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CERTAIN FRACTIONAL INTEGRAL INCLUSIONS PERTAINING TO INTERVAL–VALUED EXPONENTIAL TRIGONOMETRIC CONVEX FUNCTIONS

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Abstract. As an interesting generalization involving the interval-valued convex functions, the interval-valued exponential trigonometric convex function is firstly introduced, and their meaningful properties are then investigated. Meanwhile, certain Hermite–Hadamard- and Pachpatte-type integral inclusion relations are also developed via the newly proposed functions in interval-valued fractional calculus. In particular, an improved version of the Hermite–Hadamard's integral inclusions pertaining to the interval-valued exponential trigonometric convex functions is proposed as well. To identify the correctness of the derived inclusion relations in the study, the graphical representations for the outcomes are provided in terms of the change of the parameter α.

1. Introduction and preliminaries

The generalized convexity of functions provides a fairly powerful principle and tool, which is widely used not only in applied analysis and nonlinear analysis, but also in a crowd of problems in mathematical physics. In recent years, a great number of researchers have devoted themselves to exploring some fascinating integral inequalities by virtue of generalized convexity from different perspectives, see the published articles [4, 6, 9, 16] and the references therein. And within that, the Hermite–Hadamard's integral inequality is one of the most influential mathematical inequalities in association with convex functions, which is also widely used in many other aspects of the mathematical sciences, particularly in optimization analysis. Let us invoke it in the following.

Consider a real-valued interval $I \subseteq \mathbb{R}$. We assume that the function $f : I \to \mathbb{R}$ is convex for all $a, b \in I$ along with $a \neq b$. Then we have that

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leqslant \frac{f(a)+f(b)}{2},$$

which is referred to as the celebrated Hermite–Hadamard's inequality.

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This is an especially distinguished inequality, because it offers an estimate of the error bound for the mean value with regard to the integrable convex mapping $f:[a,b]\to\mathbb{R}$, which has aroused academic attention and research from a wealth of scholars in the mathematical analysis field. Recently, some important papers concerning different families of convex functions that are in connection with the Hermite–Hadamard-type integral inequalities have been published. For example, we can refer to Szostok [36] for higher order convex functions, to Kórus [20] for s-convex functions, to Andric and Pecaric [3] for (h,g,m)-convex functions, to Latif [22] for GA-convex and geometrically quasiconvex functions, to Niezgoda [27] for G-symmetrized convex functions, to Demır et al. [12] for trigonometrically convex functions and so on. For recent research allied to this topical subject, the interested reader may consult the published articles [1, 11, 38] and the bibliographies quoted in them.

The exponential trigonometric convex functions, as an extension of the trigonometric convex functions, was recently introduced by Breckner et al. in 2021.

DEFINITION 1.1. [17] Consider a real-valued interval $I\subseteq\mathbb{R}$. A function $f:I\to\mathbb{R}^+$ is named as the exponential trigonometric convex functions if and only if

$$f(tx+(1-t)y)\leqslant\frac{\sin\frac{\pi t}{2}}{e^{1-t}}f(x)+\frac{\cos\frac{\pi t}{2}}{e^{t}}f(y)$$

holds true for all $x,y\in I$ and $t\in[0,1]$.

In the same paper, they deduced the succeeding Hermite–Hadamard-type integral inequalities, which are in connection with the exponential trigonometric convexity.

THEOREM 1.1. [17] *Assume that $f:[a,b]\to\mathbb{R}$ is an exponential trigonometric convex function with $0\leqslant a<b$. If $f\in L^1([a,b])$, then the coming integral inequalities hold:*

$$\sqrt{\frac{e}{2}}f\left(\frac{a+b}{2}\right)\leqslant\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\leqslant\frac{2\pi+4e}{e(\pi^2+4)}[f(a)+f(b)].$$

Fractional calculus, as a fairly vigorous tool, has been shown to be a vital cornerstone not only in mathematical sciences, but also in applied sciences. The field has aroused a load of researcher's attention to address the meaningful question. As a consequence, many authors have obtained some significant integral inequalities through the efficient interaction of various methods of fractional integrals, including Ahmad et al. [2] in the study of four types of inequalities for convex functions concerning fractional integrals with exponential kernels, Mohammed and Sarikaya [24] in the study of some inequalities involving Sarikaya fractional integrals for twice differentiable functions, Set et al. [32] in the Hermite–Hadamard–Fejér-type inequality for Atangana–Baleanu fractional integral operators, and Dragomir [13] in the Hermite–Hadamard-type inequalities for generalized Riemann–Liouville fractional integrals. For more important findings related to fractional integrals, we refer the interested readers to [21, 28, 25, 31] and the bibliographies quoted in them.

In the remainder of this part, the following theories established by Moore et al. in [26] on interval analysis are significative since they can be commonly utilized throughout this study. Therefore, we assume that ω is a closed, bounded real-valued interval subset in $\mathbb{R}$, which is expressed as

$$\omega = [\underline{\omega}, \overline{\omega}] = \left\{\theta \in \mathbb{R} : \underline{\omega} \leqslant \theta \leqslant \overline{\omega}\right\}.$$

Here, the numbers $\underline{\omega}, \overline{\omega} \in \mathbb{R}$ and $\underline{\omega} \leqslant \overline{\omega}$. We denote the left endpoints of interval ω by $\underline{\omega}$, the right endpoints of interval ω by $\overline{\omega}$, respectively. If $\underline{\omega} = r = \overline{\omega}$, then the real-value interval ω is considered to be degenerated, in this condition, we take advantage of the form $\omega = r = [r,r]$. We denote ω is positive if $\underline{\omega} > 0$, or negative if $\overline{\omega} < 0$. Furthermore, we denote the sets of all closed intervals in $\mathbb{R}$ by $\mathbb{R}_I$, the sets of all positive closed intervals in $\mathbb{R}$ by $\mathbb{R}_I^+$, as well as the sets of all negative closed intervals in $\mathbb{R}$ by $\mathbb{R}_I^-$. The Hausdorff–Pompeiu distance, with respect to the intervals ω and ξ, is defined by in the following way:

$$d(\omega, \xi) = d\Big([\underline{\omega}, \overline{\omega}], [\underline{\xi}, \overline{\xi}]\Big) = \max\left\{\left|\underline{\omega} - \underline{\xi}\right|, \left|\overline{\omega} - \overline{\xi}\right|\right\}.$$

Distinctly, $(\mathbb{R}_{\mathscr{I}}, d)$ is a complete metric space.

Moore et al. considered the succeeding interval-valued Lebesgue integrable conception as well.

DEFINITION 1.2. [26] Consider a real-valued interval $I \subseteq \mathbb{R}$ and the interior I° of I. Let $\Psi(x) = [\underline{\Psi}(x), \overline{\Psi}(x)]$, $x \in I^\circ$. We call $\Psi(x)$ is Lebesgue integrable if $\overline{\Psi}(x)$ and $\underline{\Psi}(x)$ are both measurable and Lebesgue integrable in I°. What's more, we denote the interval-valued integration of the interval-valued function Ψ by $\int_a^b \Psi(x)\mathrm{d}x$, is defined as follows:

$$\int_a^b \Psi(x)\mathrm{d}x = \left[\int_a^b \underline{\Psi}(x)\mathrm{d}x, \int_a^b \overline{\Psi}(x)\mathrm{d}x\right].$$

For more axioms concerning interval-valued analysis, see the published monograph [26]. We also evoke the notion of interval-valued convexity, which was introduced by Breckner in [5].

DEFINITION 1.3. [5] Consider a convex set $I \subseteq \mathbb{R}$. An interval-valued function $\Psi : I \to \mathbb{R}_I^+$ is referred as the interval-valued convex functions if and only if the coming relation

$$\Psi(tx + (1-t)y) \supseteq t\Psi(x) + (1-t)\Psi(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

In 2021, Zhou et al. gave the concept of the interval-valued fractional integral operators together with exponential kernels as follows.

DEFINITION 1.4. [41] Suppose that $\Psi:[a,b]\to\mathbb{R}_I$ is an interval-valued function such that $\Psi(x)=\left[\underline{\Psi}(x),\overline{\Psi}(x)\right]$. Here, the real-valued functions $\underline{\Psi}(x),\overline{\Psi}(x)$ are both Riemannian integrable on the real-valued interval $[a,b]$. We denote the interval-valued left-sided and right-sided fractional integrals along with exponential kernels for the function Ψ by $\mathscr{I}^{\alpha}_{a^+}\Psi(\zeta)$ and $\mathscr{I}^{\alpha}_{b^-}\Psi(\zeta)$, respectively, are defined as the following:

$$\mathscr{I}^{\alpha}_{a^+}\Psi(\zeta)=\frac{1}{\alpha}\int_a^{\zeta}e^{\left(-\frac{1-\alpha}{\alpha}(\zeta-x)\right)}\Psi(x)\mathrm{d}x,\ \zeta>a,$$

and

$$\mathscr{I}^{\alpha}_{b^-}\Psi(\zeta)=\frac{1}{\alpha}\int_{\zeta}^{b}e^{\left(-\frac{1-\alpha}{\alpha}(x-\zeta)\right)}\Psi(x)\mathrm{d}x,\ \zeta<b,$$

with $\alpha\in(0,1)$. Obviously, we have that

$$\mathscr{I}^{\alpha}_{a^+}\Psi(\zeta)=\left[\mathscr{I}^{\alpha}_{a^+}\underline{\Psi}(\zeta),\mathscr{I}^{\alpha}_{a^+}\overline{\Psi}(\zeta)\right],$$

and

$$\mathscr{I}^{\alpha}_{b^-}\Psi(\zeta)=\left[\mathscr{I}^{\alpha}_{b^-}\underline{\Psi}(\zeta),\mathscr{I}^{\alpha}_{b^-}\overline{\Psi}(\zeta)\right].$$

Interval analysis, the branch of set value analysis devoted to the study of properties and applications of interval-valued functions, is nowadays playing an extremely significant role in the pure and applied sciences. Interval analysis was originally used to calculate the error bounds of a finite state machine numerical solution. For the last few decades, the field of interval analysis has been developed boomingly and has a great ramification in various branches of applied sciences like neural network output optimization [37], computer graphics [34] and automatic error analysis [30]. Until its development to today, there have been many scholars studied on the hot issues of various interval analysis theories. For example, Budak et al. [8] extended the Riemann–Liouville fractional integrals of an interval-valued function $F(x)$ defined on $\mathbb{R}$ to the interval-valued function $F(x,y)$ defined on $\mathbb{R}^2$. In this extension, they deduced certain Hermite–Hadamard-type fractional integral inequalities for interval-valued co-ordinated convex functions. Costa et al. [10] developed some inequalities for interval-valued functions which is based on the Kulisch–Miranker order relation. They took advantage of Aumann's and Kaleva's improper integrals to derive the Gauss's inequalities for interval functions. Liu et al. [23] studied a family of log-s-convex fuzzy-interval-valued function. They obtained certain Jensen- and Hermite–Hadamard–Fejér-type inequalities with help of this kind of function. Srivastava et al. [35] introduced the conception of interval-valued preinvex functions. The authors also gave the refinements of the Hermite–Hadamard-type inequalities with regard to the Riemann–Liouville fractional integrals. In [40], the authors presented the notion of the interval-valued harmonical h-convex function. With the aid of this conception, they acquired several Hermite–Hadamard-type inequalities for the interval Riemann integrals. In addition, some applications of interval-valued functions in optimization theory are discussed in [15] and [33]. For recent developments related to interval-valued functions, the interested reader can refer to [7, 14, 19, 18, 29] and the bibliographies quoted in them.

Motivated and inspired by the aforementioned researches, in particular, the outcomes investigated in [17], we notice that it is possible to treat and generalize these results by virtue of interval-valued theories. To achieve this objective, we introduce a class of the interval-valued exponential trigonometric convex functions. By using it, we establish some fractional integral inclusion relations having exponential kernels pertaining to the extraordinary Hermite–Hadamard- and Pachpatte-type integral inequalities, respectively. This is the main contribution of this work.

2. Main results

Let us introduce the following conception of the interval-valued exponential trigonometric convex functions.

DEFINITION 2.1. Consider $I \subseteq \mathbb{R}$ is a convex set. An interval-valued function $H(x) = \left[\underline{H}(x), \overline{H}(x)\right] : I \to \mathbb{R}_I$, is called as the interval-valued exponential trigonometric convex functions if and only if

$$H(tx + (1-t)y) \supseteq \frac{\sin\frac{\pi t}{2}}{e^{1-t}} H(x) + \frac{\cos\frac{\pi t}{2}}{e^{t}} H(y)$$

holds true for any $x, y \in I$ and $t \in [0,1]$.

Clearly, if the functions satisfy the condition $\underline{H} = \overline{H}$, then the conception of the interval-valued exponential trigonometric convex functions degenerates to the notion of the exponential-type trigonometric convex functions.

Next, we discuss some properties for interval-valued exponential trigonometric convex functions.

PROPOSITION 2.1. *An interval-valued function $H(x) = \left[\underline{H}(x), \overline{H}(x)\right] : I \to \mathbb{R}_I$ is referred as the exponential trigonometric convex functions if and only if the coming inequalities*

$$\underline{H}(tx + (1-t)y) \leqslant \frac{\sin\frac{\pi t}{2}}{e^{1-t}} \underline{H}(x) + \frac{\cos\frac{\pi t}{2}}{e^{t}} \underline{H}(y),$$

and

$$\overline{H}(tx + (1-t)y) \geqslant \frac{\sin\frac{\pi t}{2}}{e^{1-t}} \overline{H}(x) + \frac{\cos\frac{\pi t}{2}}{e^{t}} \overline{H}(y)$$

hold true for all $x, y \in I$ and $t \in [0,1]$, that is, the function $\underline{H}(x), \overline{H}(x)$ are exponential trigonometric convex function, exponential trigonometric concave function, respectively.

