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NON–UNIFORM BERRY–ESSEEN–TYPE INEQUALITIES FOR A SUPERCRITICAL BRANCHING PROCESS WITH IMMIGRATION IN A RANDOM ENVIRONMENT

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Abstract. Let W_n be the fundamental submartingale of a supercritical branching process with immigration in a random environment. In order to characterize the convergence rates of W_n, the quenched and annealed non-uniform Berry-Esseen-type inequalities are established for $W_{n+k} - W_n$ for each fixed $k \in \{1,2,\cdots,\infty\}$, which reveal the convergence rates of the corresponding central limit theorems.

1. Introduction and main results

Since the branching process was proposed, it has been widely used in many aspects, such as biophysics [20] and sociology [19]. It also has many interactions with various applied probability settings, such as fractals and Mandelbrot's cascades, perpetuities, branching random walks and branching Brownian motions. For this reason, many scholars continue to study and promote branching processes and related topics. One important extension is the branching process in a random environment (BPRE) where the offspring distribution changes with generation according to a time-dependent random environment, which considers the influence of external environment on branching mechanism; see e.g. [24, 2, 3] for earlier studies, and [28, 13, 26, 6, 29] for recent achievements. At the same time, research on branching systems with immigration reveals that not only the offspring distribution, but also the immigration distribution will affect the limit behaviours of branching processes. Related research also reflects that the idea of introducing immigration can be applied to the study of related stochastic processes. For example, Kensten et al. [16] gave a limit distribution for a random walk in a random environment with the help of a BPRE with one immigration at each generation. These facts make scholars pay more attention to the branching process with immigration in a random environment (BPIRE), which involves the dual impact of environment and immigration. A various of limit theorems and asymptotic properties, such as limit distributions, tail probabilities, large deviations and convergence rates, were studied;

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see e.g. [17, 23, 25, 7, 27, 15, 4] and the reference therein for more information. In this paper, we are interested in BPIRE and will focus on the convergence rates of the fundamental submartingale of the model by accurately describing the associated central limit theorems.

Let us introduce the models of BPRE and BPIRE as follows. The random environment, denoted by $\xi = (\xi_n)$, is an ergodic and stationary sequence of random variables indexed by time $n \in \mathbb{N} = \{0, 1, 2, \cdots\}$. It is often independent and identically distributed (i.i.d.). Each realization of ξ_n corresponds to two probability distributions on $\mathbb{N}$: one is the offspring distribution denoted by

$$p(\xi_n) = \{p_j(\xi_n); j \geqslant 0\}, \quad \text{where } p_j(\xi_n) \geqslant 0 \text{ and } \sum_j p_j(\xi_n) = 1;$$

the other is the immigration distribution denoted by

$$h(\xi_n) = \{h_j(\xi_n); j \geqslant 0\}, \quad \text{where } h_j(\xi_n) \geqslant 0 \text{ and } \sum_j h_j(\xi_n) = 1.$$

DEFINITION 1.1. The process $(\bar{Z}_n)$ is called a branching process in the random environment ξ (BPRE) if it satisfies:

$$\bar{Z}_0 = 1 \qquad \text{and} \qquad \bar{Z}_{n+1} = \sum_{i=1}^{\bar{Z}_n} X_{n,i}, \ \ n \in \mathbb{N}, \tag{1.1}$$

where given the environment ξ, all $X_{n,i}$ are independent of each other, and $X_{n,i}$ has the distribution $p(\xi_n)$.

DEFINITION 1.2. The process (Z_n) is called a branching process with immigration (Y_n) in the random environment ξ (BPIRE) if it satisfies:

$$Z_0 = 1 \qquad \text{and} \qquad Z_{n+1} = \sum_{i=1}^{Z_n} X_{n,i} + Y_n, \ \ n \in \mathbb{N}, \tag{1.2}$$

where given the environment ξ, all $X_{n,i}$ and Y_n are independent of each other, $X_{n,i}$ has the distribution $p(\xi_n)$ and Y_n has the distribution $h(\xi_n)$.

REMARK 1.1. Particularly, if $Y_n \equiv 0$ in (1.2) for all $n \in \mathbb{N}$, which means that there is no immigration, then the BPIRE (Z_n) degenerates to the BPRE $(\bar{Z}_n)$ which is defined by (1.1), namely, $Z_n = \bar{Z}_n$.

