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NORM–PARALLELISM OF HILBERT SPACE OPERATORS AND THE DAVIS–WIELANDT BEREZIN NUMBER

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Abstract. In this work, the concept of the Davis-Wielandt Berezin number is introduced. Some upper and lower bounds for the Davis-Wielandt Berezin number are introduced. A connection between norm-parallelism to the identity operator and an equality condition for the Davis-Wielandt Berezin number are also discussed. Some bounds for the Davis-Wielandt Berezin number for $n \times n$ operator matrices are established.

1. Introduction

Let $\mathscr{B}(\mathscr{H})$ be the Banach algebra of all bounded linear operators defined on a complex Hilbert space $(\mathscr{H};\langle\cdot,\cdot\rangle)$ with the identity operator $1_{\mathscr{H}}$ in $\mathscr{B}(\mathscr{H})$. When $\mathscr{H}=\mathbb{C}^n$, we identify $\mathscr{B}(\mathscr{H})$ with the algebra $\mathscr{M}_n(\mathbb{C})$ of n-by-n complex matrices.

A functional Hilbert space is the Hilbert space of complex-valued functions on some set $\Omega\subset\mathbb{C}$ that the evaluation functionals $\varphi_\lambda(f)=f(\lambda)$, $\lambda\in\Omega$ are continuous on $\mathscr{H}$. Then, by the Riesz representation theorem there is a unique element $k_\lambda\in\mathscr{H}$ such that $f(\lambda)=\langle f,k_\lambda\rangle$ for all $f\in\mathscr{H}$ and every $\lambda\in\Omega$. The function k on $\Omega\times\Omega$ defined by $k(z,\lambda)=k_\lambda(z)$ is called the reproducing kernel of $\mathscr{H}$, see [7]. It was shown that $k_\lambda(z)$ can be represented by

$$k_\lambda(z)=\sum_{n=1}^{\infty}\overline{e_n(\lambda)}e_n(z)$$

for any orthonormal basis $\{e_n\}_{n\geqslant1}$ of $\mathscr{H}$, see [52]. For example, for the Hardy-Hilbert space $H^2=H^2(\mathbb{D})$ over the unit disc $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$, $\{z^n\}_{n\geqslant1}$ is an orthonormal basis, therefore the reproducing kernel of H^2 is the function $k_\lambda(z)=\sum_{n=1}^{\infty}\overline{\lambda}_n z^n=\left(1-\overline{\lambda}z\right)^{-1}$, $\lambda\in\mathbb{D}$. Let $\widehat{k}_\lambda=\frac{k_\lambda}{\|k_\lambda\|}$ be the normalized reproducing kernel of the space

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$\mathscr{H}$. For a given a bounded linear operator T on $\mathscr{H}$, the Berezin symbol (or Berezin transform) of T is the bounded function $\widetilde{T}$ on Ω defined by

$$\widetilde{T}(\lambda)=\left\langle T\widehat{k}_{\lambda}(z),\widehat{k}_{\lambda}(z)\right\rangle,\ \lambda\in\Omega.$$

An important property of the Berezin symbol is that for all $T,S\in\mathscr{B}(\mathscr{H})$ if $\widetilde{T}(\lambda)=\widetilde{S}(\lambda)$ for all $\lambda\in\Omega$, then $T=S$ (at least when $\mathscr{H}$ consists from analytic functions, see Zhu [57]). For more details, see [11, 15, 16, 23]–[33]. So, the map $T\to\widetilde{T}$ is injective [18]. The Berezin set and the Berezin number of an operator T are defined, respectively, by

$$\operatorname{Ber}(T)=\left\{\widetilde{T}(\lambda):\lambda\in\Omega\right\}=\operatorname{Range}\left(\widetilde{T}\right),$$

and

$$\operatorname{ber}(T)=\sup\{|\gamma|:\gamma\in\operatorname{Ber}(T)\}=\sup_{\lambda\in\Omega}\left|\widetilde{T}(\lambda)\right|.$$

The Crawford Berezin number and the minimum Berezin modulus of the operator T are defined by

$$C_{\mathrm{Ber}}(T):=\inf\{|\widetilde{T}(\lambda)|:\lambda\in\Omega\}\quad\text{and}\quad m_{\mathrm{Ber}}(T):=\inf\{\|T\widehat{k}_{\lambda}\|:\lambda\in\Omega\}$$

respectively (see [24]).

The Berezin norm of an operator $T\in\mathscr{B}(\mathscr{H})$ is defined by

$$\|T\|_{\mathrm{Ber}}:=\sup_{\lambda\in\Omega}\left\|T\widehat{k}_{\lambda}\right\|.$$

Recall that the numerical range, the numerical radius and the Crawford number of $T\in\mathscr{B}(\mathscr{H})$ are defined respectively, by

$$W(T):=\{\langle Tx,x\rangle:x\in\mathscr{H}\text{ and }\|x\|=1\},$$

$$w(T):=\sup\{|\langle Tx,x\rangle|:\langle Tx,x\rangle\in W(T)\},$$

and

$$C(T):=\inf\{|\langle Tx,x\rangle|:\langle Tx,x\rangle\in W(T)\}.$$

It is well known that $w(\cdot)$ defines a norm on $\mathscr{B}(\mathscr{H})$, which is equivalent to the usual operator norm $\|.\|$. In fact, for any $T\in\mathscr{B}(\mathscr{H})$, $\frac{1}{2}\|T\|\leqslant w(T)\leqslant\|T\|$; see [17].

Clearly, $\operatorname{Ber}(T)\subset W(T)$ and $\operatorname{ber}(A)\leqslant w(T)$. For example, Karaev [33] showed that if we consider $T=\langle\cdot,z\rangle z$ in H^2, simple calculation then gives that $\widetilde{T}(\lambda)=|\lambda|^2(1-|\lambda|)$. Moreover, we have $\operatorname{Ber}(T)=\left[0,\frac{1}{4}\right]\subset[0,1]=W(T)$ and $\operatorname{ber}(T)=\frac{1}{4}<1=w(T)$. For other results concerning the Berezin symbol the reader may refer to [14], [19], [20], [42]–[49] and the references therein.

One of the most less common celebrated generalization of the numerical range and the numerical radius is the Davis-Wielandt shell and its radius of $T \in \mathscr{B}(\mathscr{H})$, which are defined as:

$$DW(T) := \{(\langle Tx,x\rangle, \langle Tx,Tx\rangle), x \in \mathscr{H}, \|x\| = 1\},$$

and

$$dw(T) = \sup_{x\in\mathscr{H}, \|x\|=1} \{\sqrt{|\langle Tx,x\rangle|^2 + \|Tx\|^4}\}. \tag{1}$$

It is easy to see that the Davis-Wielandt radius is not a norm. It has many properties that you can refer to reference [55]. The following inequality immediately comes from (1):

$$\max(w(T), \|T\|^2) \leqslant dw(T) \leqslant \sqrt{w^2(T) + \|T\|^4}$$

for any $T \in \mathscr{B}(\mathscr{H})$. Clearly, the projection of the set $DW(T)$ on the first co-ordinate is $W(T)$. One can easily check that $dw(T)$ is unitarily invariant but it does not define a norm on $\mathscr{B}(\mathscr{H})$.

The Davis-Wielandt shell and its radius were introduced and described firstly by Davis in [12] and [13] and Wielandt [51]. In fact, the Davis-Wielandt shell $DW(T)$ gives more information about the operator T and $W(T)$. For instance, in the finite dimensional case, Li and Poon proved [37] (see also [38]) that the normal property of Hilbert space operators can be completely determined by the geometrical shape of their Davis-Wielandt shells, namely, $T \in \mathscr{M}_n(\mathbb{C})$ is normal if and only if $DW(T)$ is a polyhedron in $\mathbb{C} \times \mathbb{R}$ identified with $\mathbb{R}^3$. Moreover, in finite dimensional case, the spectrum of an operator T; $\operatorname{sp}(T)$ is finite and $DW(T)$ is always closed, cf [37, Theorem 2.3]. These conditions are no longer equivalent for an infinite-dimensional operator T, cf [37, Example 2.5].

In [41], Lins et al. proved that, if $T \in \mathscr{M}_n(\mathbb{C})$ is normal, then $DW(T)$ is the convex hull of the points $\left(\operatorname{Re}(\lambda_j), \operatorname{Im}(\lambda_j), |\lambda_j|^2\right)$ $(j = 1, \cdots, n)$, for $\lambda_j \in \operatorname{sp}(T)$. Moreover, each point $\left(\operatorname{Re}(\lambda_j), \operatorname{Im}(\lambda_j), |\lambda_j|^2\right)$ is an extreme point of $DW(T)$. In particular case, if $n = 2$ i.e., $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has eigenvalues λ_1, λ_2, then $DW(T)$ degenerates to the line segment joining the points $\left(\lambda_1, |\lambda_1|^2\right)$ and $\left(\lambda_2, |\lambda_2|^2\right)$. So that $\dim DW(T) \leqslant 1$. In fact, the condition $\dim(DW(T)) \leqslant 1$ holds if and only if T is normal, with at most two distinct eigenvalues. Otherwise, $DW(T)$ is an ellipsoid (without its interior) centered at $\left(\frac{\lambda_1+\lambda_2}{2}, \frac{1}{2}\operatorname{tr}\left(|T|^2\right)\right)$. Also, it was proved that if $\dim(DW(T)) \geqslant 2$, then $DW(T)$ is always convex. A complete description of $DW(T)$ for a quadratic operator T was given in [38]. For more details see also [3], [39], [40] and [41].

In [51], Wielandt showed that the Davis-Wielandt shell is a useful tool for characterizing the eigenvalues of matrices in the set

$$\{P^*TP + Q^*SQ : P, Q \in \mathscr{M}_n(\mathbb{C}) \text{ are unitary}\}$$

for given $S,T \in \mathscr{M}_n(\mathbb{C})$.

Now, we want to introduce the concepts of the Davis-Wielandt Berezin set and the Davis-Wielandt Berezin number as follows:

$$\mathrm{Ber}_{dw}(T) = \{(\langle T\widehat{k_\lambda}, \widehat{k_\lambda}\rangle, \langle T\widehat{k_\lambda}, T\widehat{k_\lambda}\rangle), \lambda \in \Omega\},$$

and

$$\mathrm{ber}_{dw}(T) = \sup_{\lambda\in\Omega}\{\sqrt{|\langle T\widehat{k_\lambda}, \widehat{k_\lambda}\rangle|^2 + \|T\widehat{k_\lambda}\|^4}\}.$$

We can clearly see that $\mathrm{ber}_{dw}(T)$ is an generalization of $\mathrm{ber}(T)$, moreover $\mathrm{ber}_{dw}(T) \leqslant dw(T)$. It is easy to see that the Davis-Wielandt Berezin number of $T \in \mathscr{B}(\mathscr{H}(\Omega))$ satisfying the following inequality:

$$\max\left(\mathrm{ber}(T), \|T\|_{\mathrm{Ber}}^2\right) \leqslant \mathrm{ber}_{dw}(T) \leqslant \sqrt{\mathrm{ber}^2(T) + \|T\|_{\mathrm{Ber}}^4}. \tag{2}$$

In this work, the concept of the Davis-Wielandt Berezin number is introduced. Some upper and lower bounds for the Davis-Wielandt Berezin number are introduced. A connection between norm-parallelism to the identity operator and an equality condition for the Davis-Wielandt Berezin number are also discussed. Some bounds for the Davis-Wielandt Berezin number for $n \times n$ operator matrices are established.