EXAMPLE 2.1. Let $H = [e^s - 1, -s^2 + 2s + 2], s \in [0,1]$, for $x = 0, y = 1$ and $t \in [0,1]$, we have

As can be seen from the Figure 2.1, the range of the purple solid line is the region containing the green dotted line, which indicates that the function H given is in line with Definition 2.1, and the left end includes the right end.

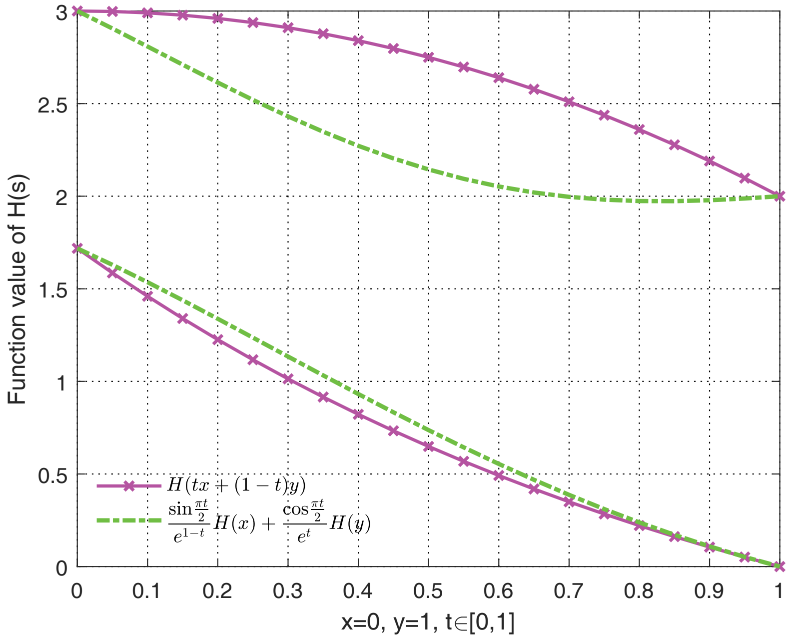

Figure 2.1: *Graphical representation for Proposition 2.1*

THEOREM 2.1. *Let $H, G : I \to \mathbb{R}_I$. If H and G are both interval-valued exponential trigonometric convex functions, then*

(i) $H+G$ is an interval-valued exponential trigonometric convex function,

(ii) For $c \in \mathbb{R}$ $(c \geqslant 0)$, cH is an interval-valued exponential trigonometric convex function.

Proof. (i) Since H, G are both interval-valued exponential trigonometric convex functions, we acquire that

$$\begin{aligned}
(H+G)(tx+(1-t)y) &= H(tx+(1-t)y)+G(tx+(1-t)y) \\
&\supseteq \frac{\sin\frac{\pi t}{2}}{e^{1-t}}H(x)+\frac{\cos\frac{\pi t}{2}}{e^{t}}H(y)+\frac{\sin\frac{\pi t}{2}}{e^{1-t}}G(x)+\frac{\cos\frac{\pi t}{2}}{e^{t}}G(y) \\
&= \Bigg[\frac{\sin\frac{\pi t}{2}}{e^{1-t}}\underline{H}(x)+\frac{\cos\frac{\pi t}{2}}{e^{t}}\underline{H}(y)+\frac{\sin\frac{\pi t}{2}}{e^{1-t}}\underline{G}(x)+\frac{\cos\frac{\pi t}{2}}{e^{t}}\underline{G}(y), \\
&\qquad \frac{\sin\frac{\pi t}{2}}{e^{1-t}}\overline{H}(x)+\frac{\cos\frac{\pi t}{2}}{e^{t}}\overline{H}(y)+\frac{\sin\frac{\pi t}{2}}{e^{1-t}}\overline{G}(x)+\frac{\cos\frac{\pi t}{2}}{e^{t}}\overline{G}(y)\Bigg] \\
&= \frac{\sin\frac{\pi t}{2}}{e^{1-t}}[H(x)+G(x)]+\frac{\cos\frac{\pi t}{2}}{e^{t}}[H(y)+G(y)].
\end{aligned}$$

(ii) Let H be an interval-valued exponential trigonometric convex function and $c \in \mathbb{R}$ $(c \geqslant 0)$, then

$$\begin{aligned}(cH)(tx+(1-t)y) &\supseteq c\left[\frac{\sin\frac{\pi t}{2}}{e^{1-t}}H(x)+\frac{\cos\frac{\pi t}{2}}{e^{t}}H(y)\right]\\ &= \frac{\sin\frac{\pi t}{2}}{e^{1-t}}(cH)(x)+\frac{\cos\frac{\pi t}{2}}{e^{t}}(cH)(y). \quad \square\end{aligned}$$

THEOREM 2.2. *Let $g: I \to J$ be a convex function. If $H: J \to \mathbb{R}_I$ is an interval-valued exponential trigonometric convex function and $\underline{H}$, $\overline{H}$ are nondecreasing and nonincreasing respectively, then $H \circ g: I \to \mathbb{R}_I$ is an interval-valued exponential trigonometric convex function.*

Proof. Taking advantage of the nondecreasing exponential trigonometric convexity of function $\underline{H}$ for any $x, y \in I$, $t \in [0,1]$, we get that

$$\begin{aligned}\underline{H}(g(tx+(1-t)y)) &\leqslant \underline{H}(tg(x)+(1-t)g(y))\\ &\leqslant \frac{\sin\frac{\pi t}{2}}{e^{1-t}}\underline{H}(g(x))+\frac{\cos\frac{\pi t}{2}}{e^{t}}\underline{H}(g(y)).\end{aligned} \tag{2.1}$$

Similarly, we have that

$$\begin{aligned}\overline{H}(g(tx+(1-t)y)) &\geqslant \overline{H}(tg(x)+(1-t)g(y))\\ &\geqslant \frac{\sin\frac{\pi t}{2}}{e^{1-t}}\overline{H}(g(x))+\frac{\cos\frac{\pi t}{2}}{e^{t}}\overline{H}(g(y)).\end{aligned} \tag{2.2}$$

By means of(2.1) and (2.2), we obtain that

$$\begin{aligned}(H\circ g)(tx+(1-t)y) &= H(g(tx+(1-t)y))\\ &\supseteq H(tg(x)+(1-t)g(y))\\ &\supseteq \frac{\sin\frac{\pi t}{2}}{e^{1-t}}H(g(x))+\frac{\cos\frac{\pi t}{2}}{e^{t}}H(g(y)).\end{aligned}$$

This completes the proof. $\square$

THEOREM 2.3. *Let $H, G: I \to \mathbb{R}_I^+$ are both interval-valued exponential trigonometric convex functions. If $\underline{H}$ and $\underline{G}$ are monotone increasing, $\overline{H}$ and $\overline{G}$ are monotone decreasing, then HG is an interval-valued exponential trigonometric convex function.*

Proof. If $x \leqslant y$ (similarly, $y \leqslant x$), then $[\underline{H}(x)-\underline{H}(y)][\underline{G}(y)-\underline{G}(x)] \leqslant 0$ which deduces that

$$\underline{H}(x)\underline{G}(y)+\underline{H}(y)\underline{G}(x) \leqslant \underline{H}(x)\underline{G}(x)+\underline{H}(y)\underline{G}(y). \tag{2.3}$$

Using the exponential trigonometric convexity of function $\underline{H}$ and $\underline{G}$ for any $x,y \in I$, $t \in [0,1]$, we obtain that

$$\begin{aligned}(\underline{H}\,\underline{G})(tx+(1-t)y) &= \underline{H}(tx+(1-t)y)\underline{G}(tx+(1-t)y)\\ &\leqslant \left[\frac{\sin\frac{\pi t}{2}}{e^{1-t}}\underline{H}(x)+\frac{\cos\frac{\pi t}{2}}{e^{t}}\underline{H}(y)\right]\left[\frac{\sin\frac{\pi t}{2}}{e^{1-t}}\underline{G}(x)+\frac{\cos\frac{\pi t}{2}}{e^{t}}\underline{G}(y)\right]\\ &= \frac{\sin^2\frac{\pi t}{2}}{e^{2(1-t)}}\underline{H}(x)\underline{G}(x)+\left[\frac{\sin\frac{\pi t}{2}}{e^{1-t}}+\frac{\cos\frac{\pi t}{2}}{e^{t}}\right][\underline{H}(x)\underline{G}(y)+\underline{H}(y)\underline{G}(x)]\\ &\quad+\frac{\cos^2\frac{\pi t}{2}}{e^{2t}}\underline{H}(y)\underline{G}(y).\end{aligned}$$

Utilizing the inequality (2.3), we get that

$$\begin{aligned}(\underline{H}\,\underline{G})(tx+(1-t)y) \leqslant\ & \frac{\sin\frac{\pi t}{2}}{e^{1-t}}\left[\frac{\sin\frac{\pi t}{2}}{e^{1-t}}+\frac{\cos\frac{\pi t}{2}}{e^{t}}\right]\underline{H}(x)\underline{G}(x)\\ &+\frac{\cos\frac{\pi t}{2}}{e^{t}}\left[\frac{\sin\frac{\pi t}{2}}{e^{1-t}}+\frac{\cos\frac{\pi t}{2}}{e^{t}}\right]\underline{H}(y)\underline{G}(y).\end{aligned}$$

Since $\frac{\sin\frac{\pi t}{2}}{e^{1-t}}+\frac{\cos\frac{\pi t}{2}}{e^{t}} \leqslant 1$, we acquire that

$$\begin{aligned}(\underline{H}\,\underline{G})(tx+(1-t)y) &\leqslant \frac{\sin\frac{\pi t}{2}}{e^{1-t}}\underline{H}(x)\underline{G}(x)+\frac{\cos\frac{\pi t}{2}}{e^{t}}\underline{H}(y)\underline{G}(y)\\ &= \frac{\sin\frac{\pi t}{2}}{e^{1-t}}(\underline{H}\,\underline{G})(x)+\frac{\cos\frac{\pi t}{2}}{e^{t}}(\underline{H}\,\underline{G})(y).\end{aligned}$$

Analogously, we obtain that

$$\begin{aligned}(\overline{H}\,\overline{G})(tx+(1-t)y) &\geqslant \frac{\sin\frac{\pi t}{2}}{e^{1-t}}\overline{H}(x)\overline{G}(x)+\frac{\cos\frac{\pi t}{2}}{e^{t}}\overline{H}(y)\overline{G}(y)\\ &= \frac{\sin\frac{\pi t}{2}}{e^{1-t}}(\underline{H}\,\underline{G})(x)+\frac{\cos\frac{\pi t}{2}}{e^{t}}(\overline{H}\,\overline{G})(y).\end{aligned}$$

Therefore, we have

$$(HG)(tx+(1-t)y) \supseteq \frac{\sin\frac{\pi t}{2}}{e^{1-t}}(HG)(x)+\frac{\cos\frac{\pi t}{2}}{e^{t}}(HG)(y).$$

This ends the proof. □

Next, we aims to establish some fractional integral inclusion relations for interval-valued exponential convex functions.

To simplify the notation, throughout the present paper we denote

$$\rho = \frac{1-\alpha}{\alpha}(b-a), \ \ 0<\alpha<1, \ \ a<b.$$

Based on the introduced interval-valued exponential trigonometric convex functions, our first main outcome in accordance with the Hermite–Hadamard-type integral inequalities is proposed in the following.

