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BOUNDEDNESS FROM BELOW OF COMPOSITION OPERATORS BETWEEN L_a^p AND L_a^q, BETWEEN L_a^p AND THE HARDY SPACE H^2, BETWEEN L_a^p AND BESOV SPACE

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Abstract. We study the relation between the composition operators C_φ with closed range on the weighted Bloch spaces and C_φ with closed range on the weighted Dirichlet spaces D_p^α. In particular, we study the boundedness from below of composition operators between L_a^p and L_a^q, between L_a^p and Hardy space, and between L_a^p and Besov space.

1. Introduction

For φ analytic self-map of the open unit disk D, the composition operator C_φ is defined by $C_\varphi(f) = f \circ \varphi$. For $z,w \in D$, let $\varphi_z(w) = \dfrac{z-w}{1-\overline{z}w}$ and, $dA(z)$ the area measure on D.

For $p > 0$, $\alpha > -1$, the weighted Dirichlet space $\mathscr{D}_p^\alpha$ is defined to be the space of analytic functions f on D such that

$$|f(0)| + \left(\int_D (1-|z|^2)^\alpha |f'(z)|^p dA(z)\right)^{\frac{1}{p}} < \infty.$$

In case $\alpha = 1$ and $p = 2$, then $\mathscr{D}_2^1 = H^2$ is the classical Hardy space. Furthermore, in case $\alpha = p$ and $1 \leqslant p < \infty$, then $\mathscr{D}_p^p = L_a^p$ is the usual Bergman space. Also, in case $\alpha = p-2$ and $1 < p < \infty$, $\mathscr{D}_p^{p-2} = B_p$ is called the Besov space. In particular, $\mathscr{D}_2^0 = \mathscr{D}$ is called the Dirichlet space. (See [19].)

For $\alpha > 0$, the weighted Bloch space $\mathscr{B}_\alpha$ is defined to be the space of analytic functions f on D such that

$$|f(0)| + \sup_{z \in D}(1-|z|^2)^\alpha \left|f'(z)\right| < \infty.$$

Note that $\mathscr{B}_1 = \mathscr{B}$ is the usual Bloch space.

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The amount $\sup_{z\in D}(1-|z|^2)^{\alpha}\left|f'(z)\right|$ is a pseudonorm, which coincides with the $\mathscr{B}_{\alpha}$-norm on the closed subspace of functions that vanish at the origin. So it coincides with the quotient norm on $\mathscr{B}_{\alpha}/\mathscr{C}$ where $\mathscr{C}$ denotes the closed subspace of constant functions. The space $BMOA$ is defined to be the space of analytic functions f on D such that

$$|f(0)|+\sup_{a\in D}\left(\int_D(1-|\varphi_a(z)|^2)|f'(z)|^2dA(z)\right)^{\frac{1}{2}}<\infty.$$

By Schwarz-Pick lemma, the operator C_{φ} is bounded on the Bloch space $\mathscr{B}$, also on $BMOA$. Furthermore, it follows from Littlewood's subordination theorem that C_{φ} is bounded on the Bergman space L_a^p for all $1\leqslant p<\infty$. In [3] P. S. Bourdon, J. A. Cima and A. L. Matheson obtained a necessary and sufficient condition for compactness of C_{φ} on $BMOA$.

To state our investigations, we give some definitions. Let X be a Banach space and let T a linear operator from X into X. An operator T is called bounded below on X if there exists a constant $C>0$ such that $\| Tf \|\geqslant C\| f \|$ for all $f\in X$. (Clearly, when a composition operator C_{φ} is defined on a space of analytic functions on D, C_{φ} is bounded below on the space if and only if C_{φ} is closed range.) Furthermore, a subset H of D is called a sampling set for the space $\mathscr{B}_{\alpha}$ if there exists a constant $C>0$ such that $\sup_{z\in D}(1-|z|^2)^{\alpha}\left|f'(z)\right|\leqslant C\sup_{z\in H}(1-|z|^2)^{\alpha}\left|f'(z)\right|$ for all $f\in\mathscr{B}_{\alpha}$. For $\varepsilon>0$, let $G_{\varepsilon}=\varphi\left(\{z\in D,\frac{(1-|z|^2)|\varphi'(z)|}{1-|\varphi(z)|^2}\geqslant\varepsilon\}\right)$. In [6], P. Ghatage, D. Zheng and N. Zorboska determined the boundedness from below of composition operators on the Bloch space using a sampling set G_{ε} for the Bloch space. Moreover, N. Zorboska ([21], [22]) characterized the boundedness from below of composition operators on the Bergman spaces. Also, H. Chen and P. Gauthier characterized the boundedness from below of composition operators on $\mathscr{B}_{\alpha}$ in [4]. Furthormore, W. Smith ([13]) studied the boundedness and compactness of composition operators between Bergman spaces and Hardy spaces.

