) = (\lambda I - T)^{-1}x \text{ for all } \lambda \in \rho(T).$$

For $T \in B(\mathscr{X})$, the local spectral radius of T at x is defined by

$$r_T(x) := \sup\{|\lambda| : \lambda \in \sigma_T(x)\}.$$

If T has the SVEP, then

$$r_T(x) = \limsup_{n\to\infty} \|T^n x\|^{\frac{1}{n}}.$$

In the following, we use the local functional calculus developed in [2, 24] which extends, in several directions, the holomorphic functional calculus developed by D. Dunford and A. E. Taylor in [8, 23].

Let $T \in B(\mathscr{X})$ have the SVEP and let $x \in X$ such that $\sigma_T(x) \subset K$, where K is a compact subset of $\mathbb{C}$. For every holomorphic function f on a neighborhood of K, the vector $f[T]x$ is defined, in [4] (see also [2]), by

$$f[T]x := \frac{1}{2\pi i}\int_\Gamma f(\mu)\hat{x}_T(\mu)d\mu.$$

For every $\lambda \in \mathbb{C}$, we denote by f_λ^n is the function given by $f_\lambda^n(\mu) = (\lambda - \mu)^{-n}$, $n = 1, 2, \ldots$. If $n = 1$, we write simply f_λ for f_λ^1.

In [3], Bermúdez, González and Martinón gave an example which shows that $f[T]$ is not well defined when T does not satisfy the SVEP. Thus, to develop the proofs of the main results, we will assume that the operator T satisfies the SVEP.

LEMMA 1. [3] *Assume that $T \in B(\mathscr{X})$ has the SVEP and let $x \in \mathscr{X}$. If $\lambda \in \rho_T(x)$, then $\hat{x}_T(\lambda) = f_\lambda[T]x$.*

Analyticity of $\hat{x}_T(.)$, Cauchy's differentiation formula and the definitions yield the following.

PROPOSITION 1. [2] Assume that $T \in B(\mathscr{X})$ has the SVEP and let $x \in \mathscr{X}$. For $\lambda \in \rho_T(x)$, we have

$$\frac{d^n \hat{x}_T(\lambda)}{d\lambda^n} = (-1)^n n! f_\lambda^{n+1}[T]x \tag{2}$$

2. Main results

In this section, we will give local versions of some definitions and we will establish some results relating these notions.

DEFINITION 1. Let $T \in B(\mathscr{X})$ and $x \in \mathscr{X}$ such that $r_T(x) \leqslant 1$. We say that T satisfies the local Ritt resolvent condition at x if there exists an analytic function $x_T(.) : \mathbb{C} \setminus \sigma_T(x) \to \mathscr{X}$ such that $(\lambda I - T)x_T(\lambda) = x$ and

$$\|x_T(\lambda)\| \leqslant \frac{M}{|\lambda - 1|} \quad \text{for all } |\lambda| > 1, \qquad [LR]$$

for some constant $M \geqslant 0$.

This concept is weaker than that of the Ritt condition $[R]$, because the subspace $\{x \in \mathscr{X} : r_T(x) \leqslant 1\}$ does not necessarily coincide with $\mathscr{X}$ or is closed.

EXAMPLE 1. Let $\mathscr{X}_1$ and $\mathscr{X}_2$ be two Banach spaces. Let $T \in B(\mathscr{X}_1)$ be an operator satisfying the $[R]$ condition and $S = 2I \in B(\mathscr{X}_2)$. Hence, the operator $L = T \oplus S \in B(\mathscr{X}_1 \oplus \mathscr{X}_2)$ does not satisfy the $[R]$ condition. Indeed, $\sigma(L) = \sigma(T) \cup \sigma(S) \nsubseteq \overline{\mathbb{D}}(0,1)$. But, for $x \in \mathscr{X}_1$, we set $f(\mu) = \left(L_{|\mathscr{X}_1} - \mu I\right)^{-1} x$ for all $\mu \in \mathbb{C} \setminus \overline{\mathbb{D}}$. Then, we get an analytic function satisfying $(L - \mu I) f(\mu) = x$ for all $\mu \in \mathbb{C} \setminus \overline{\mathbb{D}}$. Moreover, there exists $M > 0$ such that

$$\|f(\mu)\| = \left\| \left(L_{|\mathscr{X}_1} - \mu I\right)^{-1} x \right\| = \left\| (T - \mu I)^{-1} x \right\| \leqslant \frac{M}{|\mu - 1|}, \text{ for all } \mu \in \mathbb{C} \setminus \overline{\mathbb{D}}.$$