THEOREM 2.4. *If $H:[a,b]\to\mathbb{R}_I^+$ is an interval-valued exponential trigonometric convex function together with $a<b$ such that $H(x)=\left[\underline{H}(x),\overline{H}(x)\right]$, then we have:*

$$\begin{aligned}\sqrt{\frac{e}{2}}H\left(\frac{a+b}{2}\right)&\supseteq\frac{1-\alpha}{2(1-e^{-\rho})}\left[\mathscr{I}_{a^+}^{\alpha}H(b)+\mathscr{I}_{b^-}^{\alpha}H(a)\right]\\&\supseteq\frac{\rho}{1-e^{-\rho}}\mathscr{C}(\rho)\frac{H(a)+H(b)}{2},\end{aligned}\tag{2.4}$$

where

$$\mathscr{C}(\rho)=\frac{4\rho+2\pi e^{-\rho-1}+4}{4\rho^2+8\rho+\pi^2+4}+\frac{2\pi e^{-1}+e^{-\rho-1}(4e-4\rho e)}{4\rho^2-8\rho+\pi^2+4}.\tag{2.5}$$

Proof. On account of the interval-valued exponential trigonometric convexity of H, we find that

$$H\left(\frac{x+y}{2}\right)\supseteq\frac{\sin\frac{\pi}{4}}{e^{\frac{1}{2}}}H(x)+\frac{\cos\frac{\pi}{4}}{e^{\frac{1}{2}}}H(y)=\frac{1}{2}\sqrt{\frac{2}{e}}\left[H(x)+H(y)\right],\tag{2.6}$$

for any $x,y\in[a,b]$. Taking advantage of $x=\bar{\eta}a+(1-\bar{\eta})b$ and $y=(1-\bar{\eta})a+\bar{\eta}b$, $\bar{\eta}\in[0,1]$, we get that

$$H\left(\frac{x+y}{2}\right)=H\left(\frac{a+b}{2}\right)\supseteq\frac{1}{2}\sqrt{\frac{2}{e}}\left[H(\bar{\eta}a+(1-\bar{\eta})b)+H((1-\bar{\eta})a+\bar{\eta}b)\right].\tag{2.7}$$

Multiplying on each side of (2.7) by $e^{-\rho\bar{\eta}}$, and integrating the obtained outcomes with regard to $\bar{\eta}$ from 0 to 1, we deduce that

$$\begin{aligned}&\int_0^1 e^{-\rho\bar{\eta}}H\left(\frac{a+b}{2}\right)\mathrm{d}\bar{\eta}\\&\supseteq\frac{1}{2}\sqrt{\frac{2}{e}}\left[\int_0^1 e^{-\rho\bar{\eta}}H(\bar{\eta}a+(1-\bar{\eta})b)\mathrm{d}\bar{\eta}+\int_0^1 e^{-\rho\bar{\eta}}H((1-\bar{\eta})a+\bar{\eta}b)\mathrm{d}\bar{\eta}\right]\\&=\frac{1}{2}\sqrt{\frac{2}{e}}\Bigg[\int_0^1 e^{-\rho\bar{\eta}}\Big(\underline{H}(\bar{\eta}a+(1-\bar{\eta})b)+\underline{H}((1-\bar{\eta})a+\bar{\eta}b)\Big)\mathrm{d}\bar{\eta},\\&\qquad\int_0^1 e^{-\rho\bar{\eta}}\Big(\overline{H}(\bar{\eta}a+(1-\bar{\eta})b)+\overline{H}((1-\bar{\eta})a+\bar{\eta}b)\Big)\mathrm{d}\bar{\eta}\Bigg]\\&=\Bigg[\sqrt{\frac{2}{e}}\frac{1}{2(b-a)}\left(\int_a^b e^{-\frac{1-\alpha}{\alpha}(b-u)}\underline{H}(u)\mathrm{d}u+\int_a^b e^{-\frac{1-\alpha}{\alpha}(v-a)}\underline{H}(v)\mathrm{d}v\right),\\&\qquad\sqrt{\frac{2}{e}}\frac{1}{2(b-a)}\left(\int_a^b e^{-\frac{1-\alpha}{\alpha}(b-u)}\overline{H}(u)\mathrm{d}u+\int_a^b e^{-\frac{1-\alpha}{\alpha}(v-a)}\overline{H}(v)\mathrm{d}v\right)\Bigg]\\&=\sqrt{\frac{2}{e}}\frac{\alpha}{2(b-a)}\left[\mathscr{I}_{a^+}^{\alpha}H(b)+\mathscr{I}_{b^-}^{\alpha}H(a)\right].\end{aligned}\tag{2.8}$$

Also,

$$\begin{aligned}
&= \left[\int_0^1 e^{-\rho\bar{\eta}}\underline{H}\left(\frac{a+b}{2}\right)\mathrm{d}\bar{\eta}, \int_0^1 e^{-\rho\bar{\eta}}\overline{H}\left(\frac{a+b}{2}\right)\mathrm{d}\bar{\eta}\right] \\
&= \left[\frac{1-e^{-\rho}}{\rho}\underline{H}\left(\frac{a+b}{2}\right), \frac{1-e^{-\rho}}{\rho}\overline{H}\left(\frac{a+b}{2}\right)\right] \\
&= \frac{1-e^{-\rho}}{\rho}H\left(\frac{a+b}{2}\right).
\end{aligned} \tag{2.9}$$

Employing the equation (2.9) in the relation (2.8), it yields the first relation in (2.4).

For the second inclusion relation (2.4), by means of the exponential trigonometric convexity of the interval-valued function H, we acquire that

$$H\big((1-\bar{\eta})a+\bar{\eta}b\big) \supseteq \frac{\cos\frac{\pi\bar{\eta}}{2}}{e^{\bar{\eta}}}H(a)+\frac{\sin\frac{\pi\bar{\eta}}{2}}{e^{1-\bar{\eta}}}H(b), \tag{2.10}$$

and

$$H\big(\bar{\eta}a+(1-\bar{\eta})b\big) \supseteq \frac{\sin\frac{\pi\bar{\eta}}{2}}{e^{1-\bar{\eta}}}H(a)+\frac{\cos\frac{\pi\bar{\eta}}{2}}{e^{\bar{\eta}}}H(b), \tag{2.11}$$

for all $\bar{\eta}\in[0,1]$.

Adding (2.10) and (2.11), we derive that

$$H(\bar{\eta}a+(1-\bar{\eta})b)+H\big((1-\bar{\eta})a+\bar{\eta}b\big) \supseteq \left[\frac{\sin\frac{\pi\bar{\eta}}{2}}{e^{1-\bar{\eta}}}+\frac{\cos\frac{\pi\bar{\eta}}{2}}{e^{\bar{\eta}}}\right]\big[H(a)+H(b)\big]. \tag{2.12}$$

Multiplying both sides of (2.12) by $e^{-\rho\bar{\eta}}$, and integrating the resulting inclusion pertaining to $\bar{\eta}$ over $[0,1]$, we deduce that

$$\begin{aligned}
&\int_0^1 e^{-\rho\bar{\eta}}H(\bar{\eta}a+(1-\bar{\eta})b)\mathrm{d}\bar{\eta}+\int_0^1 e^{-\rho\bar{\eta}}H\big((1-\bar{\eta})a+\bar{\eta}b\big)\mathrm{d}\bar{\eta} \\
&\quad\supseteq \int_0^1 e^{-\rho\bar{\eta}}\left[\frac{\sin\frac{\pi\bar{\eta}}{2}}{e^{1-\bar{\eta}}}+\frac{\cos\frac{\pi\bar{\eta}}{2}}{e^{\bar{\eta}}}\right]\big[H(a)+H(b)\big]\mathrm{d}\bar{\eta} \\
&\quad= \left(\frac{4\rho+2\pi e^{-\rho-1}+4}{4\rho^2+8\rho+\pi^2+4}+\frac{2\pi e^{-1}+e^{-\rho-1}(4e-4\rho e)}{4\rho^2-8\rho+\pi^2+4}\right)\big[H(a)+H(b)\big].
\end{aligned} \tag{2.13}$$

Combining (2.8), (2.9) with (2.13), yields the required second relations in (2.4). As a consequence, the proof is completed. □

COROLLARY 2.1. *If we consider certain special cases in Theorem 2.4, then we acquire the succeeding findings.*

(1) *If we consider to take* $\alpha\to 1$, *i.e.* $\rho=\frac{1-\alpha}{\alpha}(b-a)\to 0$, *then we have that*

$$\lim_{\alpha\to 1}\frac{1-\alpha}{2(1-e^{-\rho})}=\frac{1}{2(b-a)},$$

and

$$\lim_{\alpha\to 1}\frac{\rho}{2(1-e^{-\rho})}\left(\frac{4\rho+2\pi e^{-\rho-1}+4}{4\rho^2+8\rho+\pi^2+4}+\frac{2\pi e^{-1}+e^{-\rho-1}(4e-4\rho e)}{4\rho^2-8\rho+\pi^2+4}\right)=\frac{2\pi e^{-1}+4}{\pi^2+4}.$$

Thus, Theorem 2.4 is converted to

$$\sqrt{\frac{e}{2}}H\left(\frac{a+b}{2}\right)\supseteq\frac{1}{b-a}\int_a^b H(x)\mathrm{d}x\supseteq\frac{2\pi e^{-1}+4}{\pi^2+4}\left[H(a)+H(b)\right].$$

(2) *If the functions satisfy the condition* $\underline{H}=\overline{H}$*, then we get the successive fractional Hermite–Hadamard's inequality for exponential trigonometric convex functions*

$$\sqrt{\frac{e}{2}}H\left(\frac{a+b}{2}\right)\leqslant\frac{1-\alpha}{(1-e^{-\rho})}\left[\mathscr{I}_{a^+}^{\alpha}H(b)+\mathscr{I}_{b^-}^{\alpha}H(a)\right]\leqslant\frac{\rho}{1-e^{-\rho}}\mathscr{C}(\rho)\frac{H(a)+H(b)}{2}.$$

(3) *If we take* $\alpha\to 1$ *and* $\underline{H}=\overline{H}$*, then Theorem 2.4 degenerates to Theorem 1.1.*

We give the undermentioned example to help readers understand the outcome obtained in Theorem 2.4.

EXAMPLE 2.2. If we take $H(s)=[2s^2,e^s+1]$, $s\in[0,1]$, $a=0$, $b=1$ and $\alpha=\frac{1}{2}$, then all hypotheses mentioned in Theorem 2.4 meet requirements.

Evidently, $\rho=\frac{1-\alpha}{\alpha}(b-a)=1$. On the one hand, we have that

$$\begin{aligned}
&\frac{1-\alpha}{2(1-e^{-\rho})}\left[\mathscr{I}_{a^+}^{\alpha}H(b)+\mathscr{I}_{b^-}^{\alpha}H(a)\right]\\
&=\frac{1}{2(1-e^{-1})}\left\{\int_0^1 e^{-(1-s)}[2s^2,e^s+1]\mathrm{d}s+\int_0^1 e^{-s}[2s^2,e^s+1]\mathrm{d}s\right\}\\
&=\frac{1}{2(1-e^{-1})}\left\{\left[2-4e^{-1},\frac{e}{2}-\frac{3e^{-1}}{2}+1\right]+\left[4-10e^{-1},2-e^{-1}\right]\right\}\\
&\approx[0.6721,2.7206].
\end{aligned}$$

On the other hand, we get that

$$\sqrt{\frac{e}{2}}H\left(\frac{a+b}{2}\right)=\sqrt{\frac{e}{2}}H\left(\frac{0+1}{2}\right)=\sqrt{\frac{e}{2}}\left[\frac{1}{2},\sqrt{e}+1\right]\approx[0.5829,3.0879],$$

and

$$\begin{aligned}
\frac{\rho}{1-e^{-\rho}}\mathscr{C}(\rho)\frac{H(a)+H(b)}{2}&=\frac{1}{2(1-e^{-1})}\left(\frac{8+2\pi e^{-2}}{16+\pi^2}\right)\left\{[0,2]+[2,1+e]\right\}\\
&\approx[0.9117,2.6067].
\end{aligned}$$

Obviously,

$$[0.5829,3.0879]\supseteq[0.6721,2.7206]\supseteq[0.9117,2.6067],$$

which exhibits the correctness of the findings yielded in Theorem 2.4.

REMARK 2.1. In Example 2.2, if the parameter α is not a fixed constant, that is $\alpha \in (0,1)$, according to Theorem 2.4, then the inclusion relation involving the parameter α is in the following:

$$\begin{aligned} \frac{2(1-e^{-\rho})}{1-\alpha}\sqrt{\frac{e}{2}}H\left(\frac{a+b}{2}\right) &\supseteq \left[\mathscr{I}_{a^+}^{\alpha}H(b)+\mathscr{I}_{b^-}^{\alpha}H(a)\right] \\ &\supseteq \frac{2\rho}{1-\alpha}\mathscr{C}(\rho)\frac{H(a)+H(b)}{2}, \end{aligned} \tag{2.14}$$

where $\mathscr{C}(\rho)$ is defined in (2.5). As a consequence, the left-, middle-, and right-side parts of the inclusion relations above can be acquired.

$$\frac{2(1-e^{-\rho})}{1-\alpha}\sqrt{\frac{e}{2}}H\left(\frac{a+b}{2}\right) = \sqrt{\frac{e}{2}}\left[\frac{1}{2}, 1+\sqrt{e}\right]\frac{2(1-e^{-\rho})}{1-\alpha},$$

$$\left[\mathscr{I}_{a^+}^{\alpha}H(b)+\mathscr{I}_{b^-}^{\alpha}H(a)\right] = \frac{1}{\alpha}\left(\int_0^1 e^{-\frac{1-\alpha}{\alpha}(1-s)}H(s)\mathrm{d}s + \int_0^1 e^{-\frac{1-\alpha}{\alpha}s}H(s)\mathrm{d}s\right),$$

and

$$\frac{2\rho}{1-\alpha}\mathscr{C}(\rho)\frac{H(a)+H(b)}{2} = \left[1, \frac{3+e}{2}\right]\frac{2\rho}{1-\alpha}\mathscr{C}(\rho).$$

Three functions with respect to the variable $\alpha \in (0,1)$, yielded by the inclusions in Theorem 2.4 on the left-, middle- and right-side portions, are plotted in Fig. 2.2. And as can be seen from the Fig. 2.2, the inclusion relations derived in Theorem 2.4 are always valid if the parameter $\alpha \in (0,1)$ is given any value.

The Hermite–Hadamard-type inclusion relations with midpoint can be expressed as interval-valued fractional integrals having exponential kernel in the following.