2. The Norm–parallelism and the Davis-Wielandt Berezin number

For $T \in \mathscr{B}(\mathscr{H})$, let $\mathbb{M}_T$ be the set of all unit vectors for which T attains its norm; i.e.,

$$\mathbb{M}_T := \{x \in H : \|x\| = 1, \|Tx\| = \|T\|\}.$$

The concept of the norm–parallelism in $\mathscr{B}(\mathscr{H})$ has been introduced by Saddik [47] and recently discussed by Zamani and Moslehian in [54]–[56]. Let $S,T \in \mathscr{B}(\mathscr{H})$, we say that T is norm-parallel to S (see [54]), in symbol $T \parallel S$, if there exists $\lambda \in \{\alpha \in \mathbb{C} : |\alpha| = 1\}$ such that

$$\|T + \lambda S\| = \|T\| + \|S\|.$$

Such property is a useful tool in solving some problems in approximation theory, as pointed out in [54]. Equivalently, it has been shown in [54] that, $T \parallel S$ if and only if there exists a sequence of unit vectors x_n in $\mathscr{H}$ such that

$$\lim_{n\to\infty} |\langle Tx_n, Sx_n\rangle| = \|T\|\,\|S\|. \tag{3}$$

From the norm properties of vectors in $\mathscr{H}$, it can be shown that [53]

$$\|b\|^2 \inf_{\gamma\in C} \|a + \gamma b\|^2 = \|a\|^2\,\|b\|^2 - |\langle a,b\rangle|^2, \qquad \forall a,b \in \mathscr{H}.$$

In particular, two vectors a and b in $\mathscr{H}$ are linearly dependent if and only if

$$\inf_{\gamma \in C} \|a + \gamma b\|^2 = 0.$$

Employing this property, a necessary and sufficient condition for $T \in \mathscr{B}(\mathscr{H})$ to be norm-parallel to $S \in \mathscr{B}(\mathscr{H})$ was proved in [53], as elaborated in the following result.

THEOREM 1. *Let $S, T \in \mathscr{B}(\mathscr{H})$ be compact operators. Then the following conditions are equivalent:*

(1) $T \parallel S$.

(2) There exists $x \in \mathbb{M}_T \cap \mathbb{M}_S$ such that for every $\xi \in \mathbb{C}$ the vectors $Tx + \xi Sx$ and Sx are linearly dependent.

Let us begin with the following primary result.

LEMMA 1. *Let $S \in \mathscr{B}(\mathscr{H}(\Omega))$.*

(1) If $\Omega \subseteq \mathbb{C}$ is closed set, then the Berezin set $\mathrm{Ber}(S)$ *is a closed subset of the numerical range $W(S)$.*

(2) If $\Omega = \mathbb{C}$, then $\mathrm{Ber}(S) = W(S)$ *and so* $\mathrm{ber}(S) = \omega(S)$.

(3) In particular, the restriction of the numerical range $W|_\Omega(S)$ onto Ω is exactly the Berezin set $\mathrm{Ber}(S)$, *and hence* $\omega|_\Omega(S) = \mathrm{ber}(S)$, *where by $W|_\Omega(S)$ i.e.*

$$W|_\Omega(S) = \left\{\langle Sx, x\rangle : x \in \mathscr{H}(\Omega) \text{ such that for some } \lambda \in \Omega,\ x = \hat{k_\lambda}\right\} = \mathrm{Ber}(S).$$

Proof. (1) Let $S \in \mathscr{B}(\mathscr{H}(\Omega))$. It is well known that $\mathrm{Ber}(S) \subseteq W(S)$. So that for any sequence of points λ_n in Ω, the normalized reproducing kernel of $\mathscr{H}(\Omega)$ is given by $\widehat{k}_{\lambda_n}$. For $\widetilde{S}(\lambda_n) \in \mathrm{Ber}(S)$, we have $\widehat{k}_{\lambda_n} \longrightarrow \widehat{k}_\lambda$ which implies that $\widetilde{S}(\lambda_n) \longrightarrow \widetilde{S}(\lambda) \in \mathrm{Ber}(S)$, as $n \to \infty$; whenever $\lambda_n \longrightarrow \lambda$.

(2) This case follows clearly by noting that for each $x \in \mathscr{H}$ with $\|x\| = 1$, there exists an associated $\lambda \in \Omega = \mathbb{C}$ such that $x_\lambda = \widehat{k}_\lambda$. Hence, $\mathrm{ber}(S) = \omega(S)$.

(3) For the restriction onto Ω we get $W|_\Omega(S) = \mathrm{Ber}(S)$, and hence $\omega|_\Omega(S) = \mathrm{ber}(S)$. □

It's convenient to note that, in the restriction case the inequality $\mathrm{ber}(S) \leqslant \omega(S)$ still holds. So that, the reader shouldn't mix up or confuse between the $\omega|_\Omega(S)$ and $\omega(S)$.

COROLLARY 1. *Let Ω be a closed subset of $\mathbb{C}$ and $S \in \mathscr{B}(\mathscr{H}(\Omega))$. If $W(S) \subset \Omega$, then we have* $\mathrm{ber}(S) = \omega(S)$.

Proof. Follows from Lemma 1. □

In the sequel, a norm–parallelism of Hilbert space operators and an equality condition for the Davis-Wielandt Berezin number is established.

THEOREM 2. *Let Ω be any closed subset of $\mathbb{C}$ and $S \in \mathscr{B}(\mathscr{H}(\Omega))$. Then the following conditions are equivalent:*

(1) $S \parallel 1_{\mathscr{H}}$.

(2) $\operatorname{ber}_{dw}(S) = \sqrt{\operatorname{ber}^2(S) + \|S\|_{\text{Ber}}^4}$.

Proof. (1)⇒(2) Assume $S \parallel 1_{\mathscr{H}}$, by (3), $S \parallel 1_{\mathscr{H}}$ if and only if there exists a sequence of unit vectors $\{\widehat{k}_\lambda^{(n)}\}$ in $\mathscr{H}(\Omega)$ for some $\lambda \in \Omega$ such that

$$\lim_{n\to\infty} \left|\left\langle S\widehat{k}_\lambda^{(n)}, \widehat{k}_\lambda^{(n)}\right\rangle\right| = \|S\|_{\text{Ber}}. \tag{4}$$

Therefore, we have

$$\left|\left\langle S\widehat{k}_\lambda^{(n)}, \widehat{k}_\lambda^{(n)}\right\rangle\right| \leqslant \left\|S\widehat{k}_\lambda^{(n)}\right\| \leqslant \|S\|_{\text{Ber}} \quad \text{and} \quad \left|\left\langle S\widehat{k}_\lambda^{(n)}, \widehat{k}_\lambda^{(n)}\right\rangle\right| \leqslant \operatorname{ber}(S) \leqslant \|S\|_{\text{Ber}}. \tag{5}$$

Hence by (4) and (5) we obtain that

$$\lim_{n\to\infty} \left\|S\widehat{k}_\lambda^{(n)}\right\| = \|S\|_{\text{Ber}} \qquad \lim_{n\to\infty} \left|\left\langle S\widehat{k}_\lambda^{(n)}, \widehat{k}_\lambda^{(n)}\right\rangle\right| = \operatorname{ber}(S). \tag{6}$$

Now, by the definition of $\operatorname{ber}_{dw}(S)$ we have

$$\sqrt{\left|\left\langle S\widehat{k}_\lambda^{(n)}, \widehat{k}_\lambda^{(n)}\right\rangle\right|^2 + \left\|S\widehat{k}_\lambda^{(n)}\right\|^4} \leqslant \operatorname{ber}_{dw}(S) \leqslant \sqrt{\operatorname{ber}^2(S) + \|S\|_{\text{Ber}}^4}, \tag{7}$$

whence (6) and (7) imply that

$$\operatorname{ber}_{dw}(S) = \sqrt{\operatorname{ber}^2(S) + \|S\|_{\text{Ber}}^4}.$$

(2)⇒(1) Assume $\operatorname{ber}_{dw}(S) = \sqrt{\operatorname{ber}^2(S) + \|S\|_{\text{Ber}}^4}$. So, by the definition of $\operatorname{ber}_{dw}(S)$, there exists a sequence of unit vectors $\{\widehat{k}_\lambda^{(n)}\}$ in $\mathscr{H}(\Omega)$, for some $\lambda \in \Omega$, such that

$$\lim_{n\to\infty} \sqrt{\left|\left\langle S\widehat{k}_\lambda^{(n)}, \widehat{k}_\lambda^{(n)}\right\rangle\right|^2 + \left\|S\widehat{k}_\lambda^{(n)}\right\|^4} = \sqrt{\operatorname{ber}^2(S) + \|S\|_{\text{Ber}}^4}.$$

Then we have (6) holds. So that let us

Claim: $\operatorname{ber}(S) = \|S\|_{\text{Ber}}$. Hence by (6) we have

$$\lim_{n\to\infty} \left|\left\langle S\widehat{k}_\lambda^{(n)}, \widehat{k}_\lambda^{(n)}\right\rangle\right| = \|S\|_{\text{Ber}}.$$

or equivalently, $S \parallel 1_{\mathscr{H}}$. Setting

$$S\widehat{k}_{\lambda}^{(n)} = \alpha_n \widehat{k}_{\lambda_1}^{(n)} + \beta_n \widehat{k}_{\lambda_2}^{(n)} \qquad \text{for some } \lambda_1, \lambda_2 \in \Omega, \tag{8}$$

such that $\left\langle \widehat{k}_{\lambda_1}^{(n)}, \widehat{k}_{\lambda_2}^{(n)} \right\rangle = 0$, $\left\| \widehat{k}_{\lambda_2}^{(n)} \right\| = 1$, and for some $\alpha_n, \beta_n \in \mathbb{C}$. Thus, from (6) and (8) we have $\alpha_n = \left\langle S\widehat{k}_{\lambda_1}^{(n)}, \widehat{k}_{\lambda_1}^{(n)} \right\rangle$, $\beta_n = \left\langle S\widehat{k}_{\lambda_1}^{(n)}, \widehat{k}_{\lambda_2}^{(n)} \right\rangle$, $\lim_{n\to\infty} |\alpha_n| = \operatorname{ber}(S)$, and

$$\lim_{n\to\infty} |\alpha_n|^2 + |\beta_n|^2 = \|S\|_{\text{Ber}}^2.$$

Let $\eta_n = \left\langle S\widehat{k}_{\lambda_2}^{(n)}, \widehat{k}_{\lambda_1}^{(n)} \right\rangle$, $\zeta_n = \left\langle S\widehat{k}_{\lambda_2}^{(n)}, \widehat{k}_{\lambda_2}^{(n)} \right\rangle$, and

$$S_n = \begin{bmatrix} \alpha_n & \eta_n \\ \beta_n & \zeta_n \end{bmatrix}.$$

Since $|\alpha_n| \leqslant \operatorname{ber}(S_n) \leqslant \operatorname{ber}(S)$, then

$$\lim_{n\to\infty} \operatorname{ber}(S_n) = \operatorname{ber}(S).$$

Moreover, we have

$$|\alpha_n|^2 \leqslant \operatorname{ber}\left(\begin{bmatrix} |\alpha_n| & \frac{\overline{\alpha_n}\eta_n + \alpha_n\overline{\beta_n}}{2} \\ \frac{\overline{\alpha_n}\beta_n + \alpha_n\overline{\eta_n}}{2} & \frac{\overline{\alpha_n}\zeta_n + \alpha_n\overline{\zeta_n}}{2} \end{bmatrix} \right) = \operatorname{ber}(\operatorname{Re}(\overline{\alpha_n}S_n)) \leqslant \operatorname{ber}(\overline{\alpha_n}S_n) \leqslant \operatorname{ber}^2(S_n).$$

Thus, $\lim_{n\to\infty} \operatorname{ber}(\operatorname{Re}(\overline{\alpha_n}S_n)) = \operatorname{ber}^2(S_n)$ and $\lim_{n\to\infty} \frac{\overline{\alpha_n}\eta_n + \alpha_n\overline{\beta_n}}{2} = 0$. It follows that

$$\lim_{n\to\infty} |\eta_n| = \lim_{n\to\infty} |\beta_n|. \tag{9}$$

On the other hand, we have

$$S_n^* S_n = \begin{bmatrix} |\alpha_n|^2 + |\beta_n|^2 & \overline{\alpha_n}\eta_n + \overline{\beta_n}\zeta_n \\ \alpha_n\overline{\eta_n} + \beta_n\overline{\zeta_n} & |\eta_n|^2 + |\zeta_n|^2 \end{bmatrix}$$

and this allows us to obtain that

$$|\alpha_n|^2 + |\beta_n|^2 \leqslant \|S_n^* S_n\|_{\text{Ber}} \leqslant \|S_n\|_{\text{Ber}}^2 \leqslant \|S\|_{\text{Ber}}^2.$$

The above inequality implies that $\lim_{n\to\infty} \|S_n^* S_n\|_{\text{Ber}} = \|S\|_{\text{Ber}}^2$, and so we get $\lim_{n\to\infty} \overline{\alpha_n}\eta_n + \overline{\beta_n}\zeta_n = 0$. This yields that

$$\lim_{n\to\infty} |\alpha_n| = \lim_{n\to\infty} |\zeta_n|. \tag{10}$$

By (9) and (10) we find that

$$\lim_{n\to\infty} |\alpha_n|^2 + |\beta_n|^2 = \lim_{n\to\infty} |\eta_n|^2 + |\zeta_n|^2 = \|S\|_{\text{Ber}}^2, \tag{11}$$

from that we get

$$\lim_{n\to\infty} S_n^* S_n = \begin{bmatrix} \|S\|_{\mathrm{Ber}}^2 & 0 \\ 0 & \|S\|_{\mathrm{Ber}}^2 \end{bmatrix}.$$

It follows that

$$\lim_{n\to\infty} \mathrm{ber}(S_n) = \|S\|_{\mathrm{Ber}}. \tag{12}$$

From (11) and (12), we conclude that $\mathrm{ber}(S) = \|S\|_{\mathrm{Ber}}$, and this proves our claim. Hence, the proof of the theorem is completely established. □

As a consequence of Theorem 2, we have the following result [53].