We introduce two probabilities. Denote by $\mathbb{P}_\xi$ the so-called *quenched law*, i.e. the conditional probability when the environment ξ is given. The total probability can be expressed as $\mathbb{P}(dx, d\xi) = \mathbb{P}_\xi(dx)\tau(d\xi)$, where τ is the law of the environment ξ. It is usually called the *annealed law*. The expectation with respective to $\mathbb{P}_\xi$ (resp. $\mathbb{P}$) will be denoted by $\mathbb{E}_\xi$ (resp. $\mathbb{E}$).

For sake of brevity, we write $p_j = p_j(\xi_0)$ and $h_j = h_j(\xi_0)$. Throughout the paper, we always assume that

$$\mathbb{P}(p_0 = 0) = 1 \qquad \text{and} \qquad \mathbb{P}(p_1 = 1) < 1, \tag{1.3}$$

which means that each individual produces at least one offspring, and the probability of producing at least two offspring is positive.

For $n \in \mathbb{N}$ and $t \in \mathbb{R}$, define

$$m_n(t) = \sum_{j=0}^{\infty} p_j(\xi_n) j^t .$$

It can be seen that $m_n(t) = \mathbb{E}_\xi X_n^t$. We write $m_n = m_n(1)$ and $X_n = X_{n,1}$. Under the assumption (1.3), one has $\mathbb{E}\log m_0 > 0$, which means that the corresponding branching process is *supercritical*. Denote

$$\Pi_0 = 1 \qquad \text{and} \qquad \Pi_n = \prod_{i=0}^{n-1} m_i, \;\; n = 1, 2, \cdots .$$

It can be seen that $\Pi_n = \mathbb{E}_\xi \bar{Z}_n$. For $n \in \mathbb{N}$, define

$$W_n = \frac{Z_n}{\Pi_n} \qquad \text{and} \qquad \bar{W}_n = \frac{\bar{Z}_n}{\Pi_n}. \tag{1.4}$$

It is well known that $\bar{W}_n$ forms a nonnegative martingale and the limit $\bar{W}_\infty = \lim_{n\to\infty} \bar{W}_n$ exists almost surely (a.s.) with $\mathbb{E}_\xi \bar{W}_\infty \leqslant 1$. As for W_n, it is known that W_n is a nonnegative submartingale, and it converges to some limit $W_\infty = \lim_{n\to\infty} W_n$ under certain moment conditions.

THEOREM 1.1. (Convergence of W_n, [27, 15]) *Assume that* $\mathbb{E}\log m_0 \in (0, \infty)$.

(a) If $\mathbb{E}\log^+ \frac{Y_0}{m_0} < \infty$, *then* $W_\infty = \lim_{n\to\infty} W_n$ *exists a.s. with values in* $[0, \infty)$.

(b) If $\mathbb{E}\log^+ \mathbb{E}_\xi (\frac{X_0}{m_0})^p < \infty$ *and* $\mathbb{E}\log^+ \mathbb{E}_\xi (\frac{Y_0}{m_0})^p < \infty$ *for some* $p > 1$, *then* $\sup_n \mathbb{E}_\xi W_n^p < \infty$ *a.s., and* W_n *converges a.s. and in* $\mathbb{P}_\xi$*-*L^p *to* W_∞ *with* $\mathbb{E}_\xi W_\infty^p \in (0, \infty)$.

(c) If $\xi = (\xi_n)$ *is i.i.d.,* $\mathbb{E}(\frac{X_0}{m_0})^p < \infty$ *and* $\mathbb{E}(\frac{Y_0}{m_0})^p < \infty$ *for some* $p > 1$, *then* $\sup_n \mathbb{E} W_n^p < \infty$, *and* W_n *converges a.s. and in* L^p *to* W_∞ *with* $\mathbb{E} W_\infty^p \in (0, \infty)$.

In this paper, we are interested in convergence rates of W_n in terms of the rate at which the ratio W_{n+k}/W_n converges to 1, for each fixed $k \in \{1, 2, \cdots, \infty\}$. Note that k can take the infinity if W_∞ exists. Obviously, if $k = \infty$, the convergence rates of W_∞/W_n to 1 reflects those of W_n to its limit W_∞; if k is finite, the rates of W_{n+k}/W_n can reveal the asymptotic changes of Z_{n+k}/Z_n which can be regarded as the population ratio across k generations. In particular, when $k = 1$, Z_{n+1}/Z_n is the ratio of the population of two neighbouring generations whose asymptotic properties were studied in many related literatures, see e.g. [1, 21, 10, 9, 18].

