Journal of
Mathematical
Inequalities
Volume 17, Number 1 (2023), 259–267
doi:10.7153/jmi-2023-17-18

FUNCTIONAL BOUNDS FOR EXTON'S DOUBLE HYPERGEOMETRIC X FUNCTION

DRAGANA JANKOV MAŠIREVIĆ AND TIBOR K. POGÁNY*

Dedicated to the memory of our great friend, teacher and colleague Dragan Jukić

(*Communicated by T. Burić*)

Abstract. Functional and uniform bounds for Exton's generalized hypergeometric X function of two variables and an associated incomplete Lipschitz–Hankel integral, as an auxiliary result, are obtained. A by-product for the Srivastava-Daoust generalized hypergeometric function of three variables is given by another derivation method. The main tools are certain representation formulae for the McKay I_ν Bessel probability distribution's cumulative distribution function established recently in [3, 5].

1. Introduction and motivation

The first results about probability distributions involving Bessel functions can be traced back to the early work of McKay [6] in 1932 who considered two classes of continuous distributions called *Bessel function distributions*.

Precisely, the random variable (rv) ξ defined on a standard probability space $(\Omega, \mathscr{F}, \mathsf{P})$, distributed according to McNolty's variant of the McKay law which is characterized by the probability distribution function (PDF) [7, p. 496, Eq. (13)]

$$f_I(x;a,b;\nu) = \frac{\sqrt{\pi}(b^2-a^2)^{\nu+1/2}}{(2a)^\nu \Gamma\left(\nu+\frac{1}{2}\right)} \mathrm{e}^{-bx} x^\nu I_\nu(ax), \qquad x \geqslant 0,$$

defined for all $\nu > -1/2$ and $b > a > 0$; McNolty reported, without derivation, several generalized PDFs of above type, among others the above listed PDF. The related cumulative distribution function (CDF) is

$$F_I(x;a,b;\nu) = \frac{\sqrt{\pi}(b^2-a^2)^{\nu+1/2}}{(2a)^\nu \Gamma\left(\nu+\frac{1}{2}\right)} \int_0^x \mathrm{e}^{-bt} t^\nu I_\nu(at)\,\mathrm{d}t, \qquad x \geqslant 0. \tag{1.1}$$

Mathematics subject classification (2020): 26D15, 33C10, 33C70, 33E20, 60E05.

Keywords and phrases: Modified Bessel functions of the first kind, McKay I_ν Bessel distribution, Grünwald–Letnikov fractional derivative, incomplete Lipschitz-Hankel integral, Exton X function, Srivastava–Daoust $\mathscr{S}$ function, functional bounding inequality.

The research of T. K. Pogány has been supported in part by the University of Rijeka, Croatia under the project number `uniri-pr-prirod-19-16`.

* Corresponding author.

We will use in our considerations these McNolty's formulae.

Derivation of new representation formulae for CDF of rv distributed according to the McKay I_ν law was one of the main goals of the mutually constructed and written article [3]. These results imply several by–products, among others. Namely, we deduce several strong functional and uniform bounds upon Exton's generalized hypergeometric X function of two variables which are the building blocks of the established CDF's. The Appendix we devote to the absolute convergence constraints upon the Exton's X functions.

2. Main results

Recently, Jankov Maširević and Pogány presented in [5] formulae for the CDF of McKay I_ν Bessel distribution; one of them is formulated in terms of the lower incomplete gamma function

$$\gamma(a,x) = \int_0^x \mathrm{e}^{-t} t^{a-1}\,\mathrm{d}t, \qquad \Re(a) > 0,$$

and Exton's double hypergeometric X function [2]

$$X^{A:B;B'}_{C:D;D'}\left[\begin{matrix}(a):(b);(b')\\(c):(d);(d')\end{matrix}\,\middle|\, x,\, y\right] = \sum_{k,n\geqslant 0} \frac{((a))_{2k+n}((b))_k((b'))_n}{((c))_{2k+n}((d))_k((d'))_n}\frac{x^k}{k!}\frac{y^n}{n!}. \tag{2.1}$$

Here (a) denotes the sequence $a_1,\cdots,a_A$, whilst $((a))_m := (a_1)_m\cdots(a_A)_m$ and the empty product equals *per definitionem* 1.