COROLLARY 2. *Let Ω be any closed subset of $\mathbb{C}$ and $S \in \mathscr{B}(\mathscr{H}(\Omega))$. The following conditions are equivalent:*

(1) $\mathrm{ber}_{dw}(S) = \sqrt{\mathrm{ber}^2(S) + \|S\|_{\mathrm{Ber}}^4}$.

(2) $\mathrm{ber}(S) = \|S\|_{\mathrm{Ber}}$.

(3) $\mathrm{ber}_{dw}(S) = \|S\|_{\mathrm{Ber}} \sqrt{1 + \|S\|_{\mathrm{Ber}}^2}$.

(4) $S^*S \leqslant \mathrm{ber}^2(S)\, 1_{\mathscr{H}}$.

Proof. The equivalence (1)$\Leftrightarrow$(2) follows from the proof of Theorem 2.

(1)$\Rightarrow$(3) This implication follows from the equivalence (1)$\Leftrightarrow$(2).

(3)$\Rightarrow$(1) Assume $\mathrm{ber}_{dw}(S) = \|S\|_{\mathrm{Ber}} \sqrt{1 + \|S\|_{\mathrm{Ber}}^2}$ for any operator $S \in \mathscr{B}(\mathscr{H}(\Omega))$. Since $\mathrm{ber}(S) \leqslant \|S\|_{\mathrm{Ber}}$, we have

$$\|S\|_{\mathrm{Ber}} \sqrt{1 + \|S\|_{\mathrm{Ber}}^2} = \mathrm{ber}_{dw}(S) \leqslant \sqrt{\mathrm{ber}^2(S) + \|S\|_{\mathrm{Ber}}^4} \leqslant \|S\|_{\mathrm{Ber}} \sqrt{1 + \|S\|_{\mathrm{Ber}}^2}$$

and so that $\mathrm{ber}_{dw}(S) = \sqrt{\mathrm{ber}^2(S) + \|S\|_{\mathrm{Ber}}^4}$.

(1)$\Leftrightarrow$(4) By the first equivalence $\mathrm{ber}_{dw}(S) = \sqrt{\mathrm{ber}^2(S) + \|S\|_{\mathrm{Ber}}^4}$ if and only if $\mathrm{ber}(S) = \|S\|_{\mathrm{Ber}}$, that is $\left\|S\widehat{k}_\lambda\right\| \leqslant \mathrm{ber}(S) \left\|\widehat{k}_\lambda\right\|$ for all $\widehat{k}_\lambda \in \mathscr{H}(\Omega)$, $\lambda \in \Omega$. This is equivalent to say that $\left\|S\widehat{k}_\lambda\right\|^2 \leqslant \mathrm{ber}^2(S) \left\|\widehat{k}_\lambda\right\|^2$, that is $\left\langle S\widehat{k}_\lambda, S\widehat{k}_\lambda \right\rangle \leqslant \left\langle \mathrm{ber}^2(S)\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle$ for all $\widehat{k}_\lambda \in \mathscr{H}(\Omega)$, i.e., $\left\langle \left(S^*S - \mathrm{ber}^2(S)\, 1_{\mathscr{H}}\right)\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \leqslant 0$ for all $\widehat{k}_\lambda \in \mathscr{H}(\Omega)$, or equivalently $S^*S \leqslant \mathrm{ber}^2(S)\, 1_{\mathscr{H}}$. □

3. Some inequalities of the Davis-Wielandt Berezin number

In order to prove our results we need a sequence of lemmas.

LEMMA 2. *Let $a,b \geqslant 0$ and $p,q > 1$ such that $\frac{1}{p}+\frac{1}{q}=1$. Then*

- $ab \leqslant \frac{a^p}{p}+\frac{b^q}{q} \leqslant (\frac{a^{pr}}{p}+\frac{b^{qr}}{q})^{\frac{1}{r}}$ *for* $r \geqslant 1$.
- *For $r=1$ we recapture the Power-Mean inequality, which reads*

$$a^{\alpha}b^{1-\alpha} \leqslant \alpha a+(1-\alpha)b \leqslant (\alpha a^p+(1-\alpha)b^p)^{\frac{1}{p}}$$

for all $\alpha \in [0,1]$, $a,b \geqslant 0$ and $p \geqslant 1$.

The next lemma follows from the spectral theorem for positive operators and Jensen inequality see [36].

LEMMA 3. (McCarty inequality) *Let $T \in \mathscr{B}(\mathscr{H})$, $T \geqslant 0$ and $x \in \mathscr{H}$ be a unit vector. Then*

- $\langle Tx,x\rangle^r \leqslant \langle T^r x,x\rangle$ *for* $r \geqslant 1$;
- $\langle T^r x,x\rangle \leqslant \langle Tx,x\rangle^r$ *for* $0<r\leqslant 1$.

The generalized mixed Schwarz inequality was introduced in [22], as follows:

LEMMA 4. [36, Theorem 1] *Let $T \in \mathscr{B}(\mathscr{H})$ and $x,y \in \mathscr{H}$ be any vectors.*

- *If f, g are non-negative continuous functions on $[0,\infty)$ which are satisfying the relation $f(t)g(t)=t(t\in[0,\infty))$, then*

$$|\langle Tx,y\rangle| \leqslant \|f(|T|)x\|\,\|g(|T^*|)y\|;$$

- *If $0 \leqslant \alpha \leqslant 1$, then*

$$|\langle Tx,y\rangle|^2 \leqslant \langle |T|^{2\alpha}x,x\rangle\langle |T^*|^{2(1-\alpha)}y,y\rangle.$$

We note that, the McCarthy inequality was extended for general Hilbert space operators in [5] and [6]. Also, the corresponding Cartesian decomposition version of Lemma 4 recently was proved in [4].

In some of our results we need the following two fundamental norm estimates, which are:

$$\|S+T\| \leqslant \frac{1}{2}\left(\|S\|+\|T\|+\sqrt{(\|S\|-\|T\|)^2+4\left\|S^{1/2}T^{1/2}\right\|^2}\right), \tag{13}$$

and

$$\left\|S^{1/2}T^{1/2}\right\| \leqslant \|ST\|^{1/2}.$$

Both estimates are valid for all positive operators $S,T \in \mathscr{B}(\mathscr{H})$. Also, it should be noted that (13) is sharper than the triangle inequality as pointed out by Kittaneh in [34].

Now, we obtain lower bounds for the Davis-Wielandt Berezin number in $\mathscr{B}(\mathscr{H}(\Omega))$.

THEOREM 3. *Let* $T \in \mathscr{B}(\mathscr{H}(\Omega))$. *Then*
(i) $\mathrm{ber}_{dw}^2(T) \geqslant \max\left(\mathrm{ber}^2(T) + C_{\mathrm{Ber}}^2(|T|^2), \|T\|_{\mathrm{Ber}}^4 + C_{\mathrm{Ber}}^2(T)\right)$;
(ii) $\mathrm{ber}_{dw}^2(T) \geqslant \max\left(\mathrm{ber}(T) C_{\mathrm{Ber}}(|T|^2), \|T\|_{\mathrm{Ber}}^2 C_{\mathrm{Ber}}(T)\right)$.

Proof. If $\widehat{k}_\lambda \in \mathscr{H}(\Omega)$ be a normalized reproducing kernel, then

$$\begin{aligned}\mathrm{ber}_{dw}^2(T) &\geqslant |\langle T\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|^2 + \|T\widehat{k}_\lambda\|^4 \\ &= |\langle T\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|^2 + \langle |T|^2\widehat{k}_\lambda, \widehat{k}_\lambda\rangle^2 \\ &\geqslant |\langle T\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|^2 + C_{\mathrm{Ber}}^2(|T|^2).\end{aligned}$$

Now, by taking the supremum over all $\lambda \in \Omega$, we get

$$\mathrm{ber}_{dw}^2(T) \geqslant \mathrm{ber}^2(T) + C_{\mathrm{Ber}}^2(|T|^2). \tag{14}$$

Also, we have

$$\begin{aligned}\mathrm{ber}_{dw}^2(T) &\geqslant |\langle T\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|^2 + \|T\widehat{k}_\lambda\|^4 \\ &\geqslant C_{\mathrm{Ber}}^2(T) + \|T\widehat{k}_\lambda\|^4.\end{aligned} \tag{15}$$

From (14) and (15), the pert (i) is hold.

For (ii), by applying arithmetic-geometric mean inequality, we have

$$\begin{aligned}\mathrm{ber}_{dw}^2(T) &\geqslant |\langle T\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|^2 + \|T\widehat{k}_\lambda\|^4 \\ &\geqslant 2|\langle T\widehat{k}_\lambda, \widehat{k}_\lambda\rangle| \|T\widehat{k}_\lambda\|^2 \\ &= 2|\langle T\widehat{k}_\lambda, \widehat{k}_\lambda\rangle| |\langle |T|^2\widehat{k}_\lambda, \widehat{k}_\lambda\rangle \\ &\geqslant 2|\langle T\widehat{k}_\lambda, \widehat{k}_\lambda\rangle| C_{\mathrm{Ber}}(|T|^2).\end{aligned}$$

By taking the supremum over $\lambda \in \Omega$, we get

$$\mathrm{ber}_{dw}^2(T) \geqslant 2\,\mathrm{ber}(T) C_{\mathrm{Ber}}(|T|^2). \tag{16}$$

Moreover,

$$\begin{aligned}\mathrm{ber}_{dw}^2(T) &\geqslant 2|\langle T\widehat{k}_\lambda, \widehat{k}_\lambda\rangle| \|T\widehat{k}_\lambda\|^2 \\ &\geqslant 2C_{\mathrm{Ber}}(T)\|T\widehat{k}_\lambda\|^2.\end{aligned}$$

Now by taking the supremum over $\lambda \in \Omega$, we get

$$\mathrm{ber}_{dw}^2(T) \geqslant 2C_{\mathrm{Ber}}(T)\|T\|_{\mathrm{Ber}}^2. \tag{17}$$

From (16) and (17), the pert (ii) holds. □

REMARK 1. You can see the inequalities obtained in Theorem 3 (i) is sharper than the lower bound obtained in (2). Because

$$\begin{aligned}\max\left(\mathrm{ber}^2(T), \|T\|_{\mathrm{Ber}}^4\right) &\leqslant \max\left(\mathrm{ber}^2(T) + C_{\mathrm{Ber}}^2(|T|^2), \|T\|_{\mathrm{Ber}}^4 + C_{\mathrm{Ber}}^2(T)\right) \\ &\leqslant \mathrm{ber}_{dw}^2(T).\end{aligned}$$

In the next theorem we obtain lower and upper bounds for $\mathrm{ber}^2_{dw}(T)$.