THEOREM 2.5. *In Theorem 2.4, if we consider the identical hypotheses, then we have the succeeding fractional integral inclusion relations:*

$$\begin{aligned} \sqrt{\frac{e}{2}}H\left(\frac{a+b}{2}\right) &\supseteq \frac{1-\alpha}{2(1-e^{-\frac{\rho}{2}})}\left[\mathscr{I}_{(\frac{a+b}{2})^+}^{\alpha}H(b)+\mathscr{I}_{(\frac{a+b}{2})^-}^{\alpha}H(a)\right] \\ &\supseteq \frac{\rho}{1-e^{-\frac{\rho}{2}}}\mathscr{E}(\rho)\frac{H(a)+H(b)}{2}, \end{aligned} \tag{2.15}$$

where

$$\begin{aligned} \mathscr{E}(\rho) = &\frac{8\rho+2^{\frac{3}{2}}\pi e^{-\frac{\rho+1}{2}}-2^{\frac{5}{2}}(\rho+1)e^{-\frac{\rho+1}{2}}+8}{4\rho^2+8\rho+\pi^2+4} \\ &+\frac{\pi(4e^{-1}-2^{\frac{3}{2}}e^{-\frac{\rho+1}{2}})-2^{\frac{5}{2}}\rho e^{-\frac{\rho+1}{2}}+2^{\frac{5}{2}}e^{-\frac{\rho+1}{2}}}{4\rho^2-8\rho+\pi^2+4}. \end{aligned}$$

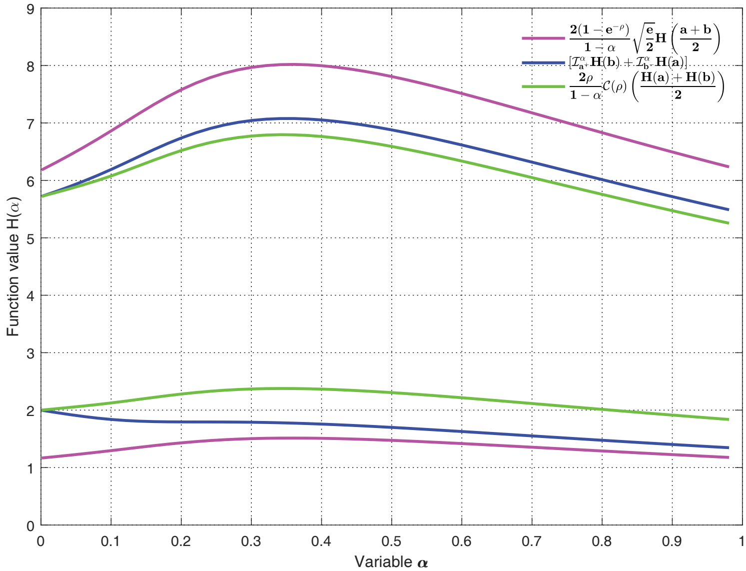

Figure 2.2: *Graphical representation for Theorem 2.4*

Proof. Starting from the inclusion relation (2.6) in Theorem 2.4 again, if we consider to let $x = \frac{\bar{\eta}}{2}a + \frac{2-\bar{\eta}}{2}b$ and $y = \frac{2-\bar{\eta}}{2}a + \frac{\bar{\eta}}{2}b, \bar{\eta} \in [0,1]$, then we acquire that

$$\begin{aligned} H\left(\frac{x+y}{2}\right) &= H\left(\frac{a+b}{2}\right) \\ &\supseteq \frac{1}{2}\sqrt{\frac{2}{e}}\left[H\left(\frac{\bar{\eta}}{2}a+\frac{2-\bar{\eta}}{2}b\right)+H\left(\frac{2-\bar{\eta}}{2}a+\frac{\bar{\eta}}{2}b\right)\right]. \end{aligned} \tag{2.16}$$

Multiplying on each side of (2.16) by $e^{-\frac{\rho}{2}\bar{\eta}}$, and integrating the derived outcomes with regard to $\bar{\eta}$ from 0 to 1, we deduce that

$$\begin{aligned} &\int_0^1 e^{-\frac{\rho}{2}\bar{\eta}} H\left(\frac{a+b}{2}\right) \mathrm{d}\bar{\eta} \\ &\quad \supseteq \frac{1}{2}\sqrt{\frac{2}{e}} \int_0^1 e^{-\frac{\rho}{2}\bar{\eta}} H\left(\frac{\bar{\eta}}{2}a+\frac{2-\bar{\eta}}{2}b\right) \mathrm{d}\bar{\eta} + \int_0^1 e^{-\frac{\rho}{2}\bar{\eta}} H\left(\frac{2-\bar{\eta}}{2}a+\frac{\bar{\eta}}{2}b\right) \mathrm{d}\bar{\eta} \\ &\quad = \frac{1}{2}\sqrt{\frac{2}{e}}\left[\int_0^1 e^{-\frac{\rho}{2}\bar{\eta}}\left(\underline{H}\left(\frac{\bar{\eta}}{2}a+\frac{2-\bar{\eta}}{2}b\right)+\underline{H}\left(\frac{2-\bar{\eta}}{2}a+\frac{\bar{\eta}}{2}b\right)\right)\mathrm{d}\bar{\eta},\right. \\ &\qquad\qquad \left.\int_0^1 e^{-\frac{\rho}{2}\bar{\eta}}\left(\overline{H}\left(\frac{\bar{\eta}}{2}a+\frac{2-\bar{\eta}}{2}b\right)+\overline{H}\left(\frac{2-\bar{\eta}}{2}a+\frac{\bar{\eta}}{2}b\right)\right)\mathrm{d}\bar{\eta}\right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\sqrt{\frac{2}{e}}\Bigg[\frac{2}{b-a}\left(\int_{\frac{a+b}{2}}^{b} e^{-\frac{1-\alpha}{\alpha}(b-u)}\underline{H}(u)\mathrm{d}u + \int_{a}^{\frac{a+b}{2}} e^{-\frac{1-\alpha}{\alpha}(v-a)}\underline{H}(v)\mathrm{d}v\right),\\
&\qquad \frac{2}{b-a}\left(\int_{\frac{a+b}{2}}^{b} e^{-\frac{1-\alpha}{\alpha}(b-u)}\overline{H}(u)\mathrm{d}u + \int_{a}^{\frac{a+b}{2}} e^{-\frac{1-\alpha}{\alpha}(v-a)}\overline{H}(v)\mathrm{d}v\right)\Bigg]\\
&= \sqrt{\frac{2}{e}}\frac{\alpha}{b-a}\left[\mathscr{I}^{\alpha}_{(\frac{a+b}{2})^+}H(b) + \mathscr{I}^{\alpha}_{(\frac{a+b}{2})^-}H(a)\right].
\end{aligned}
$$

That is,

$$
H\left(\frac{a+b}{2}\right) \supseteq \sqrt{\frac{2}{e}}\frac{(1-\alpha)}{2(1-e^{-\frac{\rho}{2}})}\left[\mathscr{I}^{\alpha}_{(\frac{a+b}{2})^+}H(b) + \mathscr{I}^{\alpha}_{(\frac{a+b}{2})^-}H(a)\right],
$$

which proves the first inclusion relation in (2.15).

For the second inclusion relation (2.15), by means of the exponential trigonometric convexity of the interval-valued function H, we find that

$$
H\left(\frac{\bar{\eta}}{2}a + \frac{2-\bar{\eta}}{2}b\right) \supseteq \frac{\sin\frac{\pi\bar{\eta}}{4}}{e^{1-\frac{\bar{\eta}}{2}}}H(a) + \frac{\cos\frac{\pi\bar{\eta}}{4}}{e^{\frac{\bar{\eta}}{2}}}H(b), \tag{2.17}
$$

and

$$
H\left(\frac{2-\bar{\eta}}{2}a + \frac{\bar{\eta}}{2}b\right) \supseteq \frac{\cos\frac{\pi\bar{\eta}}{4}}{e^{\frac{\bar{\eta}}{2}}}H(a) + \frac{\sin\frac{\pi\bar{\eta}}{4}}{e^{1-\frac{\bar{\eta}}{2}}}H(b), \tag{2.18}
$$

for all $\bar{\eta} \in [0,1]$.

Adding the relations (2.17) and (2.18), we derive that

$$
H\left(\frac{\bar{\eta}}{2}a + \frac{2-\bar{\eta}}{2}b\right) + H\left(\frac{2-\bar{\eta}}{2}a + \frac{\bar{\eta}}{2}b\right) \supseteq \left[\frac{\sin\frac{\pi\bar{\eta}}{4}}{e^{1-\frac{\bar{\eta}}{2}}} + \frac{\cos\frac{\pi\bar{\eta}}{4}}{e^{\frac{\bar{\eta}}{2}}}\right][H(a)+H(b)]. \tag{2.19}
$$

Multiplying both sides of (2.19) by $e^{-\frac{\rho}{2}\bar{\eta}}$, and integrating the presented findings pertaining to $\bar{\eta}$ over $[0,1]$, we have that

$$
\begin{aligned}
&\int_0^1 e^{-\frac{\rho}{2}\bar{\eta}}H\left(\frac{\bar{\eta}}{2}a + \frac{2-\bar{\eta}}{2}b\right)\mathrm{d}\bar{\eta} + \int_0^1 e^{-\frac{\rho}{2}\bar{\eta}}H\left(\frac{2-\bar{\eta}}{2}a + \frac{\bar{\eta}}{2}b\right)\mathrm{d}\bar{\eta}\\
&\quad \supseteq \int_0^1 e^{-\frac{\rho}{2}\bar{\eta}}\left[\frac{\sin\frac{\pi\bar{\eta}}{4}}{e^{1-\frac{\bar{\eta}}{2}}} + \frac{\cos\frac{\pi\bar{\eta}}{4}}{e^{\frac{\bar{\eta}}{2}}}\right][H(a)+H(b)]\mathrm{d}\bar{\eta}\\
&\quad = \mathscr{E}(\rho)\big[H(a)+H(b)\big],
\end{aligned}
$$

which is equivalent to

$$
\sqrt{\frac{2}{e}}\frac{(1-\alpha)}{2(1-e^{-\frac{\rho}{2}})}\left[\mathscr{I}^{\alpha}_{(\frac{a+b}{2})^+}H(b) + \mathscr{I}^{\alpha}_{(\frac{a+b}{2})^-}H(a)\right] \supseteq \sqrt{\frac{2}{e}}\frac{\rho}{2(1-e^{-\frac{\rho}{2}})}\mathscr{E}(\rho)\frac{H(a)+H(b)}{2},
$$

where,

$$\begin{aligned}
\mathscr{E}(\rho) &:= \int_0^1 e^{-\frac{\rho}{2}\bar{\eta}} \left[\frac{\sin\frac{\pi\bar{\eta}}{4}}{e^{1-\frac{\bar{\eta}}{2}}} + \frac{\cos\frac{\pi\bar{\eta}}{4}}{e^{\frac{\bar{\eta}}{2}}} \right] \mathrm{d}\bar{\eta} \\
&= \frac{8+8\rho+2^{\frac{3}{2}}\pi e^{-\frac{\rho+1}{2}} - 2^{\frac{5}{2}}(\rho+1)e^{-\frac{\rho+1}{2}}}{4\rho^2+8\rho+\pi^2+4} \\
&\quad + \frac{\pi(4e^{-1}-2^{\frac{3}{2}}e^{-\frac{\rho+1}{2}}) - 2^{\frac{5}{2}}\rho e^{-\frac{\rho+1}{2}} + 2^{\frac{5}{2}}e^{-\frac{\rho+1}{2}}}{4\rho^2-8\rho+\pi^2+4}.
\end{aligned}$$

Consequently, the second inclusion relation in Theorem 2.5 is proved. This concludes the proof. □

COROLLARY 2.2. *If we consider some special cases in Theorem 2.5, then we acquire the coming outcomes.*

(1) *If we consider to take* $\alpha \to 1$*, that is* $\rho \to 0$*,*

$$\lim_{\alpha\to 1} \frac{1-\alpha}{2(1-e^{-\frac{\rho}{2}})} = \frac{1}{b-a},$$

and

$$\lim_{\alpha\to 1} \frac{\rho}{2(1-e^{-\frac{\rho}{2}})}\mathscr{E}(\rho) = \frac{4\pi+8e}{\pi^2 e+4e},$$

then we have that

$$\sqrt{\frac{e}{2}} H\left(\frac{a+b}{2}\right) \supseteq \frac{1}{b-a}\int_a^b H(x)\mathrm{d}x \supseteq \frac{2\pi+4e}{\pi^2 e+4e}\left[H(a)+H(b)\right].$$

(2) *If the functions satisfy the condition* $\underline{H} = \overline{H}$*, then we have the succeeding fractional Hermite–Hadamard's inequality involving midpoint for exponential trigonometric convex functions*

$$\begin{aligned}
\sqrt{\frac{e}{2}} H\left(\frac{a+b}{2}\right) &\leqslant \frac{1-\alpha}{2(1-e^{-\frac{\rho}{2}})}\left[\mathscr{I}^{\alpha}_{(\frac{a+b}{2})^+}H(b) + \mathscr{I}^{\alpha}_{(\frac{a+b}{2})^-}H(a)\right] \\
&\leqslant \frac{\rho}{2(1-e^{-\frac{\rho}{2}})}\mathscr{E}(\rho)\frac{H(a)+H(b)}{2}.
\end{aligned}$$

(3) *If we take* $\alpha \to 1$ *and* $\underline{H} = \overline{H}$*, then Theorem 2.5 converts to Theorem 1.1.*

Taking advantage of the interval-valued exponential trigonometric convexity again, we put forward the following Pachpatte-type fractional integral inclusions,

THEOREM 2.6. *Suppose that the mappings* $H, F : [a,b] \to \mathbb{R}_I^+$ *are both two interval-valued exponential trigonometric convex functions together with* $a < b$ *such that* $H(x) =$

$\left[\underline{H}(x),\overline{H}(x)\right]$ as well as $F(x)=\left[\underline{F}(x),\overline{F}(x)\right]$. Then the coming inclusion relation holds true:

$$\frac{\alpha}{b-a}\left[\mathscr{I}_{a^+}^{\alpha}H(b)F(b)+\mathscr{I}_{b^-}^{\alpha}H(a)F(a)\right]\supseteq P(a,b)\mathscr{D}(\rho)+Q(a,b)\frac{\pi e^{-\rho-1}(e^{\rho}+1)}{\rho^2+\pi^2}, \tag{2.20}$$

where

$$P(a,b)=H(a)F(a)+H(b)F(b),$$

$$Q(a,b)=H(a)F(b)+H(b)F(a),$$

and

$$\mathscr{D}(\rho)=\frac{8\rho-\pi^2e^{-\rho-2}+\pi^2+2\rho^2+8}{2(\rho+2)(\rho^2+4\rho+\pi^2+4)}-\frac{e^{-\rho-2}(8e^2-8\rho e^2+\pi^2e^2+2\rho^2e^2-\pi^2e^{\rho})}{2(\rho-2)(\rho^2-4\rho+\pi^2+4)}.$$