Let T be the shift operator that $T\xi = (\xi_1, \xi_2, \cdots)$ if $\xi = (\xi_0, \xi_1, \xi_2, \cdots)$. For $k \in \{1, 2, \cdots, \infty\}$, set

$$\eta_n = \frac{\Pi_n (W_{n+k} - W_n)}{\sqrt{Z_n}\sigma(T^n\xi)} = \frac{\sqrt{Z_n}}{\sigma(T^n\xi)}\left(\frac{W_{n+k}}{W_n} - 1\right), \quad n \in \mathbb{N}, \tag{1.5}$$

where $\sigma(\xi) = \sqrt{Var_\xi(\bar{W}_k)}$. One can calculate

$$\sigma^2(\xi) = \sum_{i=0}^{k-1} \Pi_i^{-1}\left(\frac{m_i(2)}{m_i^2} - 1\right).$$

It is clear that η_n concerns with the convergence rates of W_{n+k}/W_n to 1. Our interest is the limit behaviours of the distributions of η_n, under both quenched and annealed laws. For the Galton-Watson process, Heyde [11] showed a central limit theorem on η_n which says that the distribution of η_n converges to the standard normal distribution $\mathscr{N}(0,1)$, and Heyde and Brown [12] gave an estimation of its convergence rate under a third moment condition. Such results were extended to BPRE with weaker moment condition in Wang et al. [28] by considering the annealed law, and then Huang and Liu [13] improved the corresponding convergence rate by giving the precise asymptotics of the harmonic moments of $\bar{Z}_n$. The quenched central limit theorem was shown by Zhang and Hong [29], but the convergence rate is still unclear. For the case with immigration, Gao and Zhang [8] established the central limit theorem on η_n for a branching process in a varying environment, and estimated its convergence rate by harmonic moments of Z_n. However, the rate that they obtained seemed very rough, and could not coincide with the case without immigration when applied to the Galton-Watson process. For these reasons, we aim to extend the results for the case without immigration to BPIRE, and to improve them by establishing the non-uniform Berry-Esseen-type inequalities.

For $n \in \mathbb{N}$, let

$$G_n(x;\xi) = \mathbb{P}_\xi(\eta_n \leqslant x) \qquad \text{and} \qquad G_n(x) = \mathbb{P}(\eta_n \leqslant x) \qquad (x \in \mathbb{R})$$

be the quenched and annealed distributions of η_n respectively. The main results of the paper are the following two Berry-Esseen-type inequalities which can reflect the rates of convergence in central limit theorems on η_n.

THEOREM 1.2. (Quenched Berry-Esseen-type inequality) *Assume that* (1.3) *holds,* $\mathbb{E}\log^+\mathbb{E}_\xi|\frac{\bar{W}_k - 1}{\sigma(\xi)}|^a < \infty$ *and* $\mathbb{E}\log^+\mathbb{E}_\xi|\frac{W_k - \bar{W}_k}{\sigma(\xi)}|^b < \infty$ *for some* $a > 2$ *and* $b > 0$. *Let* $\varepsilon \in (0,\ 1]$ *satisfying* $\varepsilon < b$. *Then there exists a positive random variable* $C(\xi)$ *depending on* ξ *and satisfying* $\mathbb{E}\log^+ C(\xi) < \infty$ *such that*

$$\sup_{x\in\mathbb{R}}(1+|x|)^\lambda |G_n(x;\xi) - \Phi(x)| \leqslant C(T^n\xi)\mathbb{E}_\xi Z_n^{-\delta/2} \quad a.s., \tag{1.6}$$

where $\lambda = (a \wedge b)(1 - \frac{\varepsilon}{b})$ *and* $\delta = \min\{a-2,\ \varepsilon\}$, *and* $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\mathrm{d}t$ *is the standard normal distribution function.*

THEOREM 1.3. (Annealed Berry-Esseen-type inequality) *Assume* (1.3) *and that* $\xi = (\xi_n)$ *is i.i.d.,* $\mathbb{E}\left(\mathbb{E}_\xi |\frac{\bar{W}_k - 1}{\sigma(\xi)}|^a\right)^{1+\tilde{\varepsilon}} < \infty$ *and* $\mathbb{E}\left(\mathbb{E}_\xi |\frac{W_k - \bar{W}_k}{\sigma(\xi)}|^b\right)^{1+\tilde{\varepsilon}} < \infty$ *for some* $a > 2$, $b > 0$ *and* $\tilde{\varepsilon} \in [0,1]$. *Let* $\varepsilon \in (0, 1]$ *satisfying* $\varepsilon < b$. *Then there exists a positive constant* C *such that*

$$\sup_{x \in \mathbb{R}} (1+|x|)^\lambda |G_n(x) - \Phi(x)| \leqslant C\mathbb{E}Z_n^{-\delta/2}, \tag{1.7}$$

where $\lambda = (a \wedge b)\min\{\tilde{\varepsilon},\, 1 - \frac{\varepsilon}{b}\}$ *and* $\delta = \min\{a-2,\, \varepsilon\}$, *and* $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-t^2/2}\mathrm{d}t$ *is the standard normal distribution function.*

In Theorem 1.3 and throughout the paper, C represents a general positive constant, which does not stand for the same constant and can differ from line to line.