In this paper, we study when composition operators are bounded below on the weighted Dirichlet space $\mathscr{D}_p^{\alpha}$, the weighted Bloch spaces $\mathscr{B}_{\alpha}$ and the Bergman spaces L_a^p, respectively. Moreover, we study relationship between the boundedness from below of composition operators on $\mathscr{D}_p^{\alpha}$ and it on $\mathscr{B}_{\alpha}$. As a result, we can characterized the boundedness from below of composition operators between L_a^p and H^2, between L_a^p and B_p, respectively.

2. Background material

In this section, we introduce several results to prove the main theorem. In [1] J. R. Akeroyd and P. G. Ghatage proved the following result.

THEOREM A. ([1]) *Let φ be a univalent, analytic self-map of D. Then C_{φ} is closed range on L_a^2 if and only if φ is an automorphism of D.*

In [17] we proved the following result.

THEOREM B. ([17]) *Let $\alpha > 1$. Suppose φ is a univalent self-map of D. Then the following are equivalent.*

(1) $C_\varphi : \mathscr{B}_\alpha \to \mathscr{B}_\alpha$ *is bounded below.*

(2) $C_\varphi : L_a^2 (= \mathscr{D}_2^2) \to L_a^2 (= \mathscr{D}_2^2)$ *is bounded below.*

(3) $C_\varphi : H^2 (= \mathscr{D}_2^1) \to H^2 (= \mathscr{D}_2^1)$ *is bounded below.*

(4) $C_\varphi : \mathscr{D}_2^\alpha \to \mathscr{D}_2^\alpha$ *is bounded below.*

(5) φ *is an automorphism of D.*

THEOREM C. ([17]) *Let $0 < \alpha < 1$. Suppose φ is a univalent self-map of D. Furthermore, suppose that C_φ: $\mathscr{B}_\alpha \to \mathscr{B}_\alpha$ is bounded (i.e. $\sup_{z\in D} \left(1-|z|^2\right)^\alpha \cdot \left(1-|\varphi(z)|^2\right)^{-\alpha} |\varphi'(z)| < \infty$), and that C_φ: $\mathscr{D} \to \mathscr{D}$ is bounded. Then, the following are equivalent.*

(1) $C_\varphi : \mathscr{B}_\alpha \to \mathscr{B}_\alpha$ *is bounded below for some* $0 < \alpha < 1$.

(2) $C_\varphi : \mathscr{B}_\alpha \to \mathscr{B}_\alpha$ *is bounded below for all* $0 < \alpha < 1$.

(3) $C_\varphi : L_a^2 \to L_a^2$ *is bounded below.*

(4) $C_\varphi : H^2 \to H^2$ *is bounded below.*

(5) $C_\varphi : \mathscr{D}_2^\gamma \to \mathscr{D}_2^\gamma$ *is bounded below for some* $\gamma > 1$.

(6) $C_\varphi : \mathscr{D}_2^\gamma \to \mathscr{D}_2^\gamma$ *is bounded below for all* $\gamma > 1$.

(7) $C_\varphi : \mathscr{D} \to \mathscr{D}$ *is bounded below.*

(8) $C_\varphi : BMOA \to BMOA$ *is bounded below.*

(9) $C_\varphi : \mathscr{B} \to \mathscr{B}$ *is bounded below.*

(10) φ *is an automorphism of D.*

In [4] H. Chen and P. Gauthier proved the following result with respect to the composition operators $C_\varphi : \mathscr{B}_\alpha \to \mathscr{B}_\beta$.

THEOREM D. ([4]) *Suppose $\beta \geqslant 1$ and $\alpha \leqslant \beta$. Then $C_\varphi : \mathscr{B}_\alpha \to \mathscr{B}_\beta$ is bounded, while $C_\varphi : \mathscr{B}_\alpha \to \mathscr{B}_\beta$ is not bounded below if $\alpha < \beta$.*

Moreover, in [22] N. Zorboska proved the following result that generalizes Theorem D.