Exton's X function (2.1) is a special case of the Srivastava–Daoust generalization of the Lauricella and/or the Kampé de Fériet hypergeometric functions in two variables. Accordingly, the convergence conditions [10, pp. 157–158] (compare the Appendix B) one reduces to [3]

$$\begin{array}{l}\Delta_1 = 1+2C+D-2A-B>0\\ \Delta_2 = 1+C+D'-A-B'>0\end{array},$$

under which (2.1) converges absolutely for all $x,y\in\mathbb{C}$. Another cases of convergence inside centered discs in $\mathbb{C}$ can be deduced from [10, pp. 153–157] in a straightforward way. The Exton's generalized hypergeometric function of two variables is not widely known, so functional bounds, even for its special cases $X^{1:0;0}_{1:1;0}$, $X^{1:0;0}_{0:1;1}$, $X^{1:0;1}_{1:1;1}$ which occur in this note, are of considerable interest.

REMARK 1. We point out that according to Appendix B all considered Exton X functions converge for $x\geqslant 0$.

Next, we recall the definition of the Pochhammer symbol (or rising factorial)

$$(\nu)_\lambda := \frac{\Gamma(\lambda+\nu)}{\Gamma(\nu)} = \begin{cases}1 & (\lambda=0;\nu\in\mathbb{C}\setminus\{0\})\\ \nu(\nu+1)\cdots(\nu+n-1) & (\lambda=n\in\mathbb{N};\ \nu\in\mathbb{C})\end{cases};$$

where it is understood conventionally that $(0)_0 := 1$. According to the Pochhammer symbol's use, the confluent hypergeometric function one defines as the power series

$$ {}_0F_1\left[-;b;z\right] = \sum_{n\geqslant 0} \frac{1}{(b)_n}\frac{z^n}{n!}, \qquad z \in \mathbb{C}. $$

THEOREM 1. *For all* $b > a > 0$, $\nu > -1/2$ *and all* $x \geqslant 0$ *it is*

$$ \begin{aligned} &\frac{(2\nu+1)_3}{a^2 b^{2\nu+1}}\,\gamma(2\nu+1,bx)\,{}_0F_1\left[-;\nu+1;\frac{(ax)^2}{4}\right] - \frac{\Gamma(2\nu+4)}{a^2(b^2-a^2)^{\nu+\frac12}} \\ &\qquad \leqslant x^{2\nu+3}\,X^{1:0;1}_{1:1;1}\left[\begin{matrix} 2\nu+3: & - & ;2\nu+1 \\ 2\nu+4: & \nu+2 & ;2\nu+2\end{matrix}\,\middle|\,\frac{(ax)^2}{4},-bx\right] \\ &\qquad \leqslant \frac{(2\nu+1)_3}{a^2b^{2\nu+1}}\,\gamma(2\nu+1,bx)\,{}_0F_1\left[-;\nu+1;\frac{(ax)^2}{4}\right]. \end{aligned} $$

Moreover, for $\nu \in \mathbb{N}_0$ *and* $b > a > 0$ *there holds*

$$ x^{2\nu+1}\,X^{0:1;1}_{1:0;0}\left[\begin{matrix} -:\nu+\frac12;1 \\ 2\nu+2:-;-\end{matrix}\,\middle|\,(ax)^2,bx\right] \leqslant \frac{\Gamma(2\nu+1)\,\mathrm{e}^{bx}}{(b^2-a^2)^{\nu+1/2}}, \qquad x \geqslant 0. $$

Proof. Starting with the representation formula [5, p. 156, Corollary 6]

$$ \begin{aligned} F_I(x;a,b;\nu) = \frac{(b^2-a^2)^{\nu+1/2}}{b^{2\nu+1}\Gamma(2\nu+2)}\Bigg\{&(2\nu+1)\,\gamma(2\nu+1,bx)\,{}_0F_1\left[-;\nu+1;\frac{(ax)^2}{4}\right] \\ &- \frac{a^2b^{2\nu+1}x^{2\nu+3}}{(2\nu+2)(2\nu+3)}\,X^{1:0;1}_{1:1;1}\left[\begin{matrix} 2\nu+3: & - & ;2\nu+1 \\ 2\nu+4: & \nu+2 & ;2\nu+2\end{matrix}\,\middle|\,\frac{(ax)^2}{4},-bx\right]\Bigg\}, \end{aligned} \tag{2.2} $$

we apply the bilateral inequality $0 \leqslant F_I(x;a,b;\nu) \leqslant 1$ valid for general CDFs in the range of their support set. Now, obvious steps lead to the asserted bounds.