Proof. Applying the exponential trigonometric convexities of the interval-valued functions H and F, it yields that

$$H\big((1-\bar{\eta})a+\bar{\eta}b\big)\supseteq\frac{\cos\frac{\pi\bar{\eta}}{2}}{e^{\bar{\eta}}}H(a)+\frac{\sin\frac{\pi\bar{\eta}}{2}}{e^{1-\bar{\eta}}}H(b), \tag{2.21}$$

and

$$F\big((1-\bar{\eta})a+\bar{\eta}b\big)\supseteq\frac{\cos\frac{\pi\bar{\eta}}{2}}{e^{\bar{\eta}}}F(a)+\frac{\sin\frac{\pi\bar{\eta}}{2}}{e^{1-\bar{\eta}}}F(b). \tag{2.22}$$

Multiplying both sides of (2.21) with corresponding terms of (2.22), and noticing that all these parts are non-negative, it derives that

$$\begin{aligned}&H\big((1-\bar{\eta})a+\bar{\eta}b\big)F\big((1-\bar{\eta})a+\bar{\eta}b\big)\\&\supseteq\left(\frac{\cos\frac{\pi\bar{\eta}}{2}}{e^{\bar{\eta}}}\right)^2H(a)F(a)+\left(\frac{\sin\frac{\pi\bar{\eta}}{2}}{e^{1-\bar{\eta}}}\right)^2H(b)F(b)\\&\quad+\frac{\sin\frac{\pi\bar{\eta}}{2}\cos\frac{\pi\bar{\eta}}{2}}{e}\left[H(a)F(b)+H(b)F(a)\right].\end{aligned} \tag{2.23}$$

Analogously, we get that

$$\begin{aligned}&H\big(\bar{\eta}a+(1-\bar{\eta})b\big)F\big(\bar{\eta}a+(1-\bar{\eta})b\big)\\&\supseteq\left(\frac{\sin\frac{\pi\bar{\eta}}{2}}{e^{1-\bar{\eta}}}\right)^2H(a)F(a)+\left(\frac{\cos\frac{\pi\bar{\eta}}{2}}{e^{\bar{\eta}}}\right)^2H(b)F(b)\\&\quad+\frac{\sin\frac{\pi\bar{\eta}}{2}\cos\frac{\pi\bar{\eta}}{2}}{e}\left[H(a)F(b)+H(b)F(a)\right].\end{aligned} \tag{2.24}$$

Adding (2.23) and (2.24), we deduce that

$$\begin{aligned} &H\big(\bar{\eta}a+(1-\bar{\eta})b\big)F\big(\bar{\eta}a+(1-\bar{\eta})b\big)+H\big((1-\bar{\eta})a+\bar{\eta}b\big)F\big((1-\bar{\eta})a+\bar{\eta}b\big)\\ &\quad\supseteq\left[\left(\frac{\sin\frac{\pi\bar{\eta}}{2}}{e^{1-\bar{\eta}}}\right)^2+\left(\frac{\cos\frac{\pi\bar{\eta}}{2}}{e^{\bar{\eta}}}\right)^2\right]\big[H(a)F(a)+H(b)F(b)\big]\\ &\qquad+\frac{2\sin\frac{\pi\bar{\eta}}{2}\cos\frac{\pi\bar{\eta}}{2}}{e}\big[H(a)F(b)+H(b)F(a)\big]. \end{aligned} \tag{2.25}$$

Multiplying both sides of (2.25) by $e^{-\rho\bar{\eta}}$, and integrating obtained outcomes pertaining to $\bar{\eta}$ over $[0,1]$, it acquires that

$$\begin{aligned} &\int_0^1 e^{-\rho\bar{\eta}}H\big(\bar{\eta}a+(1-\bar{\eta})b\big)F\big(\bar{\eta}a+(1-\bar{\eta})b\big)\mathrm{d}\bar{\eta}\\ &\quad+\int_0^1 e^{-\rho\bar{\eta}}H\big((1-\bar{\eta})a+\bar{\eta}b\big)F\big((1-\bar{\eta})a+\bar{\eta}b\big)\mathrm{d}\bar{\eta}\\ &\supseteq\int_0^1 e^{-\rho\bar{\eta}}\left[\left(\frac{\sin\frac{\pi\bar{\eta}}{2}}{e^{1-\bar{\eta}}}\right)^2+\left(\frac{\cos\frac{\pi\bar{\eta}}{2}}{e^{\bar{\eta}}}\right)^2\right]P(a,b)\mathrm{d}\bar{\eta}\\ &\quad+\int_0^1 e^{-\rho\bar{\eta}}\frac{2\sin\frac{\pi\bar{\eta}}{2}\cos\frac{\pi\bar{\eta}}{2}}{e}Q(a,b)\mathrm{d}\bar{\eta}. \end{aligned} \tag{2.26}$$

In the light of Definition 1.4, we have that

$$\int_0^1 e^{-\rho\bar{\eta}}H\big(\bar{\eta}a+(1-\bar{\eta})b\big)F\big(\bar{\eta}a+(1-\bar{\eta})b\big)\mathrm{d}\bar{\eta}=\frac{\alpha}{b-a}\mathscr{I}^{\alpha}_{a^+}H(b)F(b), \tag{2.27}$$

and

$$\int_0^1 e^{-\rho\bar{\eta}}H\big((1-\bar{\eta})a+\bar{\eta}b\big)F\big((1-\bar{\eta})a+\bar{\eta}b\big)\mathrm{d}\bar{\eta}=\frac{\alpha}{b-a}\mathscr{I}^{\alpha}_{b^-}H(a)F(a). \tag{2.28}$$

It is easy to check that

$$\begin{aligned} &\int_0^1 e^{-\rho\bar{\eta}}\left[\left(\frac{\sin\frac{\pi\bar{\eta}}{2}}{e^{1-\bar{\eta}}}\right)^2+\left(\frac{\cos\frac{\pi\bar{\eta}}{2}}{e^{\bar{\eta}}}\right)^2\right]\mathrm{d}\bar{\eta}\\ &\quad=\frac{8\rho-\pi^2e^{-\rho-2}+\pi^2+2\rho+8}{2(\rho+2)(\rho^2+4\rho+\pi^2+4)}-\frac{e^{-\rho-2}(8e^2-8\rho e^2+\pi^2e^2+2\rho^2e^2-\pi^2e^{\rho})}{2(\rho-2)(\rho^2-4\rho+\pi^2+4)}, \end{aligned} \tag{2.29}$$

and

$$\int_0^1 e^{-\rho\bar{\eta}}\frac{2\sin\frac{\pi\bar{\eta}}{2}\cos\frac{\pi\bar{\eta}}{2}}{e}\mathrm{d}\bar{\eta}=\frac{\pi e^{-\rho-1}(e^{\rho}+1)}{\rho^2+\pi^2}. \tag{2.30}$$

Substituting (2.27)–(2.30) into (2.26), it yields the required findings. This concludes the proof. □

COROLLARY 2.3. *If we consider certain special cases in Theorem 2.6, then we acquire the following outcomes.*

(1) *If we let* $\alpha \to 1$, *that is* $\rho \to 0$, *then we get that*

$$\lim_{\alpha\to 1}\left[\frac{8\rho-\pi^2e^{-\rho-2}+\pi^2+2\rho+8}{2(\rho+2)(\rho^2+4\rho+\pi^2+4)}-\frac{e^{-\rho-2}(8e^2-8\rho e^2+\pi^2e^2+2\rho^2e^2-\pi^2e^{\rho})}{2(\rho-2)(\rho^2-4\rho+\pi^2+4)}\right]$$
$$=\frac{\pi^2-\pi^2e^{-2}+8}{2(\pi^2+4)},$$

and

$$\lim_{\alpha\to 1}\frac{\pi e^{-\rho-1}(e^{\rho}+1)}{\rho^2+\pi^2}=\frac{2}{\pi e}.$$

Thus, Theorem 2.6 is transformed to

$$\frac{1}{b-a}\int_a^b H(x)F(x)\mathrm{d}x \supseteq \frac{\pi^2-\pi^2e^{-2}+8}{4(\pi^2+4)}\big[H(a)F(a)+H(b)F(b)\big]$$
$$+\frac{1}{\pi e}\big[H(a)F(b)+H(b)F(a)\big].$$

(2) *If we consider to let* $\underline{H}=\overline{H}$ *and* $\underline{F}=\overline{F}$, *then we have the succeeding fractional Pachpatte-type integral inequality together with exponential trigonometric convex functions*

$$\frac{\alpha}{b-a}\big[\mathscr{I}^{\alpha}_{a^+}H(b)F(b)+\mathscr{I}^{\alpha}_{b^-}H(a)F(a)\big]\leqslant P(a,b)\mathscr{D}(\rho)+Q(a,b)\frac{\pi e^{-\rho-1}(e^{\rho}+1)}{\rho^2+\pi^2}.$$

(3) *If we take* $\alpha\to 1$, $\underline{H}=\overline{H}$ *and* $\underline{F}=\overline{F}$, *then we have the following Pachpatte-type integral inequality for exponential trigonometric convex functions*

$$\frac{1}{b-a}\int_a^b H(x)F(x)\mathrm{d}x \leqslant \frac{\pi^2-\pi^2e^{-2}+8}{4(\pi^2+4)}\big[H(a)F(a)+H(b)F(b)\big]$$
$$+\frac{1}{\pi e}\big[H(a)F(b)+H(b)F(a)\big].$$

To help readers understanding the result established in Theorem 2.6, we provide the following example.

EXAMPLE 2.3. Let $H(s)=[s^2,s+1]$ and $F(s)=[2s^3,e^s+1]$, $s\in[0,1]$. If we take $a=0$, $b=1$, and $\alpha=\frac{1}{4}$, *i.e.* $\rho=\frac{1-\alpha}{\alpha}(b-a)=3$, then all hypotheses considered in Theorem 2.6 meet requirements.

On the one hand, the left-side part of (2.20) is

$$\begin{aligned}
&\frac{\alpha}{b-a}\left[\mathscr{I}_{a^+}^{\alpha}H(b)F(b)+\mathscr{I}_{b^-}^{\alpha}H(a)F(a)\right]\\
&=\int_0^1 e^{-3(1-s)}[s^2,s+1][2s^3,e^s+1]\mathrm{d}s+\int_0^1 e^{-3s}[s^2,s+1][2s^3,e^s+1]\mathrm{d}s\\
&=\left[\frac{80e^{-3}}{243}+\frac{52}{243},\frac{7e}{16}-\frac{59e^3}{144}+\frac{5}{9}\right]+\left[\frac{80}{243}-\frac{1472e^{-3}}{243},\frac{43}{36}-\frac{7e^{-3}}{9}-\frac{5e^{-2}}{4}\right]\\
&\approx[0.2580,2.711].
\end{aligned}$$

On the other hand, for the right-side part of (2.20), it follows that

$$\begin{aligned}
&P(a,b)\mathscr{D}(\rho)+Q(a,b)\frac{\pi e^{-\rho-1}(e^{\rho}+1)}{\rho^2+\pi^2}\\
&=[2,2+2(1+e)]\left(\frac{50-\pi^2e^{-5}+\pi^2}{10(25+\pi^2)}-\frac{e^{-5}(2e^2+\pi^2e^2-\pi^2e^3)}{2(1+\pi^2)}\right)\\
&\quad+[0,5+e]\frac{\pi e^{-4}(e^3+1)}{9+\pi^2}\\
&\approx[0.4115,2.4380].
\end{aligned}$$

Evidently,

$$[0.2580,2.711]\supseteq[0.4115,2.4380],$$

which elucidates the correctness of the outcomes proposed in Theorem 2.6.

REMARK 2.2. In Example 2.3, if the parameter α is not a fixed constant, that is $\alpha\in(0,1)$, according to Theorem 2.6, then the inclusion relation concerning the parameter α is in the following:

$$\begin{aligned}
&\frac{\alpha}{b-a}\left[\mathscr{I}_{a^+}^{\alpha}H(b)F(b)+\mathscr{I}_{b^-}^{\alpha}H(a)F(a)\right]\\
&=\int_0^1\left(e^{-\frac{1-\alpha}{\alpha}(1-s)}+e^{-\frac{1-\alpha}{\alpha}s}\right)[s^2,s+1][2s^3,e^s+1]\mathrm{d}s,
\end{aligned}$$

$$\begin{aligned}
&P(a,b)\mathscr{D}(\rho)+Q(a,b)\frac{\pi e^{-\rho-1}(e^{\rho}+1)}{\rho^2+\pi^2}\\
&=[2,2+2(1+e)]\\
&\times\left(\frac{8\rho-\pi^2e^{-\rho-2}+\pi^2+2\rho^2+8}{2(\rho+2)(\rho^2+4\rho+\pi^2+4)}-\frac{e^{-\rho-2}(8e^2-8\rho e^2+\pi^2e^2+2\rho^2e^2-\pi^2e^{\rho})}{2(\rho-2)(\rho^2-4\rho+\pi^2+4)}\right)\\
&+[0,5+e]\frac{\pi e^{-4}(e^3+1)}{9+\pi^2}.
\end{aligned}$$

Two functions with regard to the variable $\alpha\in(0,1)$, deduced by the inclusions in Theorem 2.6 on the left- and right-side portions, are plotted in Fig. 2.3. And as can be seen from the Fig. 2.3, the inclusion relations derived in Theorem 2.6 are always valid if the parameter $\alpha\in(0,1)$ is given for all value.