THEOREM E. ([22]) *Let $\alpha, \beta > 0$ and $\alpha \neq \beta$. Suppose $C_\varphi : \mathscr{B}_\alpha \to \mathscr{B}_\beta$ is bounded. Then $C_\varphi : \mathscr{B}_\alpha \to \mathscr{B}_\beta$ is not bounded below if $\alpha < 1 \leqslant \beta$, or $1 \leqslant \alpha < \beta$, or $\alpha > \beta$, $\beta < 1$.*

In this paper, we get several results with respect to the boundedness from below of composition operators between Bergman spaces L_a^p and Bergman spaces L_a^q, the boundedness from below of composition operators between Bergman spaces L_a^p and Hardy space.

3. The main results and the univalent case

If $\varphi(0)=a$ and $\psi=\varphi_a\circ\varphi$, then C_φ is bounded below on $\mathscr{B}_\alpha$ (or $\mathscr{D}_p^\alpha$) to $\mathscr{B}_\alpha$ (or $\mathscr{D}_p^\alpha$) if and only if C_ψ is bounded below on $\mathscr{B}_\alpha$ (or $\mathscr{D}_p^\alpha$) to $\mathscr{B}_\alpha$ (or $\mathscr{D}_p^\alpha$)(See [6] and [21]). So we assume from now on that $\varphi(0)=0$ and that C_φ is acting on the subspace of functions that vanish at the origin.

Let $\alpha>-1$. For $\forall a\in D$, the following estimate is standard ([19]).

$$\int_D \frac{(1-|z|^2)^\alpha}{|1-\overline{a}z|^\lambda}dA(z)\sim\begin{cases}(1-|a|^2)^{\alpha+2-\lambda} & (\lambda>\alpha+2)\\ \log\frac{2}{1-|a|^2} & (\lambda=\alpha+2)\\ 1 & (\lambda<\alpha+2).\end{cases}\tag{1}$$

Using the estimate (1), we have the following result.

THEOREM 1. *Let $0<p,q<+\infty$, and $\alpha,\gamma>0$. Suppose that $C_\varphi:\mathscr{D}_p^{p\alpha}\to\mathscr{D}_q^{q\gamma}$ is bounded. If $C_\varphi:\mathscr{D}_p^{p\alpha}\to\mathscr{D}_q^{q\gamma}$ is bounded below, then there exists a constant $K>0$ such that*

$$\sup_{z\in D}|(C_\varphi f)'(z)|(1-|z|^2)^\gamma\geqslant KS_{p,q,\alpha}(f)$$

for all $f\in\mathscr{B}_\alpha$, where

$$S_{p,q,\alpha}(f):=\begin{cases}\sup_{z\in D}|f'(z)|(1-|z|^2)^{\alpha+2\left(\frac{1}{p}-\frac{1}{q}\right)} & (1<q\leqslant p)\\ \sup_{z\in D}|f'(z)|(1-|z|^2)^{\alpha+2(\frac{1}{p}-1)}\left(\log\frac{2}{1-|z|^2}\right)^{-1} & (q=1<p)\\ \sup_{z\in D}|f'(z)|(1-|z|^2)^{\alpha+2(\frac{1}{p}-1)} & (0<q<1\leqslant p).\end{cases}$$

Proof. For $a\in D$ and $\forall\ f\in\mathscr{B}_\alpha$, we see that

$$F(z)=\int_0^z f'(\zeta)\varphi_a'(\zeta)d\zeta\in\mathscr{D}_p^{p\alpha}.(2)$$

In fact, using the evaluation (1), it holds that

$$\begin{aligned}\left(\int_D(1-|z|^2)^{p\alpha}|F'(z)|^pdA(z)\right)^{\frac{1}{p}}&=\left(\int_D|f'(z)|^p|\varphi_a'(z)|^p(1-|z|^2)^{p\alpha}dA(z)\right)^{\frac{1}{p}}\\&\leqslant\sup_{z\in D}(1-|z|^2)^\alpha|f'(z)|\left(\int_D|\varphi_a'(z)|^pdA(z)\right)^{\frac{1}{p}},\end{aligned}$$

and

$$\begin{aligned}\left(\int_D|\varphi_a'(z)|^pdA(z)\right)^{\frac{1}{p}}&=\left(\int_D\frac{(1-|a|^2)^p}{|1-\overline{a}z|^{2p}}dA(z)\right)^{\frac{1}{p}}\\&\sim\begin{cases}(1-|a|^2)^{\frac{2}{p}-1} & (p>1)\\ (1-|a|^2)\log\frac{2}{1-|a|^2} & (p=1)\\ (1-|a|^2) & (0<p<1).\end{cases}\end{aligned}$$