THEOREM 4. *Let $T \in \mathscr{B}(\mathscr{H}(\Omega))$. Then*

$$\frac{1}{2}\left(\mathrm{ber}^2(T+|T|)+C^2_{\mathrm{Ber}}(T-|T|^2)\right) \leqslant \mathrm{ber}^2_{dw}(T) \leqslant \frac{1}{2}\left(\mathrm{ber}^2(T+|T|)+\mathrm{ber}^2(T-|T|^2)\right).$$

Proof. If $\widehat{k}_\lambda \in \mathscr{H}(\Omega)$, then

$$\begin{aligned}
|\langle T\widehat{k}_\lambda,\widehat{k}_\lambda\rangle|^2+\|T\widehat{k}_\lambda\|^4 &= \frac{1}{2}|\langle T\widehat{k}_\lambda,\widehat{k}_\lambda\rangle+\langle T\widehat{k}_\lambda,T\widehat{k}_\lambda\rangle|^2+\frac{1}{2}|\langle T\widehat{k}_\lambda,\widehat{k}_\lambda\rangle-\langle T\widehat{k}_\lambda,T\widehat{k}_\lambda\rangle|^2\\
&= \frac{1}{2}|\langle T\widehat{k}_\lambda,\widehat{k}_\lambda\rangle+\langle |T|^2\widehat{k}_\lambda,\widehat{k}_\lambda\rangle|^2+\frac{1}{2}|\langle T\widehat{k}_\lambda,\widehat{k}_\lambda\rangle-\langle |T|^2\widehat{k}_\lambda,\widehat{k}_\lambda\rangle|^2\\
&= \frac{1}{2}|\langle T+|T|^2\widehat{k}_\lambda,\widehat{k}_\lambda\rangle|^2+\frac{1}{2}|\langle T-|T|^2\widehat{k}_\lambda,\widehat{k}_\lambda\rangle|^2\\
&\geqslant \frac{1}{2}\left(|\langle T+|T|^2\widehat{k}_\lambda,\widehat{k}_\lambda\rangle|^2+C^2_{\mathrm{Ber}}(T-|T|^2)\right).
\end{aligned}$$

By taking the supremum over all $\lambda \in \Omega$, we get

$$\mathrm{ber}^2_{dw}(T) \geqslant \frac{1}{2}\left(\mathrm{ber}^2(T+|T|^2)+C^2_{\mathrm{Ber}}(T-|T|^2)\right).$$

For finding the upper bound, we have

$$\begin{aligned}
|\langle T\widehat{k}_\lambda,\widehat{k}_\lambda\rangle|^2+\|T\widehat{k}_\lambda\|^4 &= \frac{1}{2}|\langle T\widehat{k}_\lambda,\widehat{k}_\lambda\rangle+\langle T\widehat{k}_\lambda,T\widehat{k}_\lambda\rangle|^2+\frac{1}{2}|\langle T\widehat{k}_\lambda,\widehat{k}_\lambda\rangle-\langle T\widehat{k}_\lambda,T\widehat{k}_\lambda\rangle|^2\\
&= \frac{1}{2}|\langle T\widehat{k}_\lambda,\widehat{k}_\lambda\rangle+\langle |T|^2\widehat{k}_\lambda,\widehat{k}_\lambda\rangle|^2+\frac{1}{2}|\langle T\widehat{k}_\lambda,\widehat{k}_\lambda\rangle-\langle |T|^2\widehat{k}_\lambda,\widehat{k}_\lambda\rangle|^2\\
&= \frac{1}{2}|\langle T+|T|^2\widehat{k}_\lambda,\widehat{k}_\lambda\rangle|^2+\frac{1}{2}|\langle T-|T|^2\widehat{k}_\lambda,\widehat{k}_\lambda\rangle|^2\\
&\leqslant \frac{1}{2}\left(\mathrm{ber}^2(T+|T|^2)+\mathrm{ber}^2(T-|T|^2)\right).
\end{aligned}$$

Again, by taking the supremum over all $\lambda \in \Omega$, we get

$$\mathrm{ber}^2_{dw}(T) \leqslant \frac{1}{2}\left(\mathrm{ber}^2(T+|T|^2)+\mathrm{ber}^2(T-|T|^2)\right).$$

These statements complete the proof. □

In the following theorem, the authors obtained some relation between the Davis-Wielandt Berezin number and the Berezin number.

THEOREM 5. [50] *Let $T \in \mathscr{B}(\mathscr{H}(\Omega))$. Then*

$$\mathrm{ber}^2_{dw}(T) \leqslant \mathrm{ber}^2(|T|^2-T)+2\|T\|^2_{\mathrm{Ber}}\,\mathrm{ber}(T)$$

and

$$\mathrm{ber}^2_{dw}(T) \leqslant \frac{1}{2}\,\mathrm{ber}(|T|+2|T|^4+|T^*|^2)-\frac{1}{2}\inf_\lambda(\|T\widehat{k}_\lambda\|-\|T^*\widehat{k}_\lambda\|)^2. \tag{18}$$

In the next theorem, we obtain a lower bound for square of the Davis-Wielandt Berezin number.

THEOREM 6. *Let $T \in \mathscr{B}(\mathscr{H}(\Omega))$. If λ is a nonzero complex number, and $r > 0$, such that*

$$\|T - \lambda I\|_{\mathrm{Ber}} \leqslant r. \tag{19}$$

Then

$$\mathrm{ber}_{dw}^2(T) \geqslant \lambda^{-1}(\|T\widehat{k}_\lambda\|^2 + |\lambda|^2 - r^2)\|T\widehat{k}_\lambda\|^2. \tag{20}$$

Proof. If $\widehat{k}_\lambda \in \mathscr{H}(\Omega)$, then

$$\begin{aligned}\mathrm{ber}_{dw}^2(T) &\geqslant |\langle T\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|^2 + \|T\widehat{k}_\lambda\|^4 \\ &\geqslant 2|\langle T\widehat{k}_\lambda, \widehat{k}_\lambda\rangle||\langle T\widehat{k}_\lambda, T\widehat{k}_\lambda\rangle| \qquad \text{(by the arithmetic-geometric mean)}\end{aligned} \tag{21}$$

On the other hand, from (19), we have

$$\begin{aligned}\|T\widehat{k}_\lambda\|^2 + |\lambda|^2 - 2\mathrm{Re}\bar{\lambda}\langle T\widehat{k}_\lambda, \widehat{k}_\lambda\rangle &= |\langle (T-\lambda)\widehat{k}_\lambda, (T-\lambda)\widehat{k}_\lambda\rangle| \\ &= \|T\widehat{k}_\lambda - \lambda\widehat{k}_\lambda\|^2 \\ &\leqslant \|T - \lambda I\|_{\mathrm{Ber}}^2 \\ &\leqslant r^2.\end{aligned}$$

So,

$$\|T\widehat{k}_\lambda\|^2 + |\lambda|^2 \leqslant 2|\lambda||\langle T\widehat{k}_\lambda, \widehat{k}_\lambda\rangle| + r^2. \tag{22}$$

From (21) and (22), we have

$$\mathrm{ber}_{dw}^2(T) \geqslant \lambda^{-1}(\|T\widehat{k}_\lambda\|^2 + |\lambda|^2 - r^2)\|T\widehat{k}_\lambda\|^2. \quad \square$$

REMARK 2. From (22) for any $T \in \mathscr{B}(\mathscr{H}(\Omega))$, nonzero complex number λ, and $r > 0$, we have

$$\begin{aligned}\mathrm{ber}_{dw}^2(T) - \|T\widehat{k}_\lambda\|^4 &= |\langle T\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|^2 \\ &\leqslant \|T\widehat{k}_\lambda\|^2 \\ &\leqslant 2|\lambda||\langle T\widehat{k}_\lambda, \widehat{k}_\lambda\rangle| + r^2 - |\lambda|^2 \\ &\leqslant 2|\lambda|\,\mathrm{ber}(T) + r^2 - |\lambda|^2.\end{aligned}$$

In the next theorem we obtain upper bound for the Davis-Wielandt Berezin number by stating the minimum Berezin modulus of an operator.

THEOREM 7. *Let* $T \in \mathscr{B}(\mathscr{H}(\Omega))$. *Then*

$$\operatorname{ber}_{dw}^2(T) \leqslant \sup_{\theta \in \mathbb{R}} \operatorname{ber}^2(e^{i\theta}T + |T|^2) - 2C_{\mathrm{Ber}}(T) m_{\mathrm{Ber}}^2(T). \tag{23}$$

Proof. Let $\theta \in \mathbb{R}$ such that $|\langle T\widehat{k}_\lambda, \widehat{k}_\lambda\rangle| = e^{i\theta}\langle T\widehat{k}_\lambda, \widehat{k}_\lambda\rangle$. If $\widehat{k}_\lambda \in \mathscr{H}(\Omega)$, then

$$\begin{aligned} |\langle T\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|^2 + \|T\widehat{k}_\lambda\|^4 &= \langle e^{i\theta}T\widehat{k}_\lambda, \widehat{k}_\lambda\rangle^2 + \langle |T|^2\widehat{k}_\lambda, \widehat{k}_\lambda\rangle^2 \\ &= (\langle e^{i\theta}T\widehat{k}_\lambda, \widehat{k}_\lambda\rangle + \langle |T|^2\widehat{k}_\lambda, \widehat{k}_\lambda\rangle)^2 - 2\langle e^{i\theta}T\widehat{k}_\lambda, \widehat{k}_\lambda\rangle\langle |T|^2\widehat{k}_\lambda, \widehat{k}_\lambda\rangle. \end{aligned}$$

So,

$$\begin{aligned} &|\langle T\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|^2 + \|T\widehat{k}_\lambda\|^4 + 2|\langle T\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|\langle |T|^2\widehat{k}_\lambda, \widehat{k}_\lambda\rangle \\ &\quad = |\langle T\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|^2 + \|T\widehat{k}_\lambda\|^4 + 2\langle e^{i\theta}T\widehat{k}_\lambda, \widehat{k}_\lambda\rangle\langle |T|^2\widehat{k}_\lambda, \widehat{k}_\lambda\rangle \\ &\quad = (\langle e^{i\theta}T\widehat{k}_\lambda, \widehat{k}_\lambda\rangle + \langle |T|^2\widehat{k}_\lambda, \widehat{k}_\lambda\rangle)^2 \\ &\quad = \langle (e^{i\theta}T + |T|^2)\widehat{k}_\lambda, \widehat{k}_\lambda\rangle^2 \\ &\quad \leqslant \operatorname{ber}^2(e^{i\theta}T + |T|^2) \\ &\quad \leqslant \sup_{\theta \in \mathbb{R}} \operatorname{ber}^2(e^{i\theta}T + |T|^2). \end{aligned}$$

Therefore by taking the supremum over all $\lambda \in \Omega$, we get

$$\operatorname{ber}_{dw}^2(T) + 2C_{\mathrm{Ber}}(T) m_{\mathrm{Ber}}^2(T) \leqslant \sup_{\theta \in \mathbb{R}} \operatorname{ber}^2(e^{i\theta}T + |T|^2). \quad \square$$

REMARK 3. Note that inequality (23) in Theorem 7 is sharper than inequality (18) in Theorem 5.

THEOREM 8. *Let* $T \in \mathscr{B}(\mathscr{H}(\Omega))$. *If* f, g *are nonnegative continuous functions on* $[0, \infty)$ *which are satisfying the relation* $f(t)g(t) = t$ $(t \in [0, \infty))$, *then*

$$\operatorname{ber}_{dw}^2(T) \leqslant \operatorname{ber}\left[\frac{1}{p}\left(f^{2p}(|T|) + f^{2p}(|T^*T|)\right) + \frac{1}{q}\left(g^{2q}(|T^*|) + g^{2q}(|T^*T|)\right)\right], \tag{24}$$

for any $p \geqslant q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemmas 4, 2 and 3(b), we have

$$\begin{aligned} &|\langle T\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|^2 + \|T\widehat{k}_\lambda\|^4 \\ &= |\langle T\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|^2 + \langle T^*T\widehat{k}_\lambda, \widehat{k}_\lambda\rangle^2 \\ &\leqslant \langle f^2(|T|)\widehat{k}_\lambda, \widehat{k}_\lambda\rangle\langle g^2(|T^*|)\widehat{k}_\lambda, \widehat{k}_\lambda\rangle + \langle f^2(|T^*T|)\widehat{k}_\lambda, \widehat{k}_\lambda\rangle\langle g^2(|T^*T|)\widehat{k}_\lambda, \widehat{k}_\lambda\rangle \\ &\leqslant \frac{1}{p}\langle f^2(|T|)\widehat{k}_\lambda, \widehat{k}_\lambda\rangle^p + \frac{1}{q}\langle g^2(|T^*|)\widehat{k}_\lambda, \widehat{k}_\lambda\rangle^q + \frac{1}{p}\langle f^2(|T^*T|)\widehat{k}_\lambda, \widehat{k}_\lambda\rangle^p \\ &\qquad + \frac{1}{q}\langle g^2(|T^*T|)\widehat{k}_\lambda, \widehat{k}_\lambda\rangle^q \end{aligned}$$

$$\leqslant \frac{1}{p}\langle f^{2p}(|T|)\widehat{k}_\lambda,\widehat{k}_\lambda\rangle+\frac{1}{q}\langle g^{2q}(|T^*|)\widehat{k}_\lambda,\widehat{k}_\lambda\rangle+\frac{1}{p}\langle f^{2p}(|T^*T|)\widehat{k}_\lambda,\widehat{k}_\lambda\rangle + \frac{1}{q}\langle g^{2q}(|T^*T|)\widehat{k}_\lambda,\widehat{k}_\lambda\rangle$$

$$\leqslant \langle\frac{1}{p}\left(f^{2p}(|T|)+f^{2p}(|T^*T|)\right)+\frac{1}{q}\left(g^{2q}(|T^*|)+g^{2q}(|T^*T|)\right)\widehat{k}_\lambda,\widehat{k}_\lambda\rangle$$

$$\leqslant \operatorname{ber}\left[\frac{1}{p}\left(f^{2p}(|T|)+f^{2p}(|T^*T|)\right)+\frac{1}{q}\left(g^{2q}(|T^*|)+g^{2q}(|T^*T|)\right)\right].$$