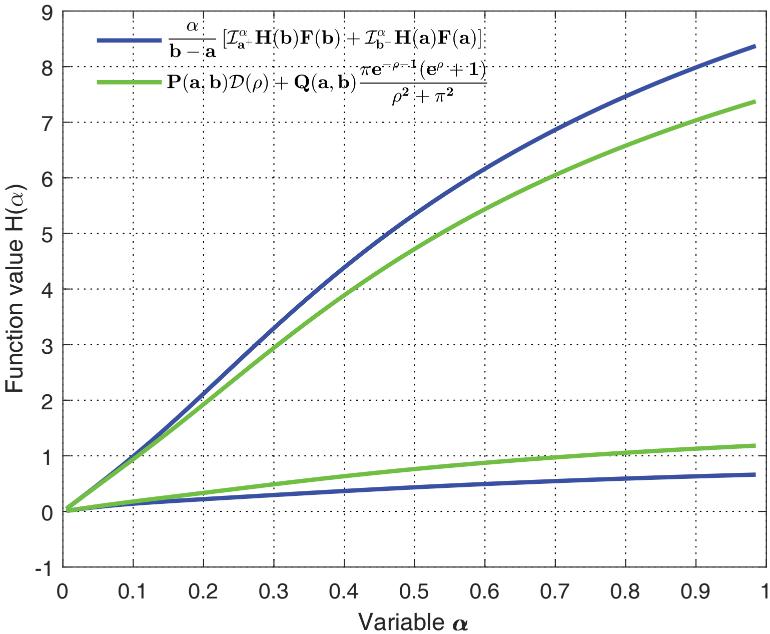

Figure 2.3: *Graphical representation for Theorem 2.6*

THEOREM 2.7. *In Theorem 2.6, if we consider the identical hypotheses, then we have the successive fractional integral inclusion relations:*

$$2H\left(\frac{a+b}{2}\right)F\left(\frac{a+b}{2}\right)\supseteq\frac{1-\alpha}{e(1-e^{-\rho})}\left[\mathscr{I}^{\alpha}_{a^+}H(b)F(b)+\mathscr{I}^{\alpha}_{b^-}H(a)F(a)\right]$$
$$+P(a,b)\frac{\pi e^{-\rho-1}(\rho e^{\rho}+\rho)}{(e-e^{-\rho+1})(\rho^2+\pi^2)}+Q(a,b)\frac{\rho}{e(1-e^{-\rho})}\mathscr{D}(\rho), \tag{2.31}$$

where $P(a,b)$, $Q(a,b)$ and $\mathscr{D}(\rho)$ are defined as in Theorem 2.6.

Proof. For $\bar{\eta}\in[0,1]$, we have that

$$\frac{a+b}{2}=\frac{(1-\bar{\eta})a+\bar{\eta}b}{2}+\frac{\bar{\eta}a+(1-\bar{\eta})b}{2}.$$

Since H, F are both two nonnegative interval-valued exponential trigonometric convex functions, we derive that

$$H\left(\frac{a+b}{2}\right)F\left(\frac{a+b}{2}\right)$$
$$=H\left(\frac{(1-\bar{\eta})a+\bar{\eta}b}{2}+\frac{\bar{\eta}a+(1-\bar{\eta})b}{2}\right)F\left(\frac{(1-\bar{\eta})a+\bar{\eta}b}{2}+\frac{\bar{\eta}a+(1-\bar{\eta})b}{2}\right)$$

$$
\begin{aligned}
&\supseteq \left(\frac{\sqrt{2}}{2\sqrt{e}}\right)^2 \left[H((1-\bar{\eta})a+\bar{\eta}b)+H(\bar{\eta}a+(1-\bar{\eta})b)\right] \\
&\quad \times \left[F((1-\bar{\eta})a+\bar{\eta}b)+F(\bar{\eta}a+(1-\bar{\eta})b)\right] \\
&= \frac{1}{2e}\left[H((1-\bar{\eta})a+\bar{\eta}b)F((1-\bar{\eta})a+\bar{\eta}b)+H(\bar{\eta}a+(1-\bar{\eta})b)F(\bar{\eta}a+(1-\bar{\eta})b)\right] \\
&\quad +\frac{1}{2e}\left[H(\bar{\eta}a+(1-\bar{\eta})b)F((1-\bar{\eta})a+\bar{\eta}b)+H((1-\bar{\eta})a+\bar{\eta}b)F(\bar{\eta}a+(1-\bar{\eta})b)\right] \\
&\supseteq \frac{1}{2e}\left[H((1-\bar{\eta})a+\bar{\eta}b)F((1-\bar{\eta})a+\bar{\eta}b)+H(\bar{\eta}a+(1-\bar{\eta})b)F(\bar{\eta}a+(1-\bar{\eta})b)\right] \\
&\quad +\frac{1}{2e}\left\{\frac{2\sin\frac{\pi\bar{\eta}}{2}\cos\frac{\pi\bar{\eta}}{2}}{e}P(a,b)+\left[\left(\frac{\sin\frac{\pi\bar{\eta}}{2}}{e^{1-\bar{\eta}}}\right)^2+\left(\frac{\cos\frac{\pi\bar{\eta}}{2}}{e^{\bar{\eta}}}\right)^2\right]Q(a,b)\right\}. \quad (2.32)
\end{aligned}
$$

Multiplying both sides of (2.32) by $e^{-\rho\bar{\eta}}$, and integrating established findings pertaining to $\bar{\eta}$ over $[0,1]$, it follows that

$$
\begin{aligned}
&\int_0^1 e^{-\rho\bar{\eta}}H\left(\frac{a+b}{2}\right)F\left(\frac{a+b}{2}\right)\mathrm{d}\bar{\eta} \\
&\quad \supseteq \frac{1}{2e}\int_0^1 e^{-\rho\bar{\eta}}\left[H((1-\bar{\eta})a+\bar{\eta}b))F((1-\bar{\eta})a+\bar{\eta}b)\right]\mathrm{d}\bar{\eta} \\
&\qquad +\frac{1}{2e}\int_0^1 e^{-\rho\bar{\eta}}\left[H(\bar{\eta}a+(1-\bar{\eta})b)F(\bar{\eta}a+(1-\bar{\eta})b)\right]\mathrm{d}\bar{\eta} \\
&\qquad +\frac{1}{2e}P(a,b)\int_0^1 e^{-\rho\bar{\eta}}\frac{2\sin\frac{\pi\bar{\eta}}{2}\cos\frac{\pi\bar{\eta}}{2}}{e}\mathrm{d}\bar{\eta} \\
&\qquad +\frac{1}{2e}Q(a,b)\int_0^1 e^{-\rho\bar{\eta}}\left[\left(\frac{\sin\frac{\pi\bar{\eta}}{2}}{e^{1-\bar{\eta}}}\right)^2+\left(\frac{\cos\frac{\pi\bar{\eta}}{2}}{e^{\bar{\eta}}}\right)^2\right]\mathrm{d}\bar{\eta}.
\end{aligned}
$$

As a consequence,

$$
\begin{aligned}
&\frac{1-e^{-\rho}}{\rho}H\left(\frac{a+b}{2}\right)F\left(\frac{a+b}{2}\right) \\
&\quad \supseteq \frac{\alpha}{2e(b-a)}\left[\mathscr{I}^{\alpha}_{a^+}H(b)F(b)+\mathscr{I}^{\alpha}_{b^-}H(a)F(a)\right]+\frac{1}{2e}P(a,b)\frac{\pi e^{-\rho-1}(e^{\rho}+1)}{\rho^2+\pi^2} \\
&\qquad +\frac{1}{2e}Q(a,b)\left[\frac{8\rho-\pi^2e^{-\rho-2}+\pi^2+2\rho+8}{2(\rho+2)(\rho^2+4\rho+\pi^2+4)}\right. \\
&\qquad \left.-\frac{e^{-\rho-2}(8e^2-8\rho e^2+\pi^2e^2+2\rho^2e^2-\pi^2e^{\rho})}{2(\rho-2)(\rho^2-4\rho+\pi^2+4)}\right],
\end{aligned}
$$

which yields the desired finding in (2.31). This ends the proof. □

COROLLARY 2.4. *If we consider several special cases in Theorem 2.7, then we acquire the succeeding findings.*

(1) *If we consider to take* $\alpha \to 1$*, that is* $\rho \to 0$*, then we obtained that*

$$\lim_{\alpha \to 1} \frac{1-\alpha}{e(1-e^{-\rho})} = \frac{1}{e(b-a)},$$

$$\lim_{\alpha \to 1} \frac{\pi e^{-\rho-1}(\rho e^{\rho}+\rho)}{(e-e^{-\rho+1})(\rho^2+\pi^2)} = \frac{2}{\pi e^2},$$

and

$$\lim_{\alpha \to 1} \frac{\rho}{e-e^{-\rho+1}} \left[\frac{8\rho-\pi^2 e^{-\rho-2}+\pi^2+2\rho+8}{2(\rho+2)(\rho^2+4\rho+\pi^2+4)} - \frac{e^{-\rho-2}(8e^2-8\rho e^2+\pi^2 e^2+2\rho e^2-\pi^2 e^{\rho})}{2(\rho-2)(\rho^2-4\rho+\pi^2+4)} \right]$$
$$= \frac{\pi^2-\pi^2 e^{-2}+8}{2e(\pi^2+4)}.$$

Thus, Theorem 2.7 is converted to

$$2H\left(\frac{a+b}{2}\right) F\left(\frac{a+b}{2}\right) \supseteq \frac{2}{e(b-a)} \int_a^b H(x)F(x)\mathrm{d}x + \frac{2}{\pi e^2}\big[H(a)F(a)+H(b)F(b)\big]$$
$$+ \frac{\pi^2-\pi^2 e^{-2}+8}{2e(\pi^2+4)}\big[H(a)F(b)+H(b)F(a)\big].$$

(2) *If the functions satisfy the conditions* $\underline{H} = \overline{H}$ *and* $\underline{F} = \overline{F}$*, then we have the successive fractional Pachpatte-type integral inequality for exponential trigonometric convex functions*

$$2H\left(\frac{a+b}{2}\right) F\left(\frac{a+b}{2}\right) \leqslant \frac{1-\alpha}{e(1-e^{-\rho})}\big[\mathscr{I}_{a^+}^{\alpha} H(b)F(b) + \mathscr{I}_{b^-}^{\alpha} H(a)F(a)\big]$$
$$+ P(a,b)\frac{\pi e^{-\rho-1}(\rho e^{\rho}+\rho)}{(e-e^{-\rho+1})(\rho^2+\pi^2)} + Q(a,b)\frac{\rho}{e(1-e^{-\rho})}\mathscr{D}(\rho).$$

(3) *If we take* $\alpha \to 1$*,* $\underline{H} = \overline{H}$ *and* $\underline{F} = \overline{F}$*, then we have the coming Pachpatte-type integral inequality for exponential trigonometric convex functions*

$$2H\left(\frac{a+b}{2}\right) F\left(\frac{a+b}{2}\right) \leqslant \frac{2}{e(b-a)} \int_a^b H(x)F(x)\mathrm{d}x + \frac{2}{\pi e^2}\big[H(a)F(a)+H(b)F(b)\big]$$
$$+ \frac{\pi^2-\pi^2 e^{-2}+8}{2e(\pi^2+4)}\big[H(a)F(b)+H(b)F(a)\big].$$

We finally establish the following interesting outcomes, which concern multiple fractional integral inclusion relations pertaining to the interval-valued exponential trigonometric convex functions.

THEOREM 2.8. *Assume that* $H:[a,b]\to\mathbb{R}_I^+$ *is an interval-valued exponential trigonometric convex function together with* $a<b$ *and* $H(x)=\left[\underline{H}(x),\overline{H}(x)\right]$. *Then one has the succeeding multiple integral inclusion relations:*

$$\begin{aligned}
eH\left(\frac{a+b}{2}\right) &\supseteq \sqrt{\frac{e}{2}}\left[H\left(\frac{3a+b}{4}\right)+H\left(\frac{a+3b}{4}\right)\right]\\
&\supseteq \frac{1-\alpha}{2(1-e^{-\frac{\rho}{2}})}\left[\mathscr{I}^{\alpha}_{a^+}H\left(\frac{a+b}{2}\right)+\mathscr{I}^{\alpha}_{\left(\frac{a+b}{2}\right)^-}H(a)\right.\\
&\qquad\left.+\mathscr{I}^{\alpha}_{\left(\frac{a+b}{2}\right)^+}H(b)+\mathscr{I}^{\alpha}_{b^-}H\left(\frac{a+b}{2}\right)\right]\\
&\supseteq \frac{\rho}{2(1-e^{-\frac{\rho}{2}})}\mathscr{K}(\rho)\left[\frac{H(a)+H(b)}{2}+H\left(\frac{a+b}{2}\right)\right]\\
&\supseteq \frac{\rho}{2(1-e^{-\frac{\rho}{2}})}\mathscr{K}(\rho)\left(1+\sqrt{\frac{2}{e}}\right)\frac{H(a)+H(b)}{2},
\end{aligned}\tag{2.33}$$

where

$$\mathscr{K}(\rho)=\frac{2\rho+2\pi e^{-\frac{\rho}{2}-1}+4}{\rho^2+4\rho+\pi^2+4}+\frac{2\pi e^{-1}+e^{-\frac{\rho}{2}-1}(4e-2\rho e)}{\rho^2-4\rho+\pi^2+4}.$$

Proof. Taking advantage of the exponential trigonometric convexity of the interval-valued function H defined on $\left[a,\frac{a+b}{2}\right]$, we have that

$$\begin{aligned}
H\left(\frac{3x+y}{4}\right) &= H\left(\frac{x+\frac{x+y}{2}}{2}\right)\\
&\supseteq \frac{\sin\frac{\pi}{4}}{e^{\frac{1}{2}}}H(x)+\frac{\cos\frac{\pi}{4}}{e^{\frac{1}{2}}}H\left(\frac{x+y}{2}\right)\\
&= \frac{1}{2}\sqrt{\frac{2}{e}}\left(H(x)+H\left(\frac{x+y}{2}\right)\right),
\end{aligned}$$

for any $x,y\in\left[a,\frac{a+b}{2}\right]$.