REMARK 1.2. In Theorems 1.2 and 1.3, we use the harmonic moments of Z_n to characterize the convergence rates in central limit theorems on η_n. Under (1.3), it is not difficult to know that the harmonic moments of Z_n decay to 0 with exponential rates. For the convenience of application, we give below more accurate results on decay rates of the harmonic moments of Z_n, and apply them with the conclusions of Theorems 1.2 and 1.3. In order to illustrate precise rates, the additional assumption (H) below is needed:

(H) There exist constants $q > 1$ and $A_2 > A_1 > 1$ such that $A_1 \leqslant m_0$ and $m_0(q) \leqslant A_2^q$ a.s., and ess sup $p_1 < 1$.

Based on (H), we can obtain the conditions for the existence of harmonic moments of W_n and further describe the decay rates of harmonic moments of Z_n more precisely. Let $r > 0$. According to [14], one has a.s.,

$$\limsup_{n\to\infty} \frac{1}{n}\log \mathbb{E}_\xi Z_n^{-r} \leqslant \begin{cases} \max\{-r\mathbb{E}\log m_0,\, \mathbb{E}\log(p_1 h_0)\}, & \text{if (H) holds,} \\ \mathbb{E}\log m_0(-r), & \text{if (H) does not hold.} \end{cases} \tag{1.8}$$

This result implies that $\mathbb{E}_\xi Z_n^{-r}$ decays to 0 with an exponential rate. Set

$$\rho_1 = \begin{cases} e^{\max\{-\frac{\delta}{2}\mathbb{E}\log m_0,\, \mathbb{E}\log(p_1 h_0)\}}, & \text{if (H) holds,} \\ e^{\mathbb{E}\log m_0(-\frac{\delta}{2})}, & \text{if (H) does not hold.} \end{cases}$$

Combining (1.6) with (1.8), we deduce that a.s., for $x \in \mathbb{R}$,

$$|G_n(x;\xi) - \Phi(x)| \leqslant C(T^n\xi)\mathbb{E}_\xi Z_n^{-\delta/2}(1+|x|)^{-\lambda} \leqslant \rho^n(1+|x|)^{-\lambda},$$

for $\rho > \rho_1$ and $0 \leqslant \lambda \leqslant \lambda_1 := (a\wedge b)(1 - \varepsilon/b)$. When $\xi = (\xi_n)$ is i.i.d., according to Huang et al. [15], one has

$$\mathbb{E}Z_n^{-r} \leqslant C\begin{cases} a_n(r), & \text{if (H) holds,} \\ (\mathbb{E}m_0(-r))^n, & \text{if (H) does not hold,} \end{cases} \tag{1.9}$$

where

$$a_n(r) = \begin{cases} (\mathbb{E}m_0^{-r})^n, & \text{if } r < r_0, \\ n(\mathbb{E}p_1h_0)^n, & \text{if } r = r_0, \\ (\mathbb{E}p_1h_0)^n, & \text{if } r > r_0, \end{cases}$$

and r_0 is the solution of the equation $\mathbb{E}m_0^{-r_0} = \mathbb{E}p_1h_0$. Set

$$\rho_2 = \begin{cases} \max\{\mathbb{E}m_0^{-\frac{\delta}{2}}, \mathbb{E}p_1h_0\}, & \text{if (H) holds,} \\ \mathbb{E}m_0(-\frac{\delta}{2}), & \text{if (H) does not hold.} \end{cases}$$

Combining (1.7) with (1.9) gives that for $x \in \mathbb{R}$,

$$|G_n(x) - \Phi(x)| \leqslant C\mathbb{E}Z_n^{-\delta/2}(1+|x|)^{-\lambda} \leqslant \rho^n(1+|x|)^{-\lambda}, \tag{1.10}$$

for $\rho > \rho_2$ and $0 \leqslant \lambda \leqslant \lambda_2 := (a \wedge b)\min\{\tilde{\varepsilon}, 1-\varepsilon/b\}$. One can observe that $\rho_1 \leqslant \rho_2$ and $\lambda_1 \geqslant \lambda_2$.