Therefore by taking the supremum over all $\lambda\in\Omega$, we get the desired result. □

COROLLARY 3. *Let $T\in\mathscr{B}(\mathscr{H}(\Omega))$. Then for all $p>1$,*

$$\operatorname{ber}^2_{dw}(T)\leqslant \operatorname{ber}\left(\frac{1}{2}(|T|^2+|T^*|^2+2|T|^4)\right). \tag{25}$$

Proof. Inequality (25) *immediately comes from inequality* (24) *by putting $f(t)=g(t)=t^{\frac{1}{2}}$, and $p=q=2$.* □

4. Further refinemented inequalities

In order to establish our main first result concerning the the Euclidean Berezin number, we need to recall the concept of generalized Euclidean Berezin number of an n-tuple operator; which was introduced by Bakherad in [44]. Namely, for an n-tuple $\mathbf{T}=(T_1,\cdots,T_n)\in\mathscr{B}(\mathscr{H}(\Omega))^n:=\mathscr{B}(\mathscr{H}(\Omega))\times\cdots\times\mathscr{B}(\mathscr{H}(\Omega))$; i.e., for $T_1,\cdots,T_n\in\mathscr{B}(\mathscr{H}(\Omega))$. The Euclidean operator radius of $T_1,\cdots,T_n$ is defined by

$$\operatorname{ber}_p(T_1,\cdots,T_n):=\sup_{\lambda\in\Omega}\left(\sum_{i=1}^n\left|\left\langle T_i\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|^p\right)^{1/p}\qquad \text{for all } \widehat{k}_\lambda\in\mathscr{H}(\Omega),\ p\geqslant 1. \tag{26}$$

The following properties of the generalized Euclidean Berezin number could be proved easily.

(1) $\operatorname{ber}_p(T_1,\cdots,T_n)=0$ if and only if $T_k=0$ for each $k=1,\cdots,n$.

(2) $\operatorname{ber}_p(\lambda T_1,\cdots,\lambda T_n)=|\lambda|\operatorname{ber}_p(T_1,\cdots,T_n)$.

(3) $\operatorname{ber}_p(X_1+Y_1,\cdots,X_n+Y_n)\leqslant\operatorname{ber}_p(X_1,\cdots,X_n)+\operatorname{ber}_p(Y_1,\cdots,Y_n)$.

(4) $\operatorname{ber}_p(X_1,\cdots,X_n)=\operatorname{ber}_p(X_1^*,\cdots,X_n^*)$.

(5) $\operatorname{ber}_p(X_1^*X_1,\cdots,X_n^*X_n)=\operatorname{ber}_p(X_1X_1^*,\cdots,X_nX_n^*)$

for every $T_k,X_k,Y_k,C\in\mathscr{B}(\mathscr{H}(\Omega))$ $(1\leqslant k\leqslant n)$ and every scalar $\lambda\in\mathbb{C}$. In case $p=2$ we refer to the Euclidean Berezin number $\operatorname{ber}_{\mathrm{e}}(\cdot,\ldots,\cdot)$.

The following relation between the Euclidean Berezin number $\operatorname{ber}_{\mathrm{e}}(Y,Y^*Y)$ and the Davis-Wielandt radius $\operatorname{ber}_{dw}(Y)$ holds for every $Y\in\mathscr{B}(\mathscr{H}(\Omega))$.

LEMMA 5. *Let* $Y \in \mathscr{B}(\mathscr{H}(\Omega))$. *Then*

$$\mathrm{ber}_{\mathrm{e}}(Y, Y^*Y) = \mathrm{ber}_{dw}(Y). \tag{27}$$

Proof. Setting $n = 2$, $T_1 = Y$ and $T_2 = Y^*Y$, $Y \in \mathscr{B}(\mathscr{H}(\Omega))$ in (5), we have

$$\begin{aligned}
\mathrm{ber}_{\mathrm{e}}(Y, Y^*Y) &:= \sup_{\lambda \in \Omega} \left(\left| \left\langle Y\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right|^2 + \left| \left\langle Y^*Y\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right|^2 \right)^{1/2} \\
&= \sup_{\lambda \in \Omega} \left\{ \sqrt{\left| \left\langle Y\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right|^2 + \left\| Y\widehat{k}_\lambda \right\|^4} \right\} \\
&= \mathrm{ber}_{dw}(Y),
\end{aligned}$$

which gives the Davis-Wielandt radius of Y, as required. □

THEOREM 9. *Let* $Y \in \mathscr{B}(\mathscr{H}(\Omega))$. *Then* $\mathrm{ber}_{dw}(Y) = \sqrt{2} \cdot \mathrm{ber}(Y)$ *if and only if* Y *is selfadjoint idempotent operator.*

Proof. To prove the 'only if part', from Lemma 5, we have $\mathrm{ber}_{\mathrm{e}}(Y, Y^*Y) = \mathrm{ber}_{dw}(Y)$ for any $Y \in \mathscr{B}(\mathscr{H})$. Clearly if Y is selfadjoint idempotent operator, then $\mathrm{ber}_{dw}(Y) = \mathrm{ber}_{\mathrm{e}}(Y, Y^*Y) = \mathrm{ber}_{\mathrm{e}}\left(Y, Y^2\right) = \mathrm{ber}_{\mathrm{e}}(Y, Y)$. On the other hand, by setting $n = 2$ and $T_1 = T_2 = Y$, in (27), we get $\mathrm{ber}_{\mathrm{e}}(Y, Y) = \sqrt{2} \cdot \mathrm{ber}(Y)$. Hence, $\mathrm{ber}_{dw}(Y) = \sqrt{2} \cdot \mathrm{ber}(Y)$. The 'if part' follows by noting that, $Y^*Y = Y^2$ if and only if Y is selfadjoint and therefore $Y^*Y = Y$, when Y is an idempotent operator, i.e., $Y^2 = Y$. □

In 2005, Kittaneh [35] proved that

$$\frac{1}{4}\|S^*S + SS^*\| \leqslant w^2(S) \leqslant \frac{1}{2}\|S^*S + SS^*\| \tag{28}$$

for a Hilbert space operator $S \in \mathscr{B}(\mathscr{H})$.

The corresponding version of the above inequality in terms of Berezin numbers can be obtained such as:

$$\frac{1}{4}\|R^*R + RR^*\|_{\mathrm{Ber}} \leqslant \mathrm{ber}^2(R) \leqslant \frac{1}{2}\|R^*R + RR^*\|_{\mathrm{Ber}} \tag{29}$$

for Hilbert space operator $R \in \mathscr{B}(\mathscr{H}(\Omega))$. The following result extends (29) for the Euclidean Berezin number.

LEMMA 6. *Let* $R_k \in \mathscr{B}(\mathscr{H}(\Omega))$ $(k = 1, \cdots, n)$. *Then*

$$\frac{1}{2^{p+1}n^{p-1}} \left\| \sum_{k=1}^{n} R_k^*R_k + R_kR_k^* \right\|_{\mathrm{Ber}}^p \leqslant \mathrm{ber}_{2p}^{2p}(R_1, \cdots, R_n) \leqslant \frac{1}{2^p} \left\| \sum_{k=1}^{n} (R_k^*R_k + R_kR_k^*)^p \right\|_{\mathrm{Ber}} \tag{30}$$

for all $p \geqslant 1$.

Proof. Let $G_k + iH_k$ be the Cartesian decomposition of R_k for all $k = 1, \cdots, n$. As in the proof of (28) in [35], we have

$$\begin{aligned}
\left|\left\langle R_k\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|^{2p} &= \left(\left\langle G_k\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle^2 + \left\langle H_k\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle^2\right)^p \\
&\geqslant \frac{1}{2^p}\left(\left|\left\langle G_k\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right| + \left|\left\langle H_k\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|\right)^{2p} \\
&\geqslant \frac{1}{2^p}\left|\left\langle G_k\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle + \left\langle H_k\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|^{2p} \\
&= \frac{1}{2^p}\left|\left\langle G_k \pm H_k\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|^{2p}.
\end{aligned}$$

Summing over j and then taking the supremum over all unit vector $\widehat{k}_\lambda \in \mathscr{H}(\Omega)$, we get

$$\begin{aligned}
\mathrm{ber}_{2p}^{2p}(R_1,\cdots,R_n) &= \sup_{\lambda\in\Omega}\sum_{j=1}^n \left|\left\langle R_k\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|^{2p} \\
&\geqslant \frac{1}{2^p}\sup_{\lambda\in\Omega}\sum_{k=1}^n \left|\left\langle G_k \pm H_k\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|^{2p} \\
&\geqslant \frac{1}{2^p}\frac{1}{n^{p-1}}\sup_{\lambda\in\Omega}\left(\sum_{k=1}^n \left|\left\langle G_k \pm H_k\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|^2\right)^p \\
&= \frac{1}{2^p}\frac{1}{n^{p-1}}\left\|\sum_{k=1}^n (G_k \pm H_k)^2\right\|_{\mathrm{Ber}}^p,
\end{aligned}$$

where we have used Jensen's inequality in the last inequality. Thus,

$$\begin{aligned}
2\,\mathrm{ber}_{2p}^{2p}(R_1,\cdots,R_n) &\geqslant \frac{1}{2^p}\frac{1}{n^{p-1}}\left\|\sum_{k=1}^n (G_k + H_k)^2\right\|_{\mathrm{Ber}}^p + \frac{1}{2^p}\frac{1}{n^{p-1}}\left\|\sum_{k=1}^n (G_k - H_k)^2\right\|_{\mathrm{Ber}}^p \\
&\geqslant \frac{1}{2^p}\frac{1}{n^{p-1}}\left\|\sum_{k=1}^n (G_k + H_k)^2 + \sum_{k=1}^n (G_k - H_k)^2\right\|_{\mathrm{Ber}}^p \\
&= \frac{1}{2^p}\frac{1}{n^{p-1}}\left\|\sum_{k=1}^n \left\{(G_k + H_k)^2 + (G_k - H_k)^2\right\}\right\|_{\mathrm{Ber}}^p \\
&= \frac{1}{n^{p-1}}\left\|\sum_{k=1}^n G_k^2 + H_k^2\right\|_{\mathrm{Ber}}^p \\
&= \frac{1}{n^{p-1}}\left\|\sum_{k=1}^n \frac{R_k^*R_k + R_kR_k^*}{2}\right\|_{\mathrm{Ber}}^p \\
&= \frac{1}{2^p n^{p-1}}\left\|\sum_{k=1}^n R_k^*R_k + R_kR_k^*\right\|_{\mathrm{Ber}}^p,
\end{aligned}$$

and hence,

$$\operatorname{ber}_{2p}^{2p}(R_1,\cdots,R_n) \geqslant \frac{1}{2^{p+1}n^{p-1}}\left\|\sum_{k=1}^{n} R_k^*R_k + R_kR_k^*\right\|_{\mathrm{Ber}}^{p},$$

which proves the left hand side of the inequality in (30).