EXAMPLE 2. Let $\alpha = (\alpha_k)_{k\in\mathbb{N}}$ be a sequence in $\ell^\infty(\mathbb{N})$ such that $\alpha_k \in (0,1)$ for all $k \neq 0$, and $\alpha_k \nearrow 1$ as $k \to \infty$, and $|\alpha_0| > 1$. Define the operator T_α on $\mathscr{X} = \ell^2(\mathbb{N})$ by

$$\begin{aligned} T_\alpha : \mathscr{X} &\longrightarrow \mathscr{X} \\ (x_k)_{k\in\mathbb{N}} &\longmapsto (\alpha_0 x_0, \alpha_1 x_1, \alpha_2 x_2, \dots). \end{aligned}$$

Since $\alpha_0 \in \sigma(T_\alpha)$. Hence T_α does not satisfy the $[R]$ condition. On the other hand, we choose $(e_k)_{k\in\mathbb{N}}$ such that e_k be the element whose k-th entry is 1, while all others vanish. For $k \neq 0$, we have $T_\alpha e_k = \alpha_k e_k$, hence $\sigma_{T_\alpha}(e_k) = \{\alpha_k\} \subset \mathbb{D}$. We set

$$e_{kT_\alpha}(\mu) = \sum_{j=0}^{\infty} \frac{\alpha_k^j}{\mu^{j+1}} e_k \text{for all } |\mu| > 1,$$

thus $(\mu I - T_\alpha)e_{kT_\alpha}(\mu) = e_k$ for each $k \neq 0$. Moreover, one can show that

$$(\mu - 1)e_{kT_\alpha}(\mu) = e_k + \sum_{j=1}^{\infty} (\alpha_k^j - \alpha_k^{j-1})\mu^{-j} e_k = e_k + (\alpha_k - 1) \sum_{j=1}^{\infty} \frac{\alpha_k^j}{\mu^{j+1}} e_k.$$

Then

$$\left\| (\mu - 1)e_{kT_\alpha}(\mu) \right\| = \left\| e_k + (\alpha_k - 1) \sum_{j=1}^{\infty} \frac{\alpha_k^j}{\mu^{j+1}} e_k \right\| \leqslant 1 + \frac{|\alpha_k - 1|}{|\mu| - |\alpha_k|},$$

for all $\mu \in \mathbb{C}$ such that $|\mu| > 1$.

Since $|\alpha_k - 1| < |\mu| - |\alpha_k|$ for all $k \neq 0$. Hence

$$\left\| e_{kT_\alpha}(\mu) \right\| \leqslant \frac{2}{|\mu - 1|}, \text{ for all } \mu \in \mathbb{C} \setminus \overline{\mathbb{D}}.$$

Therefore, T_α satisfies the $[LR]$ condition at each e_k with $k \neq 0$.

Let $\delta > 0$ and consider the set

$$K_\delta = \left\{ \lambda = 1 + re^{i\theta},\ r > 0,\ |\theta| < \frac{\pi}{2} + \delta \right\}.$$

THEOREM 1. *Let $T \in B(\mathscr{X})$ have the SVEP and let $x \in \mathscr{X}$. If there exists $C > 0$ such that*

$$\|\hat{x}_T(\lambda)\| \leqslant \frac{C}{|\lambda - 1|} \text{ for all } |\lambda| > 1, \tag{3}$$

then

$$\|\hat{x}_T(\lambda)\| \leqslant \frac{M}{|\lambda - 1|} \text{ for all } \lambda \in K_\delta, \tag{4}$$

for some strictly positive constants δ and M.