Next, consider the expression for the CDF F_I reported for the parameters $\nu \in \mathbb{N}_0$, $b > a > 0$. Precisely, for all $x \geqslant 0$ there holds [3, Theorem 4.]

$$ F_I(x;a,b;\nu) = \frac{(b^2-a^2)^{\nu+1/2}}{\Gamma(2\nu+1)}\,x^{2\nu+1}\,\mathrm{e}^{-bx}\,X^{0:1;1}_{1:0;0}\left[\begin{matrix} -:\nu+\frac12;1 \\ 2\nu+2:-;-\end{matrix}\,\middle|\,a^2x^2,bx\right]. \tag{2.3} $$

Since $F_I(x;a,b;\nu) \leqslant 1$ we conclude the second upper bound. $\square$

A more sophisticated bound can be derived by the following monotonicity result for the CDF of the McKay I_ν Bessel law with respect to the parameter ν, which is covered by [4, Theorem 2.1.]: *for all* $\min(\mu,\nu) > -1/2$ *and* $b > a > 0$ *there holds*

$$ F_I(x;a,b;\mu+\nu+1/2) < F_I(x;a,b;\nu), \qquad x \geqslant 0. \tag{2.4} $$

THEOREM 2. *For all* $\min\big(\nu,\,\mu\big) > -1/2$, $b > a > 0$ *and* $x \geqslant 0$ *we have*

$$\begin{aligned}
&\frac{\big(1-(a/b)^2\big)^{\mu+\frac12}}{\Gamma(2\mu+2\nu+2)}\,\gamma\big(2\mu+2\nu+2,bx\big)\,{}_0F_1\Big[-;\mu+\nu+\tfrac32;\frac{(ax)^2}{4}\Big]\\
&\qquad-\frac{\gamma\big(2\nu+1,bx\big)}{\Gamma(2\nu+1)}\,{}_0F_1\Big[-;\nu+1;\frac{(ax)^2}{4}\Big]\\
&\qquad\qquad\leqslant a^2b^{2\nu+1}x^{2\nu+3}\left\{\frac{[(b^2-a^2)^{\mu+\frac12}x^{2\mu+1}}{\Gamma(2\mu+2\nu+5)}X^{1:0;1}_{1:1;1}\big[\mu+\nu+\tfrac12\big]-\frac{1}{\Gamma(2\nu+4)}X^{1:0;1}_{1:1;1}\big[\nu\big]\right\},
\end{aligned}$$

where the shorthand

$$X^{1:0;1}_{1:1;1}[\nu] = X^{1:0;1}_{1:1;1}\left[\begin{matrix}2\nu+3:\,-\,;2\nu+1\\ 2\nu+4:\nu+2\,;2\nu+2\end{matrix}\,\middle|\,\frac{(ax)^2}{4},\,-bx\right].$$

Proof. Let x be positive. Inserting (2.2) into the monotonicity relation (2.4), we get by routine transformations that

$$\begin{aligned}
&\frac{\left(1-\dfrac{a^2}{b^2}\right)^{\mu+1/2}\gamma\big(2(\mu+\nu+1),bx\big)}{\Gamma(2\mu+2\nu+2)}\,{}_0F_1\Big[-;\mu+\nu+\tfrac32;\frac{(ax)^2}{4}\Big]\\
&\qquad-\frac{a^2(b^2-a^2)^{\mu+1/2}b^{2\nu+1}}{\Gamma(2\mu+2\nu+5)}\cdot x^{2\mu+2\nu+4}X^{1:0;1}_{1:1;1}\big[\mu+\nu+\tfrac12\big]\\
&\qquad<\frac{\gamma\big(2\nu+1,bx\big)}{\Gamma(2\nu+1)}\,{}_0F_1\Big[-;\nu+1;\frac{(ax)^2}{4}\Big]-\frac{a^2b^{2\nu+1}x^{2\nu+3}}{\Gamma(2\nu+4)}X^{1:0;1}_{1:1;1}\big[\nu\big],
\end{aligned}$$