Hence $F(z)=\int_0^z f'(\zeta)\varphi_a'(\zeta)d\zeta \in \mathscr{D}_p^{p\alpha}$.

Let $p \geqslant q > 1$ and $f \in \mathscr{B}_\alpha$, then (2) implies that $F \in \mathscr{D}_p^{p\alpha}$. Since $C_\varphi : \mathscr{D}_p^{p\alpha} \to \mathscr{D}_q^{q\gamma}$ is bounded below, for any $a \in D$, using subharmonicity of $|f\circ\varphi_a|^p$, there exists a constant $K>0$ such that

$$\begin{aligned}
&\left(|f'(a)|^p(1-|a|^2)^{p(\alpha-1)+2}\right)^{\frac{1}{p}}\\
&\leqslant K\left(\int_D (1-|z|^2)^{p(\alpha-1)}|f'(z)|^p(1-|\varphi_a(z)|^2)^p dA(z)\right)^{\frac{1}{p}}\\
&= K\left(\int_D |f'(z)|^p|\varphi_a'(z)|^p(1-|z|^2)^{p\alpha}dA(z)\right)^{\frac{1}{p}}\\
&= K\left(\int_D (1-|z|^2)^{p\alpha}|F'(z)|^p dA(z)\right)^{\frac{1}{p}}\\
&\leqslant K\left(\int_D (1-|z|^2)^{q\gamma}|(C_\varphi F)'(z)|^q dA(z)\right)^{\frac{1}{q}}\\
&= K\left(\int_D (1-|z|^2)^{q\gamma}|(C_\varphi f)'(z)\varphi_a'(\varphi(z))|^q dA(z)\right)^{\frac{1}{q}}\\
&\leqslant K\left(\sup_{z\in D}|(C_\varphi f)'(z)|(1-|z|^2)^\gamma\right)\left(\int_D |\varphi_a'(\varphi(z))|^q dA(z)\right)^{\frac{1}{q}}.
\end{aligned}$$

Since C_φ is bounded on L_a^q and that $\varphi_a' \in L_a^q$, for any $a\in D$, using the evaluation (1),

$$\left\{\int_D |\varphi_a'(\varphi(z))|^q dA(z)\right\}^{\frac{1}{q}} \leqslant \| C_\varphi \| \left\{\int_D |\varphi_a'(z)|^q dA(z)\right\}^{\frac{1}{q}} \approx \| C_\varphi \| \left(1-|a|^2\right)^{(2-q)\frac{1}{q}} < \infty,$$

where $\| C_\varphi \|$ is the operator norm. Hence there exists a constant $K'>0$ such that

$$\sup_{z\in D}|f'(z)|(1-|z|^2)^{\alpha+2\left(\frac{1}{p}-\frac{1}{q}\right)} \leqslant K' \sup_{z\in D}|(C_\varphi f)'(z)|(1-|z|^2)^\gamma \ \ (\forall f\in\mathscr{B}_\alpha).$$

Let $p>q=1$ and $f\in\mathscr{B}_\alpha$, then (2) implies that $F\in\mathscr{D}_p^{p\alpha}$. So we can also prove that there exists a constant $K'>0$ such that

$$\sup_{z\in D}|f'(z)|(1-|z|^2)^{\alpha+2(\frac{1}{p}-1)}\left(\log\frac{2}{1-|z|^2}\right)^{-1} \leqslant K' \sup_{z\in D}|(C_\varphi f)'(z)|(1-|z|^2)^\gamma$$

$(\forall f\in\mathscr{B}_\alpha)$.