REMARK 1.3. For BPRE, Wang et al. [28, Theorem 2.2] showed that

$$\sup_{x\in\mathbb{R}} |G_n(x) - \Phi(x)| \leqslant C\mathbb{E}Z_n^{-\delta/2} \leqslant C\left(\mathbb{E}m_0(-\frac{\delta}{2})\right)^n \tag{1.11}$$

under the moment condition that $\mathbb{E}|\frac{\bar{W}_k-1}{\sigma(\xi)}|^a < \infty$ for some $a > 2$, where $\delta = \min\{a-2, 1\}$. Note that in (1.11), the upper bound is independent of x. This kind of inequality is called *the uniform Berry-Esseen-type inequality*. If that upper bound depends on x such as (1.10), corresponding result is called *the non-uniform Berry-Esseen-type inequality*. In Theorem 1.3, we actually establish the annealed non-uniform Berry-Esseen-type inequality on the distribution of η_n for BPIRE. This result is a generalization and improvement of Wang et al. [28, Theorem 2.2] and Huang and Liu [13, Theorem 1.7]. Similarly, Theorem 1.2 shows the corresponding quenched non-uniform Berry-Esseen-type inequality. Applied to BPRE, it improves Zhang and Hong [29, Theorem 1.3] by showing the rate of convergence in the quenched central limit theorem. Meanwhile, Theorem 1.2 also weakens the moment condition and estimates a faster convergence rate at the same time in comparison with Gao and Zhang [8, Theorem 1.4]. Applied to the Galton-Watson process with $a = 3$, Theorem 1.2 implies Heyde and Brown [12, Theorem 2].

REMARK 1.4. Note that in Theorems 1.2 and 1.3, the δ depends on both a and b. This fact reveals that for the case with immigration, the convergence rate in central limit theorem is affected not only by the offspring distribution, but also by the immigration distribution. Moreover, there exists certain mutual constraint between δ and λ, and the increase of one may lead to the decrease of the other.

2. Preliminaries

In this section, we do some preliminary work. For the convenience of comprehension, we introduce the family system. We regard Z_n as the total population of the n-th generation of the family. In this sense, the condition $Z_0 = 1$ means that the family begins with an initial ancestor of generation 0. We denote $u_{n,i}$ the i-th individual of the n-th generation, $i = 1,2,\cdots,Z_n$.Then $X_{n,i}$ can be understood as the offspring number of $u_{n,i}$, and Y_n is the number of new immigrants of generation $n+1$.

For a particle u of the family, we use $\bar{W}_k^{(u)}$ (resp. $W_k^{(u)}$) to denote the corresponding martingale (resp. submartingale) defined similarly to (1.4) for the BPRE (resp. BPIRE) originating from the particle u. Fix $k \in \{1,2,\cdots,\infty\}$. According to the differences of the predecessors at the n-th generation, we decompose

$$\eta_n = \sigma(T^n\xi)^{-1} Z_n^{-1/2}\left[\sum_{i=1}^{Z_n}(\bar{W}_k^{(u_{n,i})} - 1) + D_k^{(u_{n,1})}\right], \tag{2.1}$$

where

$$D_k^{(u_{n,1})} = W_k^{(u_{n,1})} - \bar{W}_k^{(u_{n,1})}.$$

When the current n-th generation of the family is known, all the information about the immigrants from $(n+1)$-th generation to $(n+k)$-th generation will be included in $D_k^{(u_{n,1})}$, while the first term of the sum in the right hand side of (2.1) only depends on the offspring distribution. In particular, one has $D_k^{(u_{n,1})} = 0$ for BPRE, since there is no immigration. Thanks to the decomposition (2.1), we can study the limit behaviours of η_n by controlling two kinds of conditions: one is about the offspring distribution, and the other is about the immigration distribution hidden behind $D_k^{(u_{n,1})}$. Set

$$S_j^{(n)} = \sigma(T^n\xi)^{-1}\sum_{i=1}^{j}(\bar{W}_k^{(u_{n,i})} - 1) \quad \text{and} \quad U^{(n)} = \sigma(T^n\xi)^{-1} D_k^{(u_{n,1})}.$$

For $x \in \mathbb{R}$, define two distribution functions as follows:

$$\begin{aligned}\Phi_j^{(n)}(x) &= \mathbb{P}_\xi(j^{-1/2}S_j^{(n)} \leqslant x),\\ G_j^{(n)}(x) &= \mathbb{P}_\xi(j^{-1/2}S_j^{(n)} + j^{-1/2}U^{(n)} \leqslant x).\end{aligned}$$

By the formula of total probability,

$$G_n(x;\xi) - \Phi(x) = \sum_{j=1}^{\infty}\mathbb{P}_\xi(Z_n = j)(G_j^{(n)}(x) - \Phi(