Proof. Let $S = T - I$. Then, $\hat{x}_S(\lambda) = f_\lambda[T-I]x = \hat{x}_T(\lambda+1)$. Thus, by using condition (3), we have

$$\begin{aligned} &\sigma_S(x) \subset \{\lambda \in \mathbb{C} : |\lambda+1| < 1\} \cup \{0\}, \text{ and} \\ &\|\hat{x}_S(\lambda)\| \leqslant \frac{C}{|\lambda|}, \text{ for all } |\lambda+1| > 1. \end{aligned} \tag{5}$$

In particular, the above estimate is true for each $\lambda_0 \in \mathbb{C}$ such that $\Re(\lambda_0) = 0$ and $\Im(\lambda_0) \neq 0$.

Using Proposition 1, we obtain

$$\hat{x}_S(\lambda) = \sum_{n=0}^{\infty} (-1)^n (\lambda - \lambda_0)^n f_{\lambda_0}^{n+1}[S]x \tag{6}$$

whenever

$$|\lambda - \lambda_0| \left\| f_{\lambda_0}[S]x \right\| < 1. \tag{7}$$

One can show that $\hat{x}_S(\lambda)$ exists for all λ such that $\Im(\lambda) = \Im(\lambda_0)$, then $|\Re(\lambda)| < \frac{|\lambda_0|}{C}$. Indeed, by using (5), we get

$$|\lambda - \lambda_0| \left\| f_{\lambda_0}[S]x \right\| < |\lambda - \lambda_0| \frac{C}{|\lambda_0|}. \tag{8}$$

If $|\lambda - \lambda_0| \frac{C}{|\lambda_0|} < 1$ then $|\lambda - \lambda_0| < \frac{|\lambda_0|}{C}$. So, for $\Im(\lambda) = \Im(\lambda_0)$ we have $|\Re(\lambda)| < \frac{|\lambda_0|}{C}$. Since $\lambda_0 \neq 0$ is arbitrary on the imaginary axis. Thus, if we choose ζ such that $\tan\zeta = \frac{1}{C}$ then $\hat{x}_S(\lambda)$ exists for all $\lambda \in K_\zeta - 1$.

In order to obtain an appropriate estimate, fix $\delta \in (0,1)$ such that $\tan\delta = \frac{q}{C}$, for some $q \in (0,1)$. Let $\lambda \in K_\delta - 1$ with $\Re(\lambda) < 1$, and let $\lambda_0 = i\Im(\lambda)$. Then

$$\frac{|\lambda - \lambda_0|}{|\lambda_0|} . \frac{C}{q} < 1. \tag{9}$$

Thus,

$$\|\hat{x}_S(\lambda)\| \leqslant \|\hat{x}_S(\lambda_0)\| \sum_{n=0}^{\infty} q^n \leqslant \frac{C}{|\lambda_0|(1-q)} < \frac{C}{|\lambda|(1-q)\cos\delta}. \tag{10}$$

Therefore, by choosing

$$M = \frac{C}{(1-q)\cos\delta} = \frac{C}{(1-q)\sqrt{\frac{C^2}{C^2+q^2}}} = \frac{\sqrt{C^2+q^2}}{1-q} \geqslant C,$$

and going back to the operator T, we obtain the result. $\square$

DEFINITION 2. The peripheral local spectrum of $T \in B(\mathscr{X})$ at $x \in \mathscr{X}$ is the set

$$\gamma_T(x) := \{\lambda \in \sigma_T(x) : |\lambda| = r_T(x)\}.$$

Note that $\gamma_T(x) = \emptyset$ provided that $\max\{|\lambda| : \lambda \in \sigma_T(x)\} < r_T(x)$. The books P. Aiena [1], K. B. Laursen and M. M. Neumann [14] provide a rich bibliography of local spectral theory.

The local power boundedness for an operator $T \in B(\mathscr{X})$ has been studied in many works, see e.g. [4, 5, 7]. First, we give the definition of local power bounded operator.

DEFINITION 3. Let $T \in B(\mathscr{X})$ and $x \in \mathscr{X}$. T is said to be a locally power-bounded operator at x if there exists a constant $M > 0$ such that

$$\|T^n x\| \leqslant M \text{ for each } n \in \mathbb{N}.$$

In order to prove the second part of the Theorem 2, we need the following Lemma.