Let $p\geqslant 1>q>0$ and $f\in\mathscr{B}_\alpha$, then (2) implies that $F\in\mathscr{D}_p^{p\alpha}$. Thus we can also prove that there exists a constant $K'>0$ such that

$$\sup_{z\in D}|f'(z)|(1-|z|^2)^{\alpha+2(\frac{1}{p}-1)} \leqslant K' \sup_{z\in D}|(C_\varphi f)'(z)|(1-|z|^2)^\gamma \ \ (\forall f\in\mathscr{B}_\alpha).$$

This completes the proof of theorem. □

The following result generalizes Corollary 3.6 of [2].

COROLLARY 2. *Let $1<p<\infty$. If $C_\varphi : L_a^p \to L_a^p$ is bounded below, then $C_\varphi : \mathscr{B} \to \mathscr{B}$ is bounded below.*

Proof. When $1<p<\infty$, applying $\alpha=\gamma=1$ and $q=p>1$ in Theorem 1 and using the property (1), we can prove that $C_\varphi : \mathscr{B} \to \mathscr{B}$ is bounded below. □

If $\gamma=1$, then $L_a^p = \mathscr{D}_p^{p\gamma}$ and that there exists a symbol φ such that $C_\varphi : L_a^p \to L_a^p(=\mathscr{D}_p^p)$ is bounded below. If $\gamma \neq 1$, then there is no symbol φ such that $C_\varphi : L_a^p \to \mathscr{D}_p^{p\gamma}$ is bounded below.

COROLLARY 3. *Let $1<p<\infty$. If $\gamma>1$, then $C_\varphi : L_a^p \to \mathscr{D}_p^{p\gamma}$ is bounded, while there is no symbol φ such that it is bounded below. If $\gamma<1$, supposing that $C_\varphi : L_a^p \to \mathscr{D}_p^{p\gamma}$ is bounded, then there is no symbol φ such that it is bounded below.*

Proof. Let $p>1$. If $\gamma>1$, the boundedness of $C_\varphi : L_a^p \to \mathscr{D}_p^{p\gamma}$ is trivial. Theorem 1 and Theorem E imply that $C_\varphi : L_a^p \to \mathscr{D}_p^{p\gamma}$ is not bounded below. If $\gamma<1$, supposing the boundedness of $C_\varphi : L_a^p \to \mathscr{D}_p^{p\gamma}$, then Theorem 1 and Theorem E imply that $C_\varphi : L_a^p \to \mathscr{D}_p^{p\gamma}$ is not bounded below. □

If $p>q>1$ and $C_\varphi : L_a^p \to \mathscr{D}_q^{q\gamma}$, then we have the following.

COROLLARY 4. *Let $1<q<p$. If $\gamma \geqslant 1$, then $C_\varphi : L_a^p \to \mathscr{D}_q^{q\gamma}$ is bounded, while there is no symbol φ such that it is bounded below.*

Proof. If $\gamma \geqslant 1$, the boundedness of $C_\varphi : L_a^p \to \mathscr{D}_q^{q\gamma}$ follows from Hölder's inequality. And Theorem 1 and Theorem D imply that $C_\varphi : L_a^p \to \mathscr{D}_q^{q\gamma}$ is not bounded below. □

If $0<q<p=1$, if $\gamma \neq 1$, then there is no symbol φ such that $C_\varphi : L_a^1 \to \mathscr{D}_q^{q\gamma}$ is bounded below.

COROLLARY 5. *Let $0<q<1$. If $\gamma>1$, then $C_\varphi : L_a^1 \to \mathscr{D}_q^{q\gamma}$ is bounded, while there is no symbol φ such that it is bounded below. If $\gamma<1$, supposing that $C_\varphi : L_a^1 \to \mathscr{D}_q^{q\gamma}$ is bounded, then there is no symbol φ such that it is bounded below.*

Proof. Let $0<q<1$. If $\gamma>1$, the boundedness of $C_\varphi : L_a^1 \to \mathscr{D}_q^{q\gamma}$ follows from Hölder's inequality. Theorem 1 and Theorem E imply that $C_\varphi : L_a^1 \to \mathscr{D}_q^{q\gamma}$ is not bounded below. If $\gamma<1$, supposing that $C_\varphi : L_a^1 \to \mathscr{D}_q^{q\gamma}$ is bounded, Theorem 1 and Theorem E imply that $C_\varphi : L_a^1 \to \mathscr{D}_q^{q\gamma}$ is not bounded below. □

With respect to the composition operator $C_\varphi : \mathscr{D}_p^{p\alpha} \to L_a^q$, then we have the following.