which leads to

$$\begin{aligned}
&\frac{(b^2-a^2)^{\mu+\frac12}}{b^{2\mu+1}\,\Gamma(2\mu+2\nu+2)}\,\gamma\big(2\mu+2\nu+2,bx\big)\,{}_0F_1\Big[-;\mu+\nu+\tfrac32;\frac{(ax)^2}{4}\Big]\\
&\qquad-\frac{\gamma\big(2\nu+1,bx\big)}{\Gamma(2\nu+1)}\,{}_0F_1\Big[-;\nu+1;\frac{(ax)^2}{4}\Big]\\
&\qquad<\frac{a^2b^{2\nu+1}x^{2\mu+2\nu+4}X^{1:0;1}_{1:1;1}\big[\mu+\nu+\tfrac12\big]}{(b^2-a^2)^{-\mu-\frac12}\Gamma(2\mu+2\nu+5)}-\frac{a^2b^{2\nu+1}x^{2\nu+3}}{\Gamma(2\nu+4)}X^{1:0;1}_{1:1;1}\big[\nu\big].
\end{aligned}$$

However, this is equivalent to the asserted inequality. $\square$

The next derivation method depends on the incomplete Lipschitz-Hankel integral (ILHI) built by the modified Bessel functions of the first kind:

$$I_{e_{\mu,\nu}}(z;a,b) = \int_0^z \mathrm{e}^{-bt}t^{\mu}I_\nu(at)\,\mathrm{d}t,$$

where $a,b>0$, the argument and another two parameters $z,\nu,\mu\in\mathbb{C}$ and it should hold $\Re(\mu+\nu)>-1$ (for more details on ILHI consult for instance [3] and the there listed references). Obviously, the special case $I_{e_{\nu,\nu}}(x;a,b)$ occurs in (1.1) and builds the desired CDF integral. Therefore, we may deduce about the formula [3, Corollary 1]

$$F_I(x;a,b;\nu) = \frac{\sqrt{\pi}(b^2-a^2)^{\nu+1/2}}{(2a)^\nu\Gamma\left(\nu+\frac{1}{2}\right)} I_{e_{\nu,\nu}}(x;a,b)\,, \qquad x \geqslant 0,$$

which holds for all $b > a > 0, \nu > -1/2$, that is

$$F_I(x;a,b;\nu) = \frac{[(b/a)^2-1]^{\nu+\frac{1}{2}}}{2^\nu(\frac{1}{2})_\nu} I_{e_{\nu,\nu}}\left(ax;1,\frac{b}{a}\right),$$

as $\sqrt{\pi}\,(\frac{1}{2})_\nu = \Gamma(\nu+\frac{1}{2})$. By virtue of this expression we readily arrive at the uniform upper bound

$$I_{e_{\nu,\nu}}\left(ax;1,\frac{b}{a}\right) \leqslant \frac{2^\nu(\frac{1}{2})_\nu}{[(b/a)^2-1]^{\nu+\frac{1}{2}}}.$$

The Grünwald–Letnikov fractional derivative of order η with respect to the argument x of a suitable function f is defined by [9]

$$\mathbb{D}_x^\eta(f) = \lim_{h\to 0^+}\frac{1}{h^\eta}\sum_{m=0}^\infty(-1)^m\binom{\eta}{m} f\left(x+(\eta-m)h\right).$$

We recall the widely known and used formula [8]

$$\mathbb{D}_x^\eta\left(\mathrm{e}^{\alpha x}\right) = \alpha^\eta\mathrm{e}^{\alpha x}, \qquad \alpha\in\mathbb{C}.$$

Expressing $I_{e_{\mu,\nu}}(x;b)$ and the CDF F_I by the Grünwald–Letnikov derivative in terms of the Exton X function, we have obtained the next representation results valid for the general μ,ν order incomplete Lipschitz–Hankel integral.