LEMMA 2. [17] *For any $0 < \varepsilon < 1$ there exists a nonnegative $\chi_\varepsilon \in C^2[-\pi,\pi]$ such that*

$$\chi_\varepsilon(\theta) = \begin{cases} 1, & \textit{for } |\theta| \leqslant \varepsilon/2 \\ 0 & \textit{for } |\theta| \geqslant \varepsilon \end{cases}$$

and the Fourier coefficients

$$\hat{\chi}_\varepsilon(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta}\chi_\varepsilon(\theta)d\theta$$

satisfy

$$\sum_{-\infty}^{\infty} |\hat{\chi}_\varepsilon(n+1) - \hat{\chi}_\varepsilon(n)| < \varepsilon.$$

THEOREM 2. *Let $T \in B(\mathscr{X})$ have the SVEP and let $x \in \mathscr{X}$ such that $\|T^n x\| \leqslant C$, $n \in \mathbb{N}$ and $r_T(x) = 1$. Then*

$$\lim_{k \longrightarrow \infty} \|T^k(T-I)x\| = 0, \tag{11}$$

if and only if $\gamma_T(x) = \{1\}$.

Proof. Assume that $z \in \gamma_T(x)$. Then

$$\|T^k(T-I)x\| \geqslant r_{T^k(T-I)}(x) = \sup_{\lambda \in \sigma_{T^k(T-I)}(x)} |\lambda^k(\lambda-1)| \geqslant |z-1|.$$

Thus, by (11), we obtain $z = 1$.

Conversely, suppose that $\gamma_T(x) = \{1\}$. By choosing the integration path $\Gamma = \{\lambda \in \mathbb{C} : |\lambda| = re^{i\theta}, r > 1\}$ and using the local functional calculus, we get

$$T^k x = \frac{1}{2\pi i}\int_\Gamma \lambda^k \hat{x}_T(\lambda)d\lambda.$$

Then

$$r^{-(k+1)}T^k(r^{-1}T-I)x=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{i(k+1)\theta}(e^{i\theta}-1)\hat{x}_T(re^{i\theta})d\theta. \tag{12}$$

Define $B_{\varepsilon,r}(\theta)=(e^{i\theta}-1)(1-\chi_\varepsilon(\theta))\hat{x}_T(re^{i\theta})$. Consider

$$I=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{i(k+1)\theta}(e^{i\theta}-1)\chi_\varepsilon(\theta)\hat{x}_T(re^{i\theta})d\theta.$$

The "Fourier Coefficients" of $(e^{i\theta}-1)\chi_\varepsilon(\theta)\hat{x}_T(re^{i\theta})$ are given by I and they could be obtained by convolving the Coefficients of $\phi(\theta)=(e^{i\theta}-1)\chi_\varepsilon(\theta)$ with those of $\psi(\theta)=\hat{x}_T(re^{i\theta})$. Since $r_T(x)=1$, the local resolvent function is defined in $\{\lambda\in\mathbb{C}:|\lambda|>1\}$ by

$$\hat{x}_T(\lambda)=R(\lambda,T)x=\sum_{k=0}^{\infty}\frac{T^kx}{\lambda^{k+1}}.$$

By Lemma 2, $\|\hat{\phi}\|_1$ and as T is locally power-bounded at x, $\left\|\frac{T^kx}{\lambda^{k+1}}\right\|\leqslant C$ for $k=0,1,\ldots$. The estimation obtained is of the form $\left\|\hat{\phi}*\hat{\psi}\right\|_\infty\leqslant\left\|\hat{\phi}\right\|_1\|\hat{\psi}\|_\infty$, with $\left\|\hat{\phi}\right\|_1\leqslant\varepsilon$ and $\|\hat{\psi}\|_\infty\leqslant C$. Then, $\|I\|\leqslant C\varepsilon$.