COROLLARY 6. *Suppose $\alpha<1$. If $p \geqslant q>1$ or $p>1=q$ or $p \geqslant 1>q>0$, then $C_\varphi : \mathscr{D}_p^{p\alpha} \to L_a^q$ is bounded, while there is no symbol φ such that it is bounded below.*

Proof. Since $\alpha<1$, the boundedness of $C_\varphi : \mathscr{D}_p^{p\alpha} \to L_a^q$ follows from Hölder's inequality. If $p \geqslant q>1$ or $p>1=q$ or $p \geqslant 1>q>0$, then Theorem 1 and Theorem E imply that $C_\varphi : \mathscr{D}_p^{p\alpha} \to L_a^q$ is not bounded below. □

The following result generalizes Theorem B.

COROLLARY 7. *Let* $\alpha > 1$. *Suppose* $1 < p < \infty$. *Suppose* φ *is a univalent self-map of* D. *Then the following are equivalent.*

(1) $C_\varphi : \mathscr{D}_p^{p\alpha} \to \mathscr{D}_p^{p\alpha}$ *is bounded below.*

(2) $C_\varphi : \mathscr{B}_\alpha \to \mathscr{B}_\alpha$ *is bounded below.*

(3) φ *is an automorphism of* D.

Proof. It follows from Theorem 1 that (1) implies (2). The equivalence of (2) and (3) follows from Theorem B. It is trivial that (3) implies (1). □

The following result has never been proven so far.

COROLLARY 8. *Suppose that* $1 < q \leqslant 2$. *Then* $C_\varphi : H^2 \to L_a^q$ *is bounded, while there is no symbol* φ *such that it is bounded below.*

Proof. The boundedness of the composition operator $C_\varphi : H^2 \to L_a^q$ $(q \leqslant 4)$ follows from Hölder's inequality and the fact $H^2 \subset L_a^q$ $(q \leqslant 4)$ (see [5]). Applying $\alpha = \frac{1}{2}$, $\gamma = 1$ and $p = 2$, $1 < q \leqslant 2$ in Theorem 1, it follows from Theorem D that $C_\varphi : H^2 \to L_a^q$ is not bounded below. □

The following result has never been proven so far.

COROLLARY 9. *Suppose that* $2 \leqslant p < 4$, *and that* $C_\varphi : L_a^p \to H^2$ *is bounded. Then there is no symbol* φ *such that* $C_\varphi : L_a^p \to H^2$ *is bounded below.*

Proof. Suppose that $2 \leqslant p < 4$, and that $C_\varphi : L_a^p \to H^2$ is bounded. Applying $\alpha = 1$, $\gamma = \frac{1}{2}$, $2 \leqslant p < 4$ and $q = 2$ in Theorem 1, it follows from Theorem E and the fact $1 + 2(\frac{1}{p} - \frac{1}{2}) > \frac{1}{2} = \gamma$, that $C_\varphi : L_a^p \to H^2$ is not bounded below. □

The following result characterizes the boundedness from below of the composition operator $C_\varphi : L_a^p \to L_a^q$.

COROLLARY 10. *If* $1 < q < p$, *or* $0 < q \leqslant 1 < p < 2$, *then* $C_\varphi : L_a^p \to L_a^q$ *is bounded, while there is no symbol* φ *such that it is bounded below.*

Proof. If $1 < q < p$, or $0 < q \leqslant 1 < p < 2$, the boundedness of $C_\varphi : L_a^p \to L_a^q$ follows from Hölder's inequality. Applying $\alpha = \gamma = 1$ and $1 < q < p$, or $0 < q \leqslant 1 < p < 2$ in Theorem 1, it follows from Theorem D that $C_\varphi : L_a^p \to L_a^q$ is not bounded below. □

The following result generalizes Theorem C.

THEOREM 11. *Let* $0 < \alpha < 1$ *and* $1 < p < \infty$. *Suppose* φ *is a univalent self-map of* D. *Furthermore, suppose that* $C_\varphi : \mathscr{B}_\alpha \to \mathscr{B}_\alpha$ *is bounded (i.e.* $\sup_{z \in D} \left(1 - |z|^2\right)^\alpha \left(1 - |\varphi(z)|^2\right)^{-\alpha} |\varphi'(z)| < \infty$), *and that* $C_\varphi : \mathscr{D} \to \mathscr{D}$ *is bounded. Then, the following are equivalent.*

(1) $C_\varphi : L_a^p \to L_a^p$ *is bounded below.*

(2) $C_\varphi : \mathscr{D}_p^{p\gamma} \to \mathscr{D}_p^{p\gamma}$ *is bounded below for some* $\gamma > 1$.