PROPOSITION 1. [3, Theorem 3] *For all $a,b>0$, $z,\nu,\mu\in\mathbb{C}$, such that $\Re(\nu) > -1$, we have*

$$I_{e_{\mu,\nu}}(z;a,b) = (-1)^\mu\,\mathbb{D}_b^\mu\left(X^{1:0;0}_{1:1;0}\left[\begin{matrix}\nu+1:-;-\\ \nu+2:\nu+1;-\end{matrix}\,\middle|\,\frac{a^2z^2}{4},-bz\right]\right).$$

Moreover, when additionally $\Re(\mu+\nu) > -1$, there holds

$$I_{e_{\mu,\nu}}(z;a,b) = \frac{\left(\frac{a}{2}\right)^\nu z^{\mu+\nu+1}}{(\mu+\nu+1)\Gamma(\nu+1)} X^{1:0;0}_{1:1;0}\left[\begin{matrix}\mu+\nu+1:-;-\\ \mu+\nu+2:\nu+1;-\end{matrix}\,\middle|\,\frac{a^2z^2}{4},-bz\right].$$

At this point we turn back to bounding Exton's X function by virtue of the Proposition 1. The following uniform bound results close this section.

COROLLARY 1. *Let $b>a>0$, $\nu>-1/2$. Then for all $x\geqslant 0$ hold the upper bounds for the Exton X function, read as follows:*

$$\mathbb{D}_b^\nu\left(X^{1:0;0}_{1:1;0}\left[\begin{matrix}\nu+1:-;-\\ \nu+2:\nu+1;-\end{matrix}\,\middle|\,\frac{a^2z^2}{4},-bz\right]\right) \leqslant \frac{(-2a)^\nu\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}(b^2-a^2)^{\nu+1/2}},$$

$$x^{2\nu+1}X^{1:0;0}_{1:1;0}\left[\begin{matrix}2\nu+1:-;-\\ 2\nu+2:\nu+1;-\end{matrix}\,\middle|\,\frac{a^2x^2}{4},-bx\right] \leqslant \frac{\Gamma(2\nu+2)}{(b^2-a^2)^{\nu+1/2}}.$$

3. Another probabilistic approach to CDF

Firstly, we draw the reader's attention to the precise definition of Srivastava–Daoust generalized hypergeometric function $\mathscr{S}$ of three variables, which can be reached by consulting Appendix A for $n = 3$.

LEMMA 1. [1, p. 45, 2.1.7] *Let $F(x)$ be a* CDF *and $h > 0$. Then*

$$H_1(x) = \frac{1}{h}\int_x^{x+h} F(t)\,\mathrm{d}t\,; \qquad H_2(x) = \frac{1}{2h}\int_{x-h}^{x+h} F(t)\,\mathrm{d}t\,,$$

are also CDF*s.*

THEOREM 3. *For all $b > a > 0$, $\nu > -\frac{1}{2}$, we have the monotonicity property of the three variables Srivastava–Daoust function*

$$\mathscr{S}_\nu(x) = x^{2\nu+2}\,\mathscr{S}^{1:0;1;1}_{2:0;0;0}\left(\begin{matrix} [2\nu+2:1,2,1]:-;[\nu+\frac{1}{2}:1];[1:1] \\ [2\nu+3:1,2,1],[2\nu+2:0,2,1]:-;-;- \end{matrix}\,\middle|\,\begin{matrix} -bx \\ (ax)^2 \\ bx \end{matrix}\right),$$

in the following manner:

$$\mathscr{S}_\nu(x) \leqslant \mathscr{S}_\nu(x+h) \leqslant \mathscr{S}_\nu(x) + \frac{2h(\nu+1)\Gamma(2\nu+1)}{(b^2-a^2)^{\nu+\frac{1}{2}}}, \qquad h > 0;\, x \geqslant 0.$$

Proof. Consider the CDF $H_1(x)$ generated by the baseline CDF F_I when it is expressed *via* Exton X in (2.3) which is tracing back to [3, Theorem 4]. Direct calculation, using the power series description (2.3) and the Maclaurin series expansion of the exponential term give the following equality chain:

$$\begin{aligned} H_1(x) &= \frac{(b^2-a^2)^{\nu+\frac{1}{2}}}{h\Gamma(2\nu+1)}\int_x^{x+h} \mathrm{e}^{-bt}\,t^{2\nu+1}\,X^{0:1;1}_{1:0;0}\left[\begin{matrix} -:\nu+\frac{1}{2};1 \\ 2\nu+2:-;- \end{matrix}\,\middle|\, a^2t^2, bt\right]\mathrm{d}t \\ &= \frac{(b^2-a^2)^{\nu+\frac{1}{2}}}{h\Gamma(2\nu+1)}\sum_{j,m,n\geqslant 0}\frac{(\nu+\frac{1}{2})_m(1)_n(-b)^j a^{2m}b^n}{(2\nu+2)_{2m+n}\,j!\,m!\,n!}\int_x^{x+h} t^{2\nu+1+j+2m+n}\mathrm{d}t \\ &= \frac{(b^2-a^2)^{\nu+\frac{1}{2}}(x+h)^{2\nu+2}}{2h(\nu+1)\Gamma(2\nu+1)}\sum_{j,m,n\geqslant 0}\frac{(2\nu+2)_{j+2m+n}(\nu+\frac{1}{2})_m(1)_n}{(2\nu+3)_{j+2m+n}(2\nu+2)_{2m+n}} \\ &\qquad \cdot\frac{[-b(x+h)]^j}{j!}\,\frac{[a(x+h)]^{2m}}{m!}\,\frac{[b(x+h)]^n}{n!} \\ &\qquad - \frac{(b^2-a^2)^{\nu+\frac{1}{2}}x^{2\nu+2}}{2h(\nu+1)\Gamma(2\nu+1)}\sum_{j,m,n\geqslant 0}\frac{(2\nu+2)_{j+2m+n}(\nu+\frac{1}{2})_m(1)_n}{(2\nu+3)_{j+2m+n}(2\nu+2)_{2m+n}} \\ &\qquad \cdot\frac{(-bx)^j}{j!}\,\frac{(ax)^{2m}}{m!}\,\frac{(bx)^n}{n!}\,. \end{aligned}$$

In turn, the last expressions form a difference of two, weighted three–variables Srivastava–Daoust $\mathscr{S}$ functions, which results in

$$H_1(x) = \frac{(b^2-a^2)^{\nu+\frac{1}{2}}}{2h\,(\nu+1)\Gamma(2\nu+1)}\,\{\mathscr{S}_\nu(x+h)-\mathscr{S}_\nu(x)\}.$$

Now, it remains to apply the standard $0 \leqslant H_1(x) \leqslant 1; x \geqslant 0$ property to obtain the asserted inequality.

Concerning the convergence of the given function $\mathscr{S}_\nu(x)$ (see the Appendix A (4.2)), as

$$\begin{aligned}\Delta_1 &= 1+(1+0)+0-1-0 = 1\\ \Delta_2 &= 1+(2+2)+0-2-1 = 2\\ \Delta_3 &= 1+(1+1)+0-1-1 = 1,\end{aligned}$$

i.e. $\Delta_\ell > 0$ for all $\ell = 1,2,3$, which implies that our function converges absolutely for all complex arguments. $\square$

REMARK 2. The Lemma 1 contains also the CDF $H_2(x)$, its application leads to slightly different monotonicity result. However, to establish the related inequality bounds we leave to the interested reader.

4. Appendix A

Srivastava and Daoust generalized the Lauricella hypergeometric function F_D by the n-tuple power series [11, p. 454]

$$\mathscr{S}^{A:B^{(1)};\cdots;B^{(n)}}_{C:D^{(1)};\cdots;D^{(n)}}\left(\begin{matrix}[(a):\theta^{(1)},\cdots,\theta^{(n)}]:[(b^{(1)}):\varphi^{(1)}];\cdots;[(b^{(n)}):\varphi^{(n)}]\\ \\ [(c):\psi^{(1)};\cdots;\psi^{(n)}]:[(d^{(1)}):\delta^{(1)}];\cdots;[(d^{(n)}):\delta^{(n)}]\end{matrix}\;\middle|\;\begin{matrix}x_1\\ \vdots\\ x_n\end{matrix}\right)$$

$$= \sum_{\boldsymbol{m}\geqslant 0}\frac{\prod\limits_{j=1}^{A}(a_j)_{m_1\theta_j^{(1)}+\cdots+m_n\theta_j^{(n)}}\prod\limits_{j=1}^{B^{(1)}}(b_j^{(1)})_{m_1\varphi_j^{(1)}}\cdots\prod\limits_{j=1}^{B^{(n)}}(b_j^{(n)})_{m_n\varphi_j^{(n)}}}{\prod\limits_{j=1}^{C}(c_j)_{m_1\psi_j^{(1)}+\cdots+m_n\psi_j^{(n)}}\prod\limits_{j=1}^{D^{(1)}}(d_j^{(1)})_{m_1\delta_j^{(1)}}\cdots\prod\limits_{j=1}^{D^{(n)}}(d_j^{(n)})_{m_n\delta_j^{(n)}}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!} \tag{4.1}$$