Consider now

$$J=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{i(k+1)\theta}B_{\varepsilon,r}(\theta)d\theta.$$

Since $\chi_\varepsilon\in C^2$, by partial integration we obtain

$$J=\frac{-1}{2\pi i(k+1)}\int_{-\pi}^{\pi}e^{i(k+1)\theta}\frac{d}{d\theta}\left[B_{\varepsilon,r}(\theta)\right]d\theta.$$

As $1-\chi_\varepsilon$ is vanishes for $|\theta|\leqslant\varepsilon/2$, so is $B_{\varepsilon,r}(\theta)$. We claim that then there exists a $T(\varepsilon)$, such that for, say $1<r<2$ satisfies

$$\left\|\frac{d}{d\theta}B_{\varepsilon,r}(\theta)\right\|\leqslant T(\varepsilon). \tag{13}$$

Indeed, let us define the compact set

$$K_\varepsilon:=\left\{re^{i\theta}\;/\;\frac{\varepsilon}{2}\leqslant|\theta|\leqslant\pi,1\leqslant r\leqslant 2\right\}.$$

$\hat{x}_T(\lambda)$ is analytic in K_ε, and thus both $\hat{x}_T(\lambda)$ and $\frac{d}{d\theta}\hat{x}_T(\lambda)$ are bounded in K_ε. Hence (13) follows. Combining the estimates for I and J implies the inequality

$$\left\|r^{-(k+1)}T^k(r^{-1}T-I)x\right\|\leqslant C\varepsilon+\frac{C(\varepsilon)}{k+1}. \tag{14}$$

As the right hand side of (14) is independent of r. Thus, by letting $r\longrightarrow 1$, we get (11). $\square$

THEOREM 3. *Let $T \in B(\mathscr{X})$ have the SVEP and let $x \in \mathscr{X}$ such that $\|T^n x\| \leqslant C$, $n \in \mathbb{N}$ and there exists $M > 0$ such that*

$$\left\|T^k(T-I)x\right\| \leqslant \frac{M}{k}, \ k \in \mathbb{N}, \tag{15}$$

with

$$\|\hat{x}_T(\lambda)\| \leqslant \frac{L}{|\lambda-1|} \ \textit{for}\ \lambda \in K_\delta. \tag{16}$$

Then, there exists $\delta > 0$ such that

$$\sigma_T(x) \cap K_\delta \subset \emptyset.$$

Conversely, suppose that $\gamma_T(x) = \{1\}$ or $\rho_T(x) < 1$ and for some $\delta > 0$ and $L > 0$ we have

$$\|\hat{x}_T(\lambda)\| \leqslant \frac{L}{|\lambda-1|} \ \textit{for}\ \lambda \in K_\delta. \tag{17}$$

Then T is locally power-bounded at x and there exists $M > 0$ such that (15) is satisfied.

Proof. Assume that T satisfies (15) but there is no δ such as $\sigma_T(x) \cap K_\delta = \emptyset$. By Theorem 2, we have $\gamma_T(x) \subset \{1\}$ and there exists a sequence $\{\lambda_j\} \subset \sigma_T(x)$ such that $\left|\Im(\lambda_j)\right| > j(1-\Re(\lambda_j))$ for any j. This means that $\lambda_j \to 1$ and

$$\left|\lambda_j\right|^2 = [1-(1-\Re(\lambda_j))]^2 + [\Im(\lambda_j)]^2 \geqslant 1 - 2(1-\Re(\lambda_j)) \geqslant 1 - 2\frac{\left|\lambda_j - 1\right|}{\sqrt{1+j^2}}$$

By choosing k_j such that $\frac{1}{k_j+1} < \frac{|\lambda_j-1|}{\sqrt{1+j^2}} < \frac{1}{k_j}$ we get $\left|\lambda_j\right|^2 \geqslant 1 - \frac{2}{k_j}$. Then, by using (16), we obtain

$$M \geqslant k_j \left|\lambda_j\right|^{k_j} \left|\lambda_j - 1\right| \geqslant \frac{k_j}{k_{j+1}} \left(1 - \frac{2}{k_j}\right)^{\frac{k_j}{2}} \sqrt{1+j^2} \ \text{for any } j,$$

which is a contradiction.