To prove the second inequality, for every unit vector $\widehat{k}_\lambda \in \mathscr{H}(\Omega)$ we have

$$\begin{aligned}\sum_{k=1}^{n}\left|\left\langle R_k\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|^{2p} &= \sum_{k=1}^{n}\left(\left\langle G_k\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle^2 + \left\langle H_k\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle^2\right)^p \\ &\leqslant \sum_{k=1}^{n}\left(\left\langle G_k^2\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle + \left\langle H_k^2\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right)^p \\ &= \sum_{k=1}^{n}\left\langle \left(G_k^2+H_k^2\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle^p,\end{aligned}$$

which implies that

$$\begin{aligned}\sup_{\lambda\in\Omega}\sum_{k=1}^{n}\left|\left\langle R_k\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|^{2p} &= \operatorname{ber}_{2p}^{2p}(R_1,\cdots,R_1) \\ &\leqslant \sup_{\lambda\in\Omega}\sum_{k=1}^{n}\left\langle \left(G_k^2+H_k^2\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle^p \\ &\leqslant \sup_{\lambda\in\Omega}\left\langle \sum_{k=1}^{n}\left(G_k^2+H_k^2\right)^p\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle \\ &= \left\|\sum_{k=1}^{n}\left(G_k^2+H_k^2\right)^p\right\|_{\mathrm{Ber}} = \frac{1}{2^p}\left\|\sum_{k=1}^{n}\left(R_k^*R_k+R_kR_k^*\right)^p\right\|_{\mathrm{Ber}},\end{aligned}$$

which proves the right hand side of (30). □

REMARK 4. In particular, setting $n=2$ and $p=1$ in (30) we get

$$\begin{aligned}\frac{1}{4}\left\|R_1^*R_1+R_1R_1^*+R_2^*R_2+R_2R_2^*\right\|_{\mathrm{Ber}} &\leqslant \operatorname{ber}_{\mathrm{e}}^2(R_1,R_2) \\ &\leqslant \frac{1}{2}\left\|R_1^*R_1+R_1R_1^*+R_2^*R_2+R_2R_2^*\right\|_{\mathrm{Ber}}.\end{aligned}$$

Moreover, if we choose $R_1=R_2=R$, then

$$\frac{1}{2}\left\|R^*R+RR^*\right\|_{\mathrm{Ber}} \leqslant \operatorname{ber}_{\mathrm{e}}^2(R,R) \leqslant \left\|R^*R+RR^*\right\|_{\mathrm{Ber}}.$$

But $\operatorname{ber}_{\mathrm{e}}(R,R)=\sqrt{2}\operatorname{ber}(R)$, which implies that

$$\frac{1}{4}\left\|R^*R+RR^*\right\|_{\mathrm{Ber}} \leqslant \operatorname{ber}^2(R) \leqslant \frac{1}{2}\left\|R^*R+RR^*\right\|_{\mathrm{Ber}}.$$

Now, based on Lemmas 5 and 6, we can introduce our first main result, as follows:

THEOREM 10. *Let* $R \in \mathscr{B}(\mathscr{H}(\Omega))$. *Then*

$$\frac{1}{4}\left\| |R|^2+|R^*|^2+2|R|^4 \right\|_{\mathrm{Ber}} \leqslant \mathrm{ber}_{dw}^2(R) \leqslant \frac{1}{2}\left\| |R|^2+|R^*|^2+2|R|^4 \right\|_{\mathrm{Ber}}. \tag{31}$$

Proof. Setting $n=2$, $p=1$, $R_1=X$ and $R_2=Y$ in (30), we get

$$\frac{1}{4}\|X^*X+XX^*+Y^*Y+YY^*\|_{\mathrm{Ber}} \leqslant \mathrm{ber}_{\mathrm{e}}^2(X,Y) \leqslant \frac{1}{2}\|X^*X+XX^*+Y^*Y+YY^*\|_{\mathrm{Ber}}.$$

Replacing X by R and Y by R^*R, we get

$$\frac{1}{4}\left\| R^*R+RR^*+2|R|^4 \right\|_{\mathrm{Ber}} \leqslant \mathrm{ber}_{\mathrm{e}}^2(R,R^*R) \leqslant \frac{1}{2}\left\| R^*R+RR^*+2|R|^4 \right\|_{\mathrm{Ber}}.$$

But as we have shown in Lemma 5 that , $\mathrm{ber}_{\mathrm{e}}(R,R^*R)=\mathrm{ber}_{dw}(R)$, hence we have

$$\frac{1}{4}\left\| |R|^2+|R^*|^2+2|R|^4 \right\|_{\mathrm{Ber}} \leqslant \mathrm{ber}_{dw}^2(R) \leqslant \frac{1}{2}\left\| |R|^2+|R^*|^2+2|R|^4 \right\|_{\mathrm{Ber}},$$

as desired. □

The following result refines sharply the upper bound in (2).

THEOREM 11. *If* $R \in \mathscr{B}(\mathscr{H}(\Omega))$, *then*

$$\frac{1}{\sqrt{2}}\|R+R^*R\|_{\mathrm{Ber}} \leqslant \mathrm{ber}_{dw}(R) \leqslant \sqrt{\left\| \frac{1}{4}(|R|+|R^*|)^2+|R|^4 \right\|_{\mathrm{Ber}}} \leqslant \sqrt{\frac{1}{4}\left(\|R\|_{\mathrm{Ber}}+\|R^2\|_{\mathrm{Ber}}^{1/2}\right)^2+\|R\|_{\mathrm{Ber}}^4}. \tag{32}$$

Proof. Since we have

$$\begin{aligned}
\mathrm{ber}_{dw}^2(R) &= \sup_{\lambda\in\Omega}\left\{ \left|\left\langle R\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|^2+\left|\left\langle R^*R\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|^2 \right\} \\
&\geqslant \frac{1}{2}\sup_{\lambda\in\Omega}\left\{ \left|\left\langle R\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|+\left|\left\langle R^*R\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right| \right\}^2 \\
&= \frac{1}{2}\sup_{\lambda\in\Omega}\left\{ \left|\left\langle R\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle+\left\langle R^*R\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right| \right\}^2 \\
&= \frac{1}{2}\sup_{\lambda\in\Omega}\left\{ \left|\left\langle (R+R^*R)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right| \right\}^2 \\
&= \frac{1}{2}\|R+R^*R\|_{\mathrm{Ber}}^2,
\end{aligned}$$

which proves the first inequality in (32). Also, since we have

$$\begin{aligned}
\mathrm{ber}_{dw}^2(R) &= \sup_{\lambda\in\Omega}\left\{\left|\left\langle R\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|^2+\left\|R\widehat{k}_\lambda\right\|^4\right\}\\
&= \sup_{\lambda\in\Omega}\left\{\left|\left\langle R\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|^2+\left|\left\langle R^*R\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|^2\right\}\\
&\leqslant \sup_{\lambda\in\Omega}\left\{\left(\left\langle |R|\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle^{\frac{1}{2}}\left\langle |R^*|\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle^{\frac{1}{2}}\right)^2+\left\langle |R^*R|^2\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right\}\\
&\qquad\qquad\qquad\text{(by Lemmas 3 and 4)}\\
&\leqslant \sup_{\lambda\in\Omega}\left[\left\langle \frac{|R|+|R^*|}{2}\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle^2+\left\langle |R^*R|^2\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right]\\
&\leqslant \sup_{\lambda\in\Omega}\left[\left\langle \left(\frac{|R|+|R^*|}{2}\right)^2\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle+\left\langle |R^*R|^2\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right] \qquad \text{(by Lemma 3)}\\
&= \sup_{\lambda\in\Omega}\left\langle \left(\left(\frac{|R|+|R^*|}{2}\right)^2+|R^*R|^2\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\\
&= \frac{1}{4}\left\|(|R|+|R^*|)^2+4\,|R^*R|^2\right\|_{\mathrm{Ber}},
\end{aligned}$$

and this proves the second inequality in (32). Applying the triangle inequality on the above inequality, we get

$$\begin{aligned}
\mathrm{ber}_{dw}^2(R) \leqslant \frac{1}{4}\left\|(|R|+|R^*|)^2+4\,|R^*R|^2\right\|_{\mathrm{Ber}} &\leqslant \frac{1}{4}\left\|(|R|+|R^*|)^2\right\|_{\mathrm{Ber}}+\left\||R^*R|^2\right\|_{\mathrm{Ber}}\\
&= \frac{1}{4}\,\||R|+|R^*|\|_{\mathrm{Ber}}^2+\left\||R|^4\right\|_{\mathrm{Ber}}.
\end{aligned}$$

Now, applying (13) to the first term in the above inequality, we get $\||R|+|R^*|\|_{\mathrm{Ber}} \leqslant \|R\|_{\mathrm{Ber}}+\left\|R^2\right\|_{\mathrm{Ber}}^{1/2}$. Now substituting this inequality in the last inequality above, we get the third inequality in (32), and this completes the proof.

To see that the second inequality in (31) is a refinement of the second inequality in (2), assume $RR^* \leqslant R^*R \leqslant \mathrm{ber}^2(R)\,1_{\mathscr{H}}$. Thus, from (31) we have

$$\begin{aligned}
\mathrm{ber}_{dw}^2(R) &\leqslant \frac{1}{2}\left\||R|^2+|R^*|^2+2\,|R|^4\right\|_{\mathrm{Ber}}\\
&\leqslant \frac{1}{2}\left\|\mathrm{ber}^2(R)\,1_{\mathscr{H}}+\mathrm{ber}^2(R)\,1_{\mathscr{H}}+2\,\mathrm{ber}^4(R)\,1_{\mathscr{H}}\right\|_{\mathrm{Ber}}\\
&\leqslant \mathrm{ber}^2(R)+\|R\|_{\mathrm{Ber}}^4.
\end{aligned}$$

Follows by the assumption, since $\mathrm{ber}(R) = \|R\|_{\mathrm{Ber}}$ (see Corollary 2), which implies that

$$\begin{aligned}
\mathrm{ber}_{dw}(R) \leqslant \sqrt{\frac{1}{2}\left\||R|^2+|R^*|^2+2\,|R|^4\right\|_{\mathrm{Ber}}} &\leqslant \sqrt{\mathrm{ber}^2(R)+\|R\|_{\mathrm{Ber}}^4}\\
&= \|R\|_{\mathrm{Ber}}\sqrt{1+\|R\|_{\mathrm{Ber}}^2},
\end{aligned}$$

which means that the right-hand side of (31) refines the right-hand side of (2). □

EXAMPLE 1. $\Omega = \left\{(x,y) : |x|^2 + |y|^2 \leqslant 6, x,y \in \mathbb{C}\right\}$. Therefore, Ω is closed subset of $\mathbb{C}$. Consider Let $R = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$. We have $W(R) \subseteq \Omega$ with $\|R\|_{\text{Ber}} = 2.28825$ and $\text{ber}(R) = 2.08114$. The upper bound of (2) gives $\text{ber}_{dw}(R) \leqslant 5.63449$. However, by applying (31), we have $\text{ber}_{dw}(R) \leqslant 5.61938$, which implies that, the upper bound in (31) is better than the upper bound in (2).

REMARK 5. We note that, a refinement of the inequality (6) could be stated as follows:

$$\frac{1}{\sqrt{2}}\|R + R^*R\| \leqslant \text{ber}_{dw}(R) \leqslant \sqrt{w\left(\frac{1}{4}(|R| + |R^*|)^2 + |R|^4\right)}.$$

Consider R as in Example 1. Applying the above inequality, we get $\text{ber}_{dw}(R) \leqslant 5.59709$, which is better than the result obtained by (5). Furthermore, (31) gives that

$$\text{ber}_{dw}(R) \leqslant \sqrt{w\left(\frac{1}{4}(|R| + |R^*|)^2 + |R|^4\right)} \leqslant \sqrt[4]{\frac{1}{2}\|T^*T + TT^*\|},$$

where $T = \frac{1}{4}(|R| + |R^*|)^2 + |R|^4$. Employing the previous second upper bound for R in Example 1, we get the same result as those obtained by (31) and (2), even we use (13); which indeed refines (32).

5. The Davis-Wielandt radius inequalities for $n \times n$ matrix operators

Several numerical radius type inequalities improving and refining the inequality

$$\frac{1}{2}\|S\| \leqslant w(S) \leqslant \|S\| \qquad (S \in \mathscr{B}(\mathscr{H}))$$

have been recently obtained by many other authors; see for example [1]–[10], and [21]. Recently, Bakherad [8] proved the following result concerning the Berezin number of $n \times n$ operator matrices.

Let $\mathbf{S} = [S_{ij}] \in \mathscr{B}\left(\bigoplus_{i=1}^n \mathscr{H}_i(\Omega_i)\right)$ such that $S_{ij} \in \mathscr{B}\left(\mathscr{H}_j(\Omega_j), \mathscr{H}_i(\Omega_i)\right)$. Then

$$w(\mathbf{S}) \leqslant \begin{cases} \text{ber}([S_{ij}]) & i = j, \\ \left\|[S_{ij}]\right\|_{\text{Ber}}, & i \neq j, \end{cases}.$$

In the next result, we present Davis-Wielandt radius inequality for $n \times n$ matrix Operators.