(3) $C_\varphi : \mathscr{D}_p^{p\gamma} \to \mathscr{D}_p^{p\gamma}$ *is bounded below for all* $\gamma > 1$.

(4) $C_\varphi : \mathscr{B}_\alpha \to \mathscr{B}_\alpha$ *is bounded below for some* $0 < \alpha < 1$.

(5) $C_\varphi : \mathscr{B}_\alpha \to \mathscr{B}_\alpha$ *is bounded below for all* $0 < \alpha < 1$.

(6) $C_\varphi : \mathscr{B}_\gamma \to \mathscr{B}_\gamma$ *is bounded below for some* $\gamma > 1$.

(7) $C_\varphi : \mathscr{B}_\gamma \to \mathscr{B}_\gamma$ *is bounded below for all* $\gamma > 1$.

(8) $C_\varphi : \mathscr{D} \to \mathscr{D}$ *is bounded below.*

(9) $C_\varphi : \mathscr{B} \to \mathscr{B}$ *is bounded below.*

(10) φ *is an automorphism of* D.

Proof. Using Corollary 2 and Theorem C, we can prove theorem. □

The following result characterizes the boundedness from below of the composition operator $C_\varphi : B_p \to B_q$ which has never been proven so far.

COROLLARY 12. *Suppose* $2 < q \leqslant p < \infty$ *and that* $C_\varphi : B_p \to B_q$ *is bounded. If* $C_\varphi : B_p \to B_q$ *is bounded below, then* $C_\varphi : \mathscr{B}_{1-\frac{2}{q}} \to \mathscr{B}_{1-\frac{2}{q}}$ *is bounded below.*

Proof. Applying $\alpha = 1 - \dfrac{2}{p}$ and $\gamma = 1 - \dfrac{2}{q}$ in Theorem 1, we can prove that $C_\varphi : \mathscr{B}_{1-\frac{2}{q}} \to \mathscr{B}_{1-\frac{2}{q}}$ is bounded below. □

REMARK 13. Let $2 < q \leqslant p < \infty$. Suppose φ is a univalent self-map of D. Furthermore, suppose that $\sup_{z \in D} \left(1-|z|^2\right)^{1-\frac{2}{q}} \left(1-|\varphi(z)|^2\right)^{-\left(1-\frac{2}{q}\right)} |\varphi'(z)| < \infty$ and that $C_\varphi : \mathscr{D} \to \mathscr{D}$ is bounded. If $C_\varphi : B_p \to B_q$ is bounded and bounded below, then Theorem 11 and Corollary 12 imply that φ is an automorphism of the open unit disk D.

The following results characterize the boundedness from below of composition operators $C_\varphi : B_p \to L_a^q$ and $C_\varphi : L_a^p \to B_q$ which have never been proven so far.

COROLLARY 14. *Suppose* $2 < q \leqslant p < \infty$. *Then* $C_\varphi : B_p \to L_a^q$ *is bounded, while there is no symbol* φ *such that it is bounded below.*

Proof. Since $2 < q \leqslant p$, the boundedness of $C_\varphi : B_p \to L_a^q$ follows from Hölder's inequality. Applying $\alpha = 1 - \frac{2}{p}$, $\gamma = 1$ and $2 < q \leqslant p$ in Theorem 1, it follows from Theorem D that $C_\varphi : B_p \to L_a^q$ is not bounded below. □

COROLLARY 15. *Suppose* $2 < q \leqslant p < \infty$. *Suppose that* $C_\varphi : L_a^p \to B_q$ *is bounded. Then there is no symbol* φ *such that* $C_\varphi : L_a^p \to B_q$ *is bounded below.*

Proof. Since $2 < q \leqslant p$, applying $\alpha = 1$, $\gamma = 1 - \frac{2}{q}$ in Theorem 1, it follows from Theorem E that $C_\varphi : L_a^p \to B_q$ is not bounded below. □

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