For the converse statement, consider the integral

$$T^k(T-I)x = \frac{1}{2\pi i}\int_\Gamma \lambda^k(\lambda-1)\hat{x}_T(\lambda)d\lambda,$$

where Γ is any curve enclosing $\sigma_T(x)$. Choosing $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ such that Γ_2 denotes a circular arc of the form $\{\lambda = \rho e^{i\theta},\ \rho < 1$ is fixed and θ varies $\}$, Γ_1 is a line segment of the form $\left\{\lambda = 1 + \frac{1}{k} + te^{i(\frac{\pi}{2}+\delta)},\ t \geqslant 0\right\}$, and Γ_3 is symmetric with Γ_1. Since $\gamma_T(x) \subset \{1\}$, we may choose ρ such that $\|(\lambda-1)\hat{x}_T(\lambda)\|$ be uniformly bounded over Γ_2. For Γ_1 and Γ_3 it suffices to use (17) in order to have this result. The path is connected only for large enough values of k. Over Γ_2 we have

$$\left\| \frac{1}{2\pi} \int_{\Gamma_2} \lambda^k (\lambda - 1) \hat{x}_T(\lambda) d\lambda \right\| \leqslant C_1 \rho^k.$$

On the other hand, over Γ_1 there exists two positive constants, c_1 and c_2 such that $|\lambda(t)| \leqslant \left(1 + \frac{c_1}{k}\right) e^{-c_2 t}$. Then, by using (17), we get

$$\left\| \frac{1}{2\pi} \int_{\Gamma_1} \lambda^k (\lambda - 1) \hat{x}_T(\lambda) d\lambda \right\| \leqslant \frac{C}{2\pi} \int_0^\infty \left(1 + \frac{c_1}{k}\right)^k e^{-c_2 k t} dt \leqslant \frac{C e^{c_1}}{2\pi c_2 k}.$$

Analogously for the integral over Γ_1. Then we have

$$\left\| T^k (T - I) \right\| \leqslant C_2 \left(\rho^k + \frac{1}{k} \right),$$

hence (15) follows. In order to complete the proof we have to show the local power boundedness of T. We have

$$T^k x = \frac{1}{2\pi i} \int_{\Gamma} \lambda^k \hat{x}_T(\lambda) d\lambda.$$

By evaluating the integral over Γ_2. We have

$$\frac{1}{2\pi} \int_{\Gamma_2} |\lambda|^k \left\| \hat{x}_T(\lambda) \right\| |d\lambda| \leqslant C_3 \rho^k.$$

Further, over Γ_1 (and analogously over Γ_3) we have

$$\begin{aligned}
&\frac{1}{2\pi} \int_{\Gamma_1} \frac{|\lambda^k|}{|\lambda - 1|} |\lambda - 1| \|\hat{x}_T(\lambda)\| |d\lambda| \\
&\leqslant \frac{C}{2\pi} \int_{\Gamma_1} \frac{|\lambda^k|}{|\lambda - 1|} |d\lambda| \\
&\leqslant \frac{C}{2\pi} \int_0^\infty e^{c_1} \frac{e^{-c_2 k t}}{\left| \frac{1}{k} + t e^{i\left(\frac{\pi}{2} + \delta\right)} \right|} dt \\
&= \frac{C e^{c_1}}{2\pi} \int_0^\infty \frac{e^{-c_2 \tau}}{\left| 1 + \tau e^{i\left(\frac{\pi}{2} + \delta\right)} \right|} d\tau =: C_4
\end{aligned}$$

Therefore, $\left\| T^k x \right\| \leqslant C$ with $C = 2C_4 + C_3 \rho^k$ which complete the proof. □

THEOREM 4. *Let $T \in B(\mathscr{X})$ have the SVEP and let $x \in \mathscr{X}$ such that $\|T^n x\| \leqslant C$, $n \in \mathbb{N}$ and $\sigma_T(x) \cap \Gamma \subset \{1\}$, where Γ denotes the unit circle. Then the following statements are equivalent*

(i) There exists $M < \infty$ such that $\|T^n (T - I) x\| \leqslant \frac{M}{n+1}$, $n \geqslant 0$;

(ii) *there exists* $K < \infty$ *such that* $\left\|(T-I)e^{t(T-I)}x\right\| \leqslant K\frac{1-e^{-t}}{t}, t > 0$;

(iii) *there exists* $K < \infty$ *such that* $\left\|(T-I)f_\lambda^{n+1}[T]x\right\| \leqslant \frac{K}{n}\left[\frac{1}{(\lambda-1)^n} - \frac{1}{\lambda^n}\right]$, $n \geqslant 1$, $\lambda > 1$;

(iv) *there exists* $B < \infty$, *and* $\delta > 0$ *such that* $\|\hat{x}_T(\lambda)\| \leqslant \frac{B}{|\lambda-1|}$, *for any* $\lambda \in K_\delta$.