where $\boldsymbol{m} := (m_1,\cdots,m_n)$ and the parameters satisfy

$$\theta_1^{(1)},\cdots,\theta_A^{(1)},\cdots,\delta_1^{(n)},\cdots,\delta_{D^{(n)}}^{(n)} > 0.$$

For convenience, we write (a) to denote the sequence of A parameters $a_1,\cdots,a_A$, with similar interpretations for $(b'),(b^{(1)}),\cdots,(d^{(n)})$. Empty products should be interpreted as unity. Srivastava and Daoust [10, pp. 157–158] reported that the series in (4.1) converges absolutely for all $x_1,\cdots,x_n \in \mathbb{C}$ when

$$\Delta_\ell = 1+\sum_{j=1}^{C}\psi_j^{(\ell)}+\sum_{j=1}^{D^{(\ell)}}\delta_j^{(\ell)}-\sum_{j=1}^{A}\theta_j^{(\ell)}-\sum_{j=1}^{B^{(\ell)}}\varphi_j^{(\ell)} > 0, \qquad \ell = \overline{1,n}. \tag{4.2}$$

In the case $\Delta_\ell = 0, \ell = \overline{1,n}$, the convergence constraints of the series is described by the relation [10, p. 157, Eqs. (5.2-5.3); p. 158 (**ii**)]. Finally, when all $\Delta_\ell < 0$, $\mathscr{S}^{A:B';\cdots;B^{(n)}}_{C:D';\cdots;D^{(n)}}(x_1,\cdots,x_n)$ diverges except at the origin, that is, this series is formal.

5. Appendix B

Exton's X function (2.1) is a special case of the Srivastava–Daoust function in two variables defined according to (4.1) by the double power series [10, p. 151]

$$\mathscr{S}^{A:B;B'}_{C:D;D'}\left(\begin{matrix}[(a):\theta;\theta']:[(b):\varphi];[(b'):\varphi'] \\ [(c):\psi;\psi']:[(d):\delta];[(d'):\delta']\end{matrix}\middle| x,y\right) = \sum_{k,n\geqslant 0} \frac{\prod_{j=1}^{A}(a_j)_{k\theta_j+n\theta'_j}\prod_{j=1}^{B}(b_j)_{k\varphi_j}\prod_{j=1}^{B'}(b'_j)_{n\varphi'_j}}{\prod_{j=1}^{C}(c_j)_{k\psi_j+n\psi'_j}\prod_{j=1}^{D}(d_j)_{k\delta_j}\prod_{j=1}^{D'}(d'_j)_{n\delta'_j}}\frac{x^k}{k!}\frac{y^n}{n!}. \tag{5.1}$$

The reduced absolute convergence constraints of (5.1) for all $x,y\in\mathbb{C}$ read

$$\Delta_1 = 1+\sum_{j=1}^{C}\psi_j+\sum_{j=1}^{D}\delta_j-\sum_{j=1}^{A}\theta_j-\sum_{j=1}^{B}\varphi_j > 0,$$

$$\Delta_2 = 1+\sum_{j=1}^{C}\psi'_j+\sum_{j=1}^{D'}\delta'_j-\sum_{j=1}^{A}\theta'_j-\sum_{j=1}^{B'}\varphi'_j > 0.$$

Another cases of convergence analysis inside disks in $\mathbb{C}$ can be found in [10, pp. 153–157].