Making use of $x=\bar{\eta}a+(1-\bar{\eta})\frac{a+b}{2}$ and $y=(1-\bar{\eta})a+\bar{\eta}\frac{a+b}{2}$, $\bar{\eta}\in[0,1]$, we derive that

$$\begin{aligned}
H\left(\frac{x+y}{2}\right) &= H\left(\frac{3a+b}{4}\right)\\
&\supseteq \frac{1}{2}\sqrt{\frac{2}{e}}\left[H\left(\bar{\eta}a+(1-\bar{\eta})\frac{a+b}{2}\right)+H\left((1-\bar{\eta})a+\bar{\eta}\frac{a+b}{2}\right)\right].
\end{aligned}\tag{2.34}$$

Multiplying on each side of (2.34) by $e^{-\frac{\rho}{2}\bar{\eta}}$, and integrating obtained findings with regard to $\bar{\eta}$ from 0 to 1, it follows that

$$\begin{aligned}
&\int_0^1 e^{-\frac{\rho}{2}\bar{\eta}} H\left(\frac{3a+b}{4}\right)\mathrm{d}\bar{\eta}\\
&\supseteq \frac{1}{2}\sqrt{\frac{2}{e}}\left[\int_0^1 e^{-\frac{\rho}{2}\bar{\eta}} H\left(\bar{\eta}a+(1-\bar{\eta})\frac{a+b}{2}\right)\mathrm{d}\bar{\eta}\right.\\
&\qquad\left.+\int_0^1 e^{-\frac{\rho}{2}\bar{\eta}} H\left((1-\bar{\eta})a+\bar{\eta}\frac{a+b}{2}\right)\mathrm{d}\bar{\eta}\right]\\
&=\frac{1}{2}\sqrt{\frac{2}{e}}\left[\int_0^1 e^{-\frac{\rho}{2}\bar{\eta}}\left(\underline{H}\left(\bar{\eta}a+(1-\bar{\eta})\frac{a+b}{2}\right)+\underline{H}\left((1-\bar{\eta})a+\bar{\eta}\frac{a+b}{2}\right)\right)\mathrm{d}\bar{\eta},\right.\\
&\qquad\qquad\left.\int_0^1 e^{-\frac{\rho}{2}\bar{\eta}}\left(\overline{H}\left(\bar{\eta}a+(1-\bar{\eta})\frac{a+b}{2}\right)+\overline{H}\left((1-\bar{\eta})a+\bar{\eta}\frac{a+b}{2}\right)\right)\mathrm{d}\bar{\eta}\right]\\
&=\left[\sqrt{\frac{2}{e}}\frac{1}{b-a}\left(\int_a^{\frac{a+b}{2}} e^{-\frac{1-\alpha}{\alpha}\left(\frac{a+b}{2}-u\right)}\underline{H}(u)\mathrm{d}u+\int_a^{\frac{a+b}{2}} e^{-\frac{1-\alpha}{\alpha}(v-a)}\underline{H}(v)\mathrm{d}v\right),\right.\\
&\qquad\left.\sqrt{\frac{2}{e}}\frac{1}{b-a}\left(\int_a^{\frac{a+b}{2}} e^{-\frac{1-\alpha}{\alpha}\left(\frac{a+b}{2}-u\right)}\overline{H}(u)\mathrm{d}u+\int_a^{\frac{a+b}{2}} e^{-\frac{1-\alpha}{\alpha}(v-a)}\overline{H}(v)\mathrm{d}v\right)\right]\\
&=\sqrt{\frac{2}{e}}\frac{\alpha}{b-a}\left[\mathscr{I}^{\alpha}_{a^+}H\left(\frac{a+b}{2}\right)+\mathscr{I}^{\alpha}_{\left(\frac{a+b}{2}\right)^-}H(a)\right].
\end{aligned}\tag{2.35}$$

Also,

$$\begin{aligned}
\int_0^1 e^{-\frac{\rho}{2}\bar{\eta}} H\left(\frac{3a+b}{4}\right)\mathrm{d}\bar{\eta}&=\left[\int_0^1 e^{-\frac{\rho}{2}\bar{\eta}}\underline{H}\left(\frac{3a+b}{4}\right)\mathrm{d}\bar{\eta},\int_0^1 e^{-\frac{\rho}{2}\bar{\eta}}\overline{H}\left(\frac{3a+b}{4}\right)\mathrm{d}\bar{\eta}\right]\\
&=\left[\frac{2\left(1-e^{-\frac{\rho}{2}}\right)}{\rho}\underline{H}\left(\frac{3a+b}{4}\right),\frac{2\left(1-e^{-\frac{\rho}{2}}\right)}{\rho}\overline{H}\left(\frac{3a+b}{4}\right)\right]\\
&=\frac{2\left(1-e^{-\frac{\rho}{2}}\right)}{\rho}H\left(\frac{3a+b}{4}\right).
\end{aligned}\tag{2.36}$$

Employing the exponential trigonometric convexity of the interval-valued function H again, we have that

$$H\left((1-\bar{\eta})a+\bar{\eta}\frac{a+b}{2}\right)\supseteq\frac{\cos\frac{\pi\bar{\eta}}{2}}{e^{\bar{\eta}}}H(a)+\frac{\sin\frac{\pi\bar{\eta}}{2}}{e^{1-\bar{\eta}}}H\left(\frac{a+b}{2}\right),\tag{2.37}$$

and

$$H\left(\bar{\eta}a+(1-\bar{\eta})\frac{a+b}{2}\right) \supseteq \frac{\sin\frac{\pi\bar{\eta}}{2}}{e^{1-\bar{\eta}}}H(a)+\frac{\cos\frac{\pi\bar{\eta}}{2}}{e^{\bar{\eta}}}H\left(\frac{a+b}{2}\right), \tag{2.38}$$

for all $\bar{\eta}\in[0,1]$.

Adding (2.37) and (2.38), it yields that

$$\begin{aligned} &H\left(\bar{\eta}a+(1-\bar{\eta})\frac{a+b}{2}\right)+H\left((1-\bar{\eta})a+\bar{\eta}\frac{a+b}{2}\right) \\ &\quad\supseteq \left[\frac{\sin\frac{\pi\bar{\eta}}{2}}{e^{1-\bar{\eta}}}+\frac{\cos\frac{\pi\bar{\eta}}{2}}{e^{\bar{\eta}}}\right]\left[H(a)+H\left(\frac{a+b}{2}\right)\right]. \end{aligned} \tag{2.39}$$

Multiplying both sides of (2.39) by $e^{-\frac{\rho}{2}\bar{\eta}}$, and integrating deduced outcomes pertaining to $\bar{\eta}$ over $[0,1]$, we derive that

$$\begin{aligned} &\int_0^1 e^{-\frac{\rho}{2}\bar{\eta}}H\left(\bar{\eta}a+(1-\bar{\eta})\frac{a+b}{2}\right)\mathrm{d}\bar{\eta}+\int_0^1 e^{-\frac{\rho}{2}\bar{\eta}}H\left((1-\bar{\eta})a+\bar{\eta}\frac{a+b}{2}\right)\mathrm{d}\bar{\eta} \\ &\quad\supseteq \int_0^1 e^{-\frac{\rho}{2}\bar{\eta}}\left[\frac{\sin\frac{\pi\bar{\eta}}{2}}{e^{1-\bar{\eta}}}+\frac{\cos\frac{\pi\bar{\eta}}{2}}{e^{\bar{\eta}}}\right]\left[H(a)+H\left(\frac{a+b}{2}\right)\right]\mathrm{d}\bar{\eta} \\ &\quad= \left(\frac{2\rho+2\pi e^{-\frac{\rho}{2}-1}+4}{\rho^2+4\rho+\pi^2+4}+\frac{2\pi e^{-1}+e^{-\frac{\rho}{2}-1}(4e-2\rho e)}{\rho^2-4\rho+\pi^2+4}\right) \\ &\qquad\times\left[H(a)+H\left(\frac{a+b}{2}\right)\right]. \end{aligned} \tag{2.40}$$

Combining (2.35), (2.36) and (2.40), we get the following results

$$\begin{aligned} \sqrt{\frac{e}{2}}H\left(\frac{3a+b}{4}\right) &\supseteq \frac{1-\alpha}{2(1-e^{-\frac{\rho}{2}})}\left[\mathscr{I}^{\alpha}_{a^+}H\left(\frac{a+b}{2}\right)+\mathscr{I}^{\alpha}_{(\frac{a+b}{2})^-}H(a)\right] \\ &\supseteq \frac{\rho}{2(1-e^{-\frac{\rho}{2}})}\left(\frac{2\rho+2\pi e^{-\frac{\rho}{2}-1}+4}{\rho^2+4\rho+\pi^2+4}+\frac{2\pi e^{-1}+e^{-\frac{\rho}{2}-1}(4e-2\rho e)}{\rho^2-4\rho+\pi^2+4}\right) \\ &\quad\times\left[\frac{H(a)+H(\frac{a+b}{2})}{2}\right]. \end{aligned} \tag{2.41}$$

Similarly, in terms of the exponential trigonometric convexity of the interval-valued function H defined on $\left[\frac{a+b}{2}, b\right]$, we obtain that

$$\begin{aligned}\sqrt{\frac{e}{2}} H\left(\frac{a+3b}{4}\right) &\supseteq \frac{1-\alpha}{2(1-e^{-\frac{\rho}{2}})}\left[\mathscr{I}^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}} H(b) + \mathscr{I}^{\alpha}_{b^{-}} H\left(\frac{a+b}{2}\right)\right] \\ &\supseteq \frac{\rho}{2(1-e^{-\frac{\rho}{2}})}\left(\frac{2\rho + 2\pi e^{-\frac{\rho}{2}-1} + 4}{\rho^2 + 4\rho + \pi^2 + 4} + \frac{2\pi e^{-1} + e^{-\frac{\rho}{2}-1}(4e - 2\rho e)}{\rho^2 - 4\rho + \pi^2 + 4}\right) \\ &\quad \times \left[\frac{H\left(\frac{a+b}{2}\right) + H(b)}{2}\right].\end{aligned} \tag{2.42}$$

Using the relations (2.41) and (2.42), it follows that

$$\begin{aligned}&\sqrt{\frac{e}{2}}\left[H\left(\frac{3a+b}{4}\right) + H\left(\frac{a+3b}{4}\right)\right] \\ &\supseteq \frac{1-\alpha}{2(1-e^{-\frac{\rho}{2}})}\left[\mathscr{I}^{\alpha}_{a^{+}} H\left(\frac{a+b}{2}\right) + \mathscr{I}^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}} H(a) + \mathscr{I}^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}} H(b) + \mathscr{I}^{\alpha}_{b^{-}} H\left(\frac{a+b}{2}\right)\right] \\ &\supseteq \frac{\rho}{2(1-e^{-\frac{\rho}{2}})}\mathscr{K}(\rho)\left[\frac{H(a)+H(b)}{2} + H\left(\frac{a+b}{2}\right)\right].\end{aligned} \tag{2.43}$$

This yields the second and third integral inclusion relations in (2.33).