THEOREM 12. *Let* $\mathbf{T} = [T_{ij}] \in \mathscr{B}\left(\bigoplus_{i=1}^n \mathscr{H}_i(\Omega_i)\right)$. *Then*

$$\text{ber}_{dw}(\mathbf{T}) \leqslant w([t_{ij}]), \tag{33}$$

where

$$t_{ij} = \begin{cases} \operatorname{ber}(T_{ii}) + \|T_{ii}\|_{\mathrm{Ber}}^2, & j = i \\ \|T_{ij}\|_{\mathrm{Ber}} + \|T_{ij}\|_{\mathrm{Ber}}^2, & j \neq i \end{cases}.$$

Proof. Let $\mathscr{H}(\Omega) = \bigoplus_{i=1}^n \mathscr{H}_i(\Omega_i)$. For every $\lambda = (\lambda_1, \cdots, \lambda_n) \in \Omega_1 \times \cdots \times \Omega_n$, let a unit vector $\widehat{\mathbf{k}}_\lambda = \left[k_{\lambda_1} \cdots k_{\lambda_n}\right]^T \in \mathscr{H}(\Omega)$. Then we have

$$\begin{aligned} \operatorname{ber}_{dw}(\mathbf{T}) &= \sup_{(\lambda_1,\cdots,\lambda_n)\in\Omega_1\times\cdots\times\Omega_n} \sqrt{\left|\left\langle \mathbf{T}\widehat{\mathbf{k}}_\lambda, \widehat{\mathbf{k}}_\lambda \right\rangle\right|^2 + \left|\left\langle \mathbf{T}^*\mathbf{T}\widehat{\mathbf{k}}_\lambda, \widehat{\mathbf{k}}_\lambda \right\rangle\right|^2} \\ &\leqslant \sup_{(\lambda_1,\cdots,\lambda_n)\in\Omega_1\times\cdots\times\Omega_n} \left\{\left|\left\langle \mathbf{T}\widehat{\mathbf{k}}_\lambda, \widehat{\mathbf{k}}_\lambda \right\rangle\right| + \left|\left\langle \mathbf{T}^*\mathbf{T}\widehat{\mathbf{k}}_\lambda, \widehat{\mathbf{k}}_\lambda \right\rangle\right|\right\} \\ &\qquad (\text{since } \sqrt{a+b} \leqslant \sqrt{a} + \sqrt{b}). \end{aligned}$$

But since

$$\begin{aligned} \left|\left\langle \mathbf{T}\widehat{\mathbf{k}}_\lambda, \widehat{\mathbf{k}}_\lambda \right\rangle\right| &= \left|\sum_{i,j=1}^n \left\langle T_{ij}k_{\lambda_j}, k_{\lambda_i} \right\rangle\right| \\ &\leqslant \sum_{i,j=1}^n \left|\left\langle T_{ij}k_{\lambda_j}, k_{\lambda_i} \right\rangle\right| \\ &= \sum_{i=1}^n \left|\left\langle T_{ii}k_{\lambda_i}, k_{\lambda_i} \right\rangle\right| + \sum_{\substack{i,j=1 \\ j\neq i}}^n \left|\left\langle T_{ij}k_{\lambda_j}, k_{\lambda_i} \right\rangle\right| \\ &\leqslant \sum_{i=1}^n \mathbf{ber}(T_{ii}) \left\|k_{\lambda_i}\right\|^2 + \sum_{\substack{i,j=1 \\ j\neq i}}^n \left\|T_{ij}\right\|_{\mathrm{Ber}} \left\|k_{\lambda_j}\right\| \left\|k_{\lambda_i}\right\| \\ &= \sum_{i=1}^n t_{ij} \left\|k_{\lambda_j}\right\| \left\|k_{\lambda_i}\right\|. \end{aligned} \tag{34}$$

Similarly, we have

$$\begin{aligned} \left|\left\langle \mathbf{T}^*\mathbf{T}\mathbf{x}, \mathbf{x} \right\rangle\right| &= \left|\sum_{i,j=1}^n \left\langle T_{ij}^*T_{ij}k_{\lambda_j}, k_{\lambda_i} \right\rangle\right| \\ &\leqslant \sum_{i=1}^n \operatorname{ber}(T_{ii}^*T_{ii}) \left\|k_{\lambda_i}\right\|^2 + \sum_{j\neq i}^n \left\|T_{ij}^*T_{ij}\right\|_{\mathrm{Ber}} \left\|k_{\lambda_i}\right\| \left\|k_{\lambda_j}\right\|. \end{aligned} \tag{35}$$

Adding (34) and (35), we get

$$\begin{aligned} \operatorname{ber}_{dw}(\mathbf{T}) &\leqslant \sup_{(\lambda_1,\cdots,\lambda_n)\in\Omega_1\times\cdots\times\Omega_n} \left\{\left|\left\langle \mathbf{T}\widehat{\mathbf{k}}_\lambda, \widehat{\mathbf{k}}_\lambda \right\rangle\right| + \left|\left\langle \mathbf{T}^*\mathbf{T}\widehat{\mathbf{k}}_\lambda, \widehat{\mathbf{k}}_\lambda \right\rangle\right|\right\} \\ &\leqslant \sum_{i=1}^n \left(\operatorname{ber}(T_{ii}) + \operatorname{ber}(T_{ii}^*T_{ii})\right) \left\|k_{\lambda_i}\right\|^2 + \sum_{j\neq i}^n \left(\left\|T_{ij}\right\|_{\mathrm{Ber}} + \left\|T_{ij}^*T_{ij}\right\|_{\mathrm{Ber}}\right) \left\|k_{\lambda_i}\right\| \left\|k_{\lambda_j}\right\| \end{aligned}$$

$$= \sum_{i=1}^{n} \left(\mathrm{ber}\left(T_{ii}\right) + \|T_{ii}\|_{\mathrm{Ber}}^{2}\right) \|k_{\lambda_i}\|^2 + \sum_{j\neq i}^{n} \left(\left\|T_{ij}\right\|_{\mathrm{Ber}} + \left\|T_{ij}\right\|_{\mathrm{Ber}}^{2}\right) \|k_{\lambda_i}\| \left\|k_{\lambda_j}\right\|$$
$$\leqslant \sum_{i,j=1}^{n} t_{ij} \left\|k_{\lambda_i}\right\| \left\|k_{\lambda_j}\right\|$$
$$= \left\langle [t_{ij}]\,\mathbf{x}, \mathbf{x}\right\rangle,$$

where $\mathbf{x} = \left(\left\|k_{\lambda_1}\right\| \ \left\|k_{\lambda_2}\right\| \ldots \left\|k_{\lambda_1}\right\| \right)^T$ with $\|\mathbf{x}\| = 1$. Therefore

$$\mathrm{ber}_{dw}\left(\mathbf{T}\right) = \sup_{(\lambda_1,\cdots,\lambda_n)\in\Omega_1\times\cdots\times\Omega_n} \left\{\left|\left\langle \mathbf{T}\widehat{\mathbf{k}}_{\lambda}, \widehat{\mathbf{k}}_{\lambda}\right\rangle\right| + \left|\left\langle \mathbf{T}^{*}\mathbf{T}\widehat{\mathbf{k}}_{\lambda}, \widehat{\mathbf{k}}_{\lambda}\right\rangle\right|\right\} \leqslant \omega\left([t_{ij}]\right)$$

Thus, we obtain the right-hand side inequality in (33), and this completes the proof. □

COROLLARY 4. *Let* $\mathbf{T} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \in \mathscr{B}\left(\mathscr{H}_1\left(\Omega_1\right) \oplus \mathscr{H}_2\left(\Omega_2\right)\right)$. *Then*

$$\mathrm{ber}_{dw}\left(\mathbf{T}\right) \leqslant \frac{1}{2}\left(a + d + \sqrt{(a-d)^2 + (b+c)^2}\right),$$

where,

$$a = \mathrm{ber}\left(T_{11}\right) + \|T_{11}\|_{\mathrm{Ber}}^{2},\ b = \|T_{12}\|_{\mathrm{Ber}} + \|T_{12}\|_{\mathrm{Ber}}^{2},\ c = \|T_{21}\|_{\mathrm{Ber}} + \|T_{21}\|_{\mathrm{Ber}}^{2},$$

and $d = \mathrm{ber}\left(T_{22}\right) + \|T_{22}\|_{\mathrm{Ber}}^{2}$.

Proof. Take $n = 2$ in Theorem 12. Let a, b, c, d be as defined above. Then

$$\mathrm{ber}_{dw}\left(\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}\right) \leqslant w\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$
$$= r\left(\begin{bmatrix} a & \frac{b+c}{2} \\ \frac{b+c}{2} & d \end{bmatrix}\right)$$
$$= \frac{1}{2}\left(a + d + \sqrt{(a-d)^2 + (b+c)^2}\right)$$

as required. □

COROLLARY 5. *Let* $\begin{bmatrix} T_{11} & 0 \\ 0 & T_{22} \end{bmatrix} \in \mathscr{B}\left(\mathscr{H}_1\left(\Omega_1\right) \oplus \mathscr{H}_2\left(\Omega_2\right)\right)$, *then*

$$\mathrm{ber}_{dw}\left(\begin{bmatrix} T_{11} & 0 \\ 0 & T_{22} \end{bmatrix}\right) \leqslant \max\left\{\mathrm{ber}\left(T_{11}\right) + \|T_{11}\|_{\mathrm{Ber}}^{2}, \mathrm{ber}\left(T_{22}\right) + \|T_{22}\|_{\mathrm{Ber}}^{2}\right\}.$$

In special case, if $\mathscr{H}_1\left(\Omega_1\right) = \mathscr{H}_2\left(\Omega_2\right)$ *and* $T_{11} = T_{22} = T$, *then*

$$\mathrm{ber}_{dw}\left(\begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix}\right) \leqslant \mathrm{ber}_{dw}\left(T\right) + \|T\|_{\mathrm{Ber}}^{2}.$$

Proof. From Corollary 4, we have

$$\begin{aligned}\operatorname{ber}_{dw}\left(\begin{bmatrix} T_{11} & 0 \\ 0 & T_{22}\end{bmatrix}\right) &\leqslant \max\left\{\operatorname{ber}(T_{11}) + w(T_{11}^*T_{11}), \operatorname{ber}(T_{22}) + \operatorname{ber}(T_{22}^*T_{22})\right\} \\ &= \max\left\{\operatorname{ber}(T_{11}) + \operatorname{ber}\left(|T_{11}|^2\right), \operatorname{ber}(T_{22}) + \operatorname{ber}\left(|T_{22}|^2_{\mathrm{Ber}}\right)\right\} \\ &\leqslant \max\left\{\operatorname{ber}(T_{11}) + \|T_{11}\|^2_{\mathrm{Ber}}, \operatorname{ber}(T_{22}) + \|T_{22}\|^2_{\mathrm{Ber}}\right\},\end{aligned}$$

as required. □

COROLLARY 6. *Let* $\mathbf{T} = \begin{bmatrix} T & S \\ S & T\end{bmatrix} \in \mathscr{B}(\mathscr{H}(\Omega) \oplus \mathscr{H}(\Omega))$. *Then*

$$\operatorname{ber}_{dw}(\mathbf{T}) \leqslant \operatorname{ber}(T) + \|T\|^2_{\mathrm{Ber}} + \|S\|_{\mathrm{Ber}} + \|S\|^2_{\mathrm{Ber}}.$$

Proof. From Corollary 4, we have $T_{11} = T_{22} = T$ and $T_{12} = T_{21} = S$, therefore

$$a = \operatorname{ber}(T) + \|T\|^2_{\mathrm{Ber}} = d, \qquad b = \|S\|_{\mathrm{Ber}} + \|S\|^2_{\mathrm{Ber}} = c.$$

Thus,

$$\operatorname{ber}_{dw}\left(\begin{bmatrix} T & S \\ S & T\end{bmatrix}\right) \leqslant a + b = \operatorname{ber}(T) + \|T\|^2_{\mathrm{Ber}} + \|S\|_{\mathrm{Ber}} + \|S\|^2_{\mathrm{Ber}},$$

as required. □

A refinement of Theorem 12 is formulated as follows:

THEOREM 13. *Let* $\mathbf{T} = [T_{ij}] \in \mathscr{B}\left(\bigoplus_{i=1}^n \mathscr{H}_i(\Omega_i)\right)$ *such that* $T_{ij} \in \mathscr{B}(\mathscr{H}_j(\Omega_j), \mathscr{H}_i(\Omega_i))$. *Then*

$$\frac{1}{\sqrt{2}}\|\mathbf{T} + \mathbf{T}^*\mathbf{T}\| \leqslant \operatorname{ber}_{dw}(\mathbf{T}) \leqslant w^{1/2}([t_{ij}]), \tag{36}$$

where

$$t_{ij} = n \cdot \begin{cases} \operatorname{ber}^2(T_{ii}) + \|T_{ii}\|^4_{\mathrm{Ber}}, & j = i \\ \|T_{ij}\|^2_{\mathrm{Ber}} + \|T_{ij}\|^4_{\mathrm{Ber}}, & j \neq i \end{cases}.$$

Proof. Let $\widehat{\mathbf{k}}_\lambda = \left[k_{\lambda_1} \cdots k_{\lambda_n}\right]^T \in \bigoplus_{i=1}^n \mathscr{H}_i(\Omega_i)$ with $\|\widehat{\mathbf{k}}_\lambda\| = \sum_{i=1}^n \left\|k_{\lambda_i}\right\|^2 = 1$. Then we have

$$\begin{aligned}\operatorname{ber}_{dw}(\mathbf{T}) &= \sup_{(\lambda_1,\cdots,\lambda_n)\in\Omega_1\times\cdots\times\Omega_n}\left\{\sqrt{\left|\left\langle \mathbf{T}\widehat{\mathbf{k}}_\lambda, \widehat{\mathbf{k}}_\lambda\right\rangle\right|^2 + \left\|\mathbf{T}\widehat{\mathbf{k}}_\lambda\right\|^4}\right\} \\ &= \sup_{(\lambda_1,\cdots,\lambda_n)\in\Omega_1\times\cdots\times\Omega_n}\sqrt{\left|\left\langle \mathbf{T}\widehat{\mathbf{k}}_\lambda, \widehat{\mathbf{k}}_\lambda\right\rangle\right|^2 + \left|\left\langle \mathbf{T}^*\mathbf{T}\widehat{\mathbf{k}}_\lambda, \widehat{\mathbf{k}}_\lambda\right\rangle\right|^2}.\end{aligned}$$

But since

$$\begin{aligned}\left|\left\langle \mathbf{T}\widehat{\mathbf{k}}_\lambda,\widehat{\mathbf{k}}_\lambda\right\rangle\right|^2 &= \left|\sum_{i,j=1}^n \left\langle T_{ij}k_{\lambda_j},k_{\lambda_i}\right\rangle\right|^2 \\ &\leqslant n\cdot\sum_{i,j=1}^n \left|\left\langle T_{ij}k_{\lambda_j},k_{\lambda_i}\right\rangle\right|^2 \qquad \text{(by Jensen's inequality)} \\ &\leqslant n\cdot\sum_{i=1}^n |\langle T_{ii}k_i,k_i\rangle|^2 + n\cdot\sum_{j\neq i}^n \left|\left\langle T_{ij}k_{\lambda_j},k_{\lambda_i}\right\rangle\right|^2 \\ &\leqslant n\cdot\sum_{i=1}^n \mathrm{ber}^2(T_{ii})\left\|k_{\lambda_i}\right\|^4 + n\cdot\sum_{j\neq i}^n \left\|T_{ij}\right\|_{\mathrm{Ber}}^2\left\|k_{\lambda_i}\right\|^2\left\|k_{\lambda_j}\right\|^2 \\ &\leqslant n\cdot\sum_{i=1}^n \mathrm{ber}^2(T_{ii})\left\|k_{\lambda_i}\right\|^2 + n\cdot\sum_{j\neq i}^n \left\|T_{ij}\right\|_{\mathrm{Ber}}^2\left\|k_{\lambda_i}\right\|\left\|k_{\lambda_j}\right\|, \end{aligned} \tag{37}$$

the last inequality holds, since $\|k_{\lambda_i}\|^4 \leqslant \|k_{\lambda_i}\|^2 \leqslant 1$ and $\|k_{\lambda_i}\|^2 \leqslant \|k_{\lambda_i}\| \leqslant 1$ for all $\lambda_i \in \Omega_i$, $i=1,\cdots,n$. Similarly, we have

$$\begin{aligned}\left|\left\langle \mathbf{T}^*\mathbf{T}\widehat{\mathbf{k}}_\lambda,\widehat{\mathbf{k}}_\lambda\right\rangle\right|^2 &= \left|\sum_{i,j=1}^n \left\langle T_{ij}^*T_{ij}k_{\lambda_j},k_{\lambda_i}\right\rangle\right|^2 \\ &\leqslant n\cdot\sum_{i=1}^n \mathrm{ber}^2(T_{ii}^*T_{ii})\left\|k_{\lambda_i}\right\|^2 + n\cdot\sum_{j\neq i}^n \left\|T_{ij}^*T_{ij}\right\|_{\mathrm{Ber}}^2\left\|k_{\lambda_i}\right\|\left\|k_{\lambda_j}\right\|. \end{aligned} \tag{38}$$

Now adding (37) and (38), we get

$$\begin{aligned}&\left|\left\langle \mathbf{T}\widehat{\mathbf{k}}_\lambda,\widehat{\mathbf{k}}_\lambda\right\rangle\right|^2 + \left|\left\langle \mathbf{T}^*\mathbf{T}\widehat{\mathbf{k}}_\lambda,\widehat{\mathbf{k}}_\lambda\right\rangle\right|^2 \\ &= n\cdot\sum_{i=1}^n \left(\mathrm{ber}^2(T_{ii}) + \mathrm{ber}^2(T_{ii}^*T_{ii})\right)\left\|k_{\lambda_i}\right\|^2 + n\cdot\sum_{j\neq i}^n \left(\left\|T_{ij}\right\|_{\mathrm{Ber}}^2 + \left\|T_{ij}^*T_{ij}\right\|_{\mathrm{Ber}}^2\right)\left\|k_{\lambda_i}\right\|\left\|k_{\lambda_j}\right\| \\ &= n\cdot\sum_{i=1}^n \left(\mathrm{ber}^2(T_{ii}) + \|T_{ii}\|_{\mathrm{Ber}}^4\right)\left\|k_{\lambda_i}\right\|^2 + \sum_{j\neq i}^n \left(\left\|T_{ij}\right\|_{\mathrm{Ber}}^2 + \left\|T_{ij}\right\|_{\mathrm{Ber}}^4\right)\left\|k_{\lambda_i}\right\|\left\|k_{\lambda_j}\right\| \\ &\leqslant n\cdot\sum_{i,j=1}^n t_{ij}\left\|k_{\lambda_i}\right\|\left\|k_{\lambda_j}\right\| \\ &= n\cdot\left\langle [t_{ij}]\,y,y\right\rangle, \end{aligned}$$

where $y = \left(\|k_{\lambda_1}\|\ \|k_{\lambda_2}\|\ \cdots\ \|k_{\lambda_n}\|\right)^T$. Taking the supremum over unit vectors $\widehat{\mathbf{k}}_\lambda \in \bigoplus_{i=1}^n \mathscr{H}_i(\Omega_i)$, we obtain the right-hand side inequality. To prove the left hand side inequality we note that

$$\begin{aligned}\mathrm{ber}_{dw}^2(\mathbf{T}) &= \sup_{(\lambda_1,\cdots,\lambda_n)\in\Omega_1\times\cdots\times\Omega_n}\left\{\left|\left\langle \mathbf{T}\widehat{\mathbf{k}}_\lambda,\widehat{\mathbf{k}}_\lambda\right\rangle\right|^2 + \left|\left\langle \mathbf{T}^*\mathbf{T}\widehat{\mathbf{k}}_\lambda,\widehat{\mathbf{k}}_\lambda\right\rangle\right|^2\right\} \\ &\geqslant \frac{1}{2}\sup_{(\lambda_1,\cdots,\lambda_n)\in\Omega_1\times\cdots\times\Omega_n}\left\{\left|\left\langle \mathbf{T}\widehat{\mathbf{k}}_\lambda,\widehat{\mathbf{k}}_\lambda\right\rangle\right| + \left|\left\langle \mathbf{T}^*\mathbf{T}\widehat{\mathbf{k}}_\lambda,\widehat{\mathbf{k}}_\lambda\right\rangle\right|\right\}^2\end{aligned}$$

$$\geqslant \frac{1}{2} \sup_{(\lambda_1,\cdots,\lambda_n)\in\Omega_1\times\cdots\times\Omega_n} \left\{ \left| \left\langle (\mathbf{T}+\mathbf{T}^*\mathbf{T})\widehat{\mathbf{k}}_\lambda, \widehat{\mathbf{k}}_\lambda \right\rangle \right|^2 \right\}$$
$$= \frac{1}{2} \|\mathbf{T}+\mathbf{T}^*\mathbf{T}\|_{\mathrm{Ber}}^2,$$

as required. □

COROLLARY 7. *Let* $\mathbf{T} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \in \mathscr{B}(\mathscr{H}_1(\Omega_1) \oplus \mathscr{H}_2(\Omega_2))$. *Then*

$$\mathrm{ber}_{dw}(\mathbf{T}) \leqslant \sqrt{a+d+\sqrt{(a-d)^2+(b+c)^2}}, \tag{39}$$

where,

$$a = \mathrm{ber}^2(T_{11}) + \|T_{11}\|_{\mathrm{Ber}}^4,\ b = \|T_{12}\|_{\mathrm{Ber}}^2 + \|T_{12}\|_{\mathrm{Ber}}^4,\ c = \|T_{21}\|_{\mathrm{Ber}}^2 + \|T_{21}\|_{\mathrm{Ber}}^4,$$

and $d = \mathrm{ber}^2(T_{22}) + \|T_{22}\|_{\mathrm{Ber}}^4$.

Proof. Take $n = 2$ in Theorem 13. Let a,b,c,d be as defined above. Then

$$\begin{aligned}
\mathrm{ber}_{dw}^2\left(\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}\right) &\leqslant 2w\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \\
&= 2r\left(\begin{bmatrix} a & \frac{b+c}{2} \\ \frac{b+c}{2} & d \end{bmatrix}\right) \\
&= a+d+\sqrt{(a-d)^2+(b+c)^2},
\end{aligned}$$

which proves the required inequality. □

COROLLARY 8. *Let* $\begin{bmatrix} T_{11} & 0 \\ 0 & T_{22} \end{bmatrix} \in \mathscr{B}(\mathscr{H}_1(\Omega) \oplus \mathscr{H}_2(\Omega))$. *Then*

$$\mathrm{ber}_{dw}\left(\begin{bmatrix} T_{11} & 0 \\ 0 & T_{22} \end{bmatrix}\right) \leqslant \sqrt{2}\max\left\{\sqrt{\mathrm{ber}^2(T_{11}) + \|T_{11}\|_{\mathrm{Ber}}^4}, \sqrt{\mathrm{ber}^2(T_{22}) + \|T_{22}\|_{\mathrm{Ber}}^4}\right\}.$$

In special case, if $\mathscr{H}_1(\Omega) = \mathscr{H}_2(\Omega)$ *and* $T_{11} = T_{22} = T$, *then*

$$\mathrm{ber}_{dw}\left(\begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix}\right) \leqslant \sqrt{2}\left(\mathrm{ber}^2(T) + \|T\|_{\mathrm{Ber}}^4\right)^{1/2}.$$

Proof. Form Corollary 7, we have

$$\begin{aligned}
\mathrm{ber}_{dw}^2\left(\begin{bmatrix} T_{11} & 0 \\ 0 & T_{22} \end{bmatrix}\right) &\leqslant 2\max\left\{\mathrm{ber}^2(T_{11}) + \mathrm{ber}^2(T_{11}^*T_{11}), \mathrm{ber}^2(T_{22}) + \mathrm{ber}^2(T_{22}^*T_{22})\right\} \\
&= 2\max\left\{\mathrm{ber}^2(T_{11}) + \mathrm{ber}^2\left(|T_{11}|^2\right), \mathrm{ber}^2(T_{22}) + \mathrm{ber}^2\left(|T_{22}|^2\right)\right\} \\
&\leqslant 2\max\left\{\mathrm{ber}^2(T_{11}) + \|T_{11}\|_{\mathrm{Ber}}^4, \mathrm{ber}^2(T_{22}) + \|T_{22}\|_{\mathrm{Ber}}^4\right\},
\end{aligned}$$

which gives the desired result. □

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