We are interested in the convergence regions of the Exton's $X^{1:0;0}_{1:1;0}$, $X^{0:1;1}_{1:0;0}$, $X^{1:0;1}_{1:1;1}$ occuring in our results and contain the parameters in any particular case. These series are generated by

$$X^{A:B;B'}_{C:D;D'}\left[\begin{matrix}a:b;b' \\ c:d;d'\end{matrix}\middle| x,y\right] = \mathscr{S}^{A:B;B'}_{C:D;D'}\left(\begin{matrix}[a:2;1]:[b:1];[b':1] \\ [c:2;1]:[d:1];[d':1]\end{matrix}\middle| x,y\right),$$

where $A,B,B',C,D,D'\in\{0,1\}$ and in the cases when any of parameters a,b,b',c,d,d' vanish, the empty products should be interpreted as unity. The constraints for the Srivastava–Daoust function (5.1) guaranteeing the absolute convergence of the $\mathscr{S}$ for $x,y\in\mathbb{C}$ reduce for our three functions to

- **a.** $X^{1:0;0}_{1:1;0}$. Here $\Delta_1(\mathbf{a}) = 2; \Delta_2(\mathbf{a}) = 1$. The series converges for any $x \geqslant 0$.
- **b.** $X^{0:1;1}_{1:0;0}$. Now, $\Delta_1(\mathbf{b}) = 1; \Delta_2(\mathbf{b}) = 1$. This ensures the convergence of the second series for all $x \geqslant 0$ as well.
- **c.** $X^{1:0;1}_{1:1;1}$. The remaining case is also clear, being $\Delta_1(\mathbf{c}) = 2; \Delta_2(\mathbf{c}) = 1$, which means absolute convergence for any $x \geqslant 0$.

We point out that for all three cases of X function which occur in related bounding inequalities we have that both Δ_1, Δ_2 are positive, which ensures the convergence for all considered values of the argument $x \geqslant 0$.

REFERENCES

[1] J. BOGNÁR, J. MOGYORÓDI, A. PRÉKOPA, A. RÉNYI AND D. SZÁSZ, *Exercises in Probability Theory*, Fourth corrected edition, Typotex Kiadó, Budapest, 2001. (in Hungarian).

[2] H. EXTON, *Reducible double hypergeometric functions and associated integrals*, Anais da Faculdade de Ciências. Universidade Do Porto, **LXIII** (1982), No. 1–4, 137–143.

[3] K. GÓRSKA, A. HORZELA, D. JANKOV MAŠIREVIĆ AND T. K. POGÁNY, *Observations on the McKay I_ν Bessel distribution*, J. Math. Anal. Appl., **516** (2022), No. 1, Paper No. 126481, 14 pp.

[4] D. JANKOV MAŠIREVIĆ AND T. K. POGÁNY, *Functional bounds for Horn function Φ_2 generated by CDF of McKay I_ν Bessel distribution*, Rad Hrvat. Akad. Znan. Umjet. Mat. Znan., (2022), (to appear).

[5] D. JANKOV MAŠIREVIĆ AND T. K. POGÁNY, *On new formulae for cumulative distribution function for McKay Bessel distribution*, Comm. Statist. Theory Methods, **50** (2021), No. 1, 143–160.

[6] A. T. MCKAY, *A Bessel function distribution*, Biometrika, **24** (1932), No. 1–2, 39–44.

[7] F. MCNOLTY, *Some probability density functions and their characteristic functions*, Math. Comp., **27** (1973), No. 123, 495–504.

[8] M. D. ORTIGUEIRA, *A coherent approach to non-integer order derivatives*, Signal Processing, **86** (2006), 2505–2515.

[9] S. G. SAMKO, A. A. KILBAS AND O. I. MARICHEV, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, New York, 1993.

[10] H. M. SRIVASTAVA AND M. C. DAOUST, *A note on the convergence of Kampé de Fériet double hypergeometric series*, Math. Nachr., **53** (1972), 151–159.

[11] H. M. SRIVASTAVA AND M. C. DAOUST, *Certain generalized Neumann expansions associated with the Kampé De Fériet function*, Nederl. Akad. Wetensch. Proc. Ser. A **72** = Indag. Math., **31** (1969), 449–457.

(Received July 12, 2022)

Dragana Jankov Maširević
Department of Mathematics
University of Osijek
31000 Osijek, Croatia
e-mail: djankov@mathos.hr

Tibor K. Pogány
Institute of Applied Mathematics
Óbuda University
1034 Budapest, Hungary
and
Faculty of Maritime Studies
University of Rijeka
51000 Rijeka, Studentska 2, Croatia
e-mail: pogany.tibor@nik.uni-obuda.hr
tibor.poganj@uniri.hr