By virtue of the exponential trigonometric convexity of interval-valued function H again, we acquire that

$$\begin{aligned}H\left(\frac{a+b}{2}\right) &= H\left(\frac{\frac{3a+b}{4} + \frac{a+3b}{4}}{2}\right) \\ &\supseteq \frac{\sin\frac{\pi}{4}}{e^{\frac{1}{2}}} H\left(\frac{3a+b}{4}\right) + \frac{\cos\frac{\pi}{4}}{e^{\frac{1}{2}}} H\left(\frac{a+3b}{4}\right) \\ &= \frac{1}{2}\sqrt{\frac{2}{e}}\left[H\left(\frac{3a+b}{4}\right) + H\left(\frac{a+3b}{4}\right)\right],\end{aligned} \tag{2.44}$$

and

$$H\left(\frac{a+b}{2}\right) \supseteq \frac{\sin\frac{\pi}{4}}{e^{\frac{1}{2}}} H(a) + \frac{\cos\frac{\pi}{4}}{e^{\frac{1}{2}}} H(b) = \sqrt{\frac{2}{e}}\frac{H(a)+H(b)}{2}, \tag{2.45}$$

Applying the relations (2.44) and (2.45) to (2.43), we get the first and fourth inclusion relations in (2.33), respectively. Therefore, the proof of Theorem 2.8, is accomplished. □

COROLLARY 2.5. *If we consider some special cases in Theorem 2.8, then we obtain the successive findings.*

(1) *If we consider to take* $\alpha \to 1$, *that is* $\rho = \frac{1-\alpha}{\alpha}(b-a) \to 0$, *then we deduce that*

$$\lim_{\alpha \to 1} \frac{1-\alpha}{2(1-e^{-\frac{\rho}{2}})} = \frac{1}{b-a},$$

and

$$\begin{aligned}
&\lim_{\alpha \to 1} \frac{\rho}{2(1-e^{-\frac{\rho}{2}})} \left(\frac{2\rho + 2\pi e^{-\frac{\rho}{2}-1} + 4}{\rho^2 + 4\rho + \pi^2 + 4} + \frac{2\pi e^{-1} + e^{-\frac{\rho}{2}-1}(4e - 2\rho e)}{\rho^2 - 4\rho + \pi^2 + 4} \right) \\
&\quad = \frac{2(2\pi e^{-1} + 4)}{\pi^2 + 4},
\end{aligned}$$

Thus, Theorem 2.8 is transformed to

$$\begin{aligned}
\frac{e}{2} H\left(\frac{a+b}{2}\right) &\supseteq \frac{1}{2}\sqrt{\frac{e}{2}} \left[H\left(\frac{3a+b}{4}\right) + H\left(\frac{a+3b}{4}\right) \right] \\
&\supseteq \frac{1}{b-a} \int_a^b H(x)\mathrm{d}x \\
&\supseteq \frac{2\pi e^{-1} + 4}{\pi^2 + 4} \left[\frac{H(a) + H(b)}{2} + H\left(\frac{a+b}{2}\right) \right] \\
&\supseteq \frac{2\pi e^{-1} + 4}{\pi^2 + 4} \left(1 + \sqrt{\frac{2}{e}}\right) \frac{H(a) + H(b)}{2},
\end{aligned}$$

which are the refinement results of Theorem 2.4 with $\alpha \to 1$.

(2) *If we consider to let* $\underline{H} = \overline{H}$, *then we get that*

$$\begin{aligned}
eH\left(\frac{a+b}{2}\right) &\leqslant \sqrt{\frac{e}{2}} \left[H\left(\frac{3a+b}{4}\right) + H\left(\frac{a+3b}{4}\right) \right] \\
&\leqslant \frac{1-\alpha}{2(1-e^{-\frac{\rho}{2}})} \left[\mathscr{I}^{\alpha}_{a^+} H\left(\frac{a+b}{2}\right) + \mathscr{I}^{\alpha}_{\left(\frac{a+b}{2}\right)^-} H(a) \right. \\
&\quad \left. + \mathscr{I}^{\alpha}_{\left(\frac{a+b}{2}\right)^+} H(b) + \mathscr{I}^{\alpha}_{b^-} H\left(\frac{a+b}{2}\right) \right] \\
&\leqslant \frac{\rho}{2(1-e^{-\frac{\rho}{2}})} \mathscr{K}(\rho) \left[\frac{H(a) + H(b)}{2} + H\left(\frac{a+b}{2}\right) \right] \\
&\leqslant \frac{\rho}{2(1-e^{-\frac{\rho}{2}})} \mathscr{K}(\rho) \left(1 + \sqrt{\frac{2}{e}}\right) \frac{H(a) + H(b)}{2}.
\end{aligned}$$

(3) *If we take* $\alpha \to 1$ *and* $\underline{H} = \overline{H}$, *then we have that*

$$\begin{aligned}
\frac{e}{2}H\left(\frac{a+b}{2}\right) &\leqslant \frac{1}{2}\sqrt{\frac{e}{2}}\left[H\left(\frac{3a+b}{4}\right)+H\left(\frac{a+3b}{4}\right)\right] \\
&\leqslant \frac{1}{b-a}\int_a^b H(x)\mathrm{d}x \\
&\leqslant \frac{2\pi e^{-1}+4}{\pi^2+4}\left[\frac{H(a)+H(b)}{2}+H\left(\frac{a+b}{2}\right)\right] \\
&\leqslant \frac{2\pi e^{-1}+4}{\pi^2+4}\left(1+\sqrt{\frac{2}{e}}\right)\frac{H(a)+H(b)}{2},
\end{aligned}$$

which are the refinement results with respect to Theorem 1.1.

To end this section, we present the following example to reveal the correctness of the results obtained in Theorem 2.8.

EXAMPLE 2.4. If we consider to take $H(s) = \left[2s^4, s+1\right]$, $s \in [0,1]$, $a = 0$, $b = 1$ and $\alpha = \frac{1}{3}$, then all hypotheses considered in Theorem 2.8 satisfy requirements. One can obtain that

$$eH\left(\frac{a+b}{2}\right) = e\left[\frac{1}{8},\frac{3}{2}\right] \approx [0.3398, 4.0774],$$

$$\begin{aligned}
\sqrt{\frac{e}{2}}\left[H\left(\frac{3a+b}{4}\right)+H\left(\frac{a+3b}{4}\right)\right] &= \sqrt{\frac{e}{2}}\left(\left[\frac{1}{128},\frac{5}{4}\right]+\left[\frac{81}{128},\frac{7}{4}\right]\right) \\
&\approx [0.7469, 3.4975],
\end{aligned}$$

$$\begin{aligned}
&\frac{1-\alpha}{2(1-e^{-\frac{\rho}{2}})}\left[\mathscr{I}^{\alpha}_{a^+}H\left(\frac{a+b}{2}\right)+\mathscr{I}^{\alpha}_{\left(\frac{a+b}{2}\right)^-}H(a)+\mathscr{I}^{\alpha}_{\left(\frac{a+b}{2}\right)^+}H(b)+\mathscr{I}^{\alpha}_{b^-}H\left(\frac{a+b}{2}\right)\right] \\
&= \frac{1}{1-e^{-1}}\left\{\int_0^{\frac{1}{2}} e^{-2\left(\frac{1}{2}-s\right)}[2s^4, s+1]\mathrm{d}s+\int_0^{\frac{1}{2}} e^{-2s}[2s^4, s+1]\mathrm{d}s\right. \\
&\quad \left.+\int_{\frac{1}{2}}^1 e^{-2(1-s)}[2s^4, s+1]\mathrm{d}s+\int_{\frac{1}{2}}^1 e^{-2\left(s-\frac{1}{2}\right)}[2s^4, s+1]\mathrm{d}s\right\} \\
&= \frac{1}{1-e^{-1}}\left[\frac{3e}{2}-\frac{121e^{-1}}{8}+2, 3-3e^{-1}\right] \approx [0.8052, 3.0000],
\end{aligned}$$

$$\begin{aligned}
&\frac{\rho}{2(1-e^{-\frac{\rho}{2}})}\mathscr{K}(\rho)\left[\frac{H(a)+H(b)}{2}+H\left(\frac{a+b}{2}\right)\right] \\
&\quad = \frac{1}{1-e^{-1}}\left(\frac{6+2\pi e^{-2}}{16+\pi^2}+\frac{2e^{-1}}{\pi}\right)\left[\frac{9}{8}, 3\right] \approx [1.0257, 2.7351],
\end{aligned}$$

$$\frac{\rho}{2(1-e^{-\frac{\rho}{2}})}\mathscr{K}(\rho)\left(1+\sqrt{\frac{2}{e}}\right)\frac{H(a)+H(b)}{2}$$
$$=\frac{1}{1-e^{-1}}\left(1+\sqrt{\frac{2}{e}}\right)\left(\frac{6+2\pi e^{-2}}{16+\pi^2}+\frac{2e^{-1}}{\pi}\right)\left[1,\frac{3}{2}\right]\approx[1.6937,2.5406].$$

Therefore,

$$[0.3398,4.0774]\supseteq[0.7469,3.4975]\supseteq[0.8052,3.0000]$$
$$\supseteq[1.0257,2.7351]\supseteq[1.6937,2.5406],$$

which elucidates the correctness of the outcomes presented in Theorem 2.8.

REMARK 2.3. In Example 2.4, if the parameter α is not a fixed constant, that is $\alpha\in(0,1)$, according to Theorem 2.8, then the inclusion relation pertaining to the parameter α is in the following:

$$\sqrt{\frac{e}{2}}\left[H\left(\frac{3a+b}{4}\right)+H\left(\frac{a+3b}{4}\right)\right]$$
$$\supseteq\frac{1-\alpha}{2(1-e^{-\frac{\rho}{2}})}\left[\mathscr{I}^{\alpha}_{a^+}H\left(\frac{a+b}{2}\right)\right.$$
$$\left.+\mathscr{I}^{\alpha}_{\left(\frac{a+b}{2}\right)^-}H(a)+\mathscr{I}^{\alpha}_{\left(\frac{a+b}{2}\right)^+}H(b)+\mathscr{I}^{\alpha}_{b^-}H\left(\frac{a+b}{2}\right)\right]$$
$$\supseteq\frac{\rho}{2(1-e^{-\frac{\rho}{2}})}\mathscr{K}(\rho)\left[\frac{H(a)+H(b)}{2}+H\left(\frac{a+b}{2}\right)\right].$$

Therefore, the left-, middle-, and right-side parts of the above inclusions can be acquired.

$$\frac{2(1-e^{-\frac{\rho}{2}})}{\rho}\sqrt{\frac{e}{2}}\left[H\left(\frac{3a+b}{4}\right)+H\left(\frac{a+3b}{4}\right)\right]=\frac{2(1-e^{-\frac{\rho}{2}})}{\rho}\sqrt{\frac{e}{2}}\left[\frac{41}{64},3\right],$$

$$\frac{\alpha}{b-a}\left[\mathscr{I}^{\alpha}_{a^+}H\left(\frac{a+b}{2}\right)+\mathscr{I}^{\alpha}_{\left(\frac{a+b}{2}\right)^-}H(a)+\mathscr{I}^{\alpha}_{\left(\frac{a+b}{2}\right)^+}H(b)+\mathscr{I}^{\alpha}_{b^-}H\left(\frac{a+b}{2}\right)\right]$$
$$=\left\{\int_0^{\frac{1}{2}}e^{-\frac{1-\alpha}{\alpha}\left(\frac{1}{2}-s\right)}[2s^4,s+1]\mathrm{d}s+\int_0^{\frac{1}{2}}e^{-\frac{1-\alpha}{\alpha}s}[2s^4,s+1]\mathrm{d}s\right.$$
$$\left.+\int_{\frac{1}{2}}^1 e^{-\frac{1-\alpha}{\alpha}(1-s)}[2s^4,s+1]\mathrm{d}s+\int_{\frac{1}{2}}^1 e^{-\frac{1-\alpha}{\alpha}\left(s-\frac{1}{2}\right)}[2s^4,s+1]\mathrm{d}s\right\},$$

and

$$\mathscr{K}(\rho)\left[\frac{H(a)+H(b)}{2}+H\left(\frac{a+b}{2}\right)\right]$$
$$=\left[\frac{9}{8},3\right]\frac{2\rho+2\pi e^{-\frac{\rho}{2}-1}+2}{\rho^2+4\rho+\pi^2+4}+\frac{2\pi e^{-1}+e^{-\frac{\rho}{2}-1}(4e-2\rho e)}{\rho^2-4\rho+\pi^2+4}.$$

Three functions pertaining to the variable $\alpha \in (0,1)$, yielded by the inclusions in Theorem 2.8 on the left-, middle- and right-side portions, are plotted in Fig. 2.4. And as can be seen from the Fig. 2.4, the inclusion relations deduced in Theorem 2.8 are always valid if the parameter $\alpha \in (0,1)$ is given for all value.

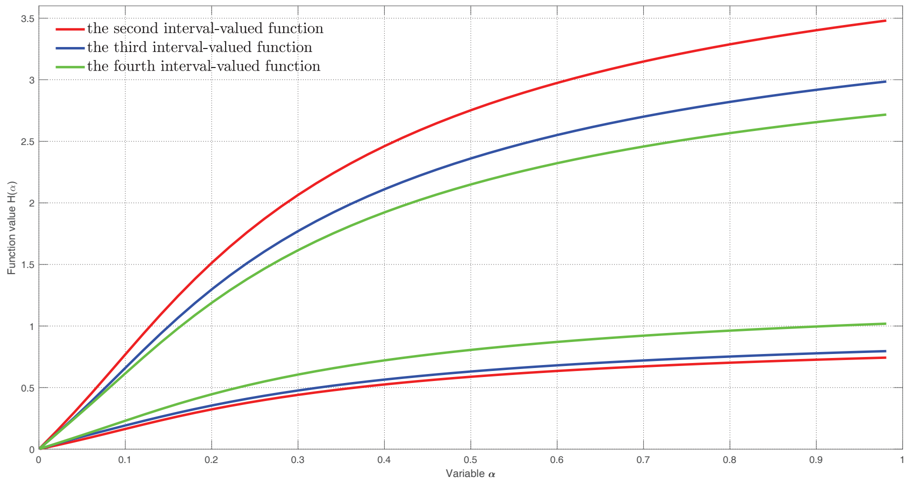

Figure 2.4: *Graphical representation for Theorem* 2.8

3. Conclusions

To the best of our knowledge, this is the first paper concerning to the fractional inclusion relations involving the interval-valued exponential trigonometric convexity. Herein, we deduce the interval-valued fractional integral inclusions in association with the Hermite–Hadamard- and Pachpatte-type inequality for the newly introduced family of functions. In particular, we come up with an improved version of the Hermite–Hadamard-type integral inclusions pertaining to the interval-valued exponential trigonometric convex functions. These integral inclusion relations addressed in the present study are substantial generalizations of the outcomes obtained by Kadakal et al. in [17] (2021). We would like to emphasize that interval analysis has a wide range of applications in applied mathematics, especially in the field of optimality analysis, see the published articles [15, 33, 39]. To a certain extent, the important area of the interval-valued analysis research, which is linked to fractional integral operators, deserves further exploration.

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