Proof. $(i) \Longrightarrow (ii)$ For $t > 0$, we have

$$\left\|(T-I)e^{tT}x\right\| \leqslant \sum_0^\infty \|T^n(T-I)x\| \frac{t^n}{n!} \leqslant \frac{M}{t}\sum_1^\infty \frac{t^n}{n!} = M\frac{e^t-1}{t}.$$

$(ii) \Longrightarrow (iii)$ We define

$$f_x(\lambda) = \int_0^\infty e^{-\lambda t}e^{tT}x\mathrm{d}t.$$

Since $\left\|e^{tT}x\right\| \leqslant e^{\omega t}$ for some $\omega > 0$ and all $t \geqslant 0$ (see [20, pages 1-3]), $f_x(\lambda)$ is defined for every $\lambda \in \mathbb{C}$ such that $\Re(\lambda) > \omega$. One can show that, for $h > 0$

$$\frac{e^{hT}-I}{h}f_x(\lambda) = \frac{e^{\lambda h}-1}{h}\int_0^\infty e^{-\lambda t}e^{tT}x\mathrm{d}t - \frac{e^{\lambda h}}{h}\int_0^h e^{-\lambda t}e^{tT}x\mathrm{d}t,$$

and by letting $h \longrightarrow 0$, we get

$$Tf_x(\lambda) = \lambda f_x(\lambda) - x,$$

which implies that $(\lambda I - T)f_x(\lambda) = x$ for every $|\lambda| > 1$. As T has the SVEP, $f_x(\lambda) = \hat{x}_T(\lambda)$. It is easy to see that

$$\frac{\mathrm{d}^n\hat{x}_T(\lambda)}{\mathrm{d}\lambda^n} = (-1)^n\int_0^\infty t^n e^{-(\lambda-1)t}e^{t(T-I)}x\mathrm{d}t.$$

Thus, by Proposition 1, for $n \geqslant 1$, $\lambda > 0$

$$f_\lambda^{n+1}[T]x = \frac{1}{n!}\int_0^\infty t^n e^{-(\lambda-1)t}e^{t(T-I)}x\mathrm{d}t.$$

By multiplying with $T-I$ and from relation (ii) we get (iii).

$(iii) \Longrightarrow (ii)$ It suffices to substitute $\lambda := (n+1)/t$, in

$$\left\|(T-I)f_1^{n+1}\left[\frac{1}{\lambda}T\right]x\right\| \leqslant \frac{K\lambda}{n}\left[\frac{1}{\left(1-\frac{1}{\lambda}\right)^n} - 1\right],$$

and by making $n \to \infty$, we get (ii) [20, Theorem 8.3] (indeed, for all real number t there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$, $\|\frac{t}{n+1}T\| < 1$).

$(ii) \Longrightarrow (iv)$ By

$$\left\|e^{tT}x\right\| \leqslant C\sum_{n}\frac{t^n}{n!} = Ce^t$$

it follows that $\left\|e^{t(T-I)}x\right\| \leqslant C$, for $t > 0$. By estimating $\left\|e^{z(T-I)}x\right\|$ uniformly in a sector that surrounds the positive axis $t > 0$, which made it possible to change the integration path in

$$\hat{x}_T(\lambda) = \int_0^\infty e^{-(\lambda-1)t}e^{t(T-I)}x\mathrm{d}t, \text{ for } \lambda > 1$$

to the path $z = re^{i\theta}$ for θ small enough. Therefore, the proof of (ii) is the same as in [20, Proof. pp. 62–63].

$(iv) \Longrightarrow (i)$ Direct and immediate application of Theorem 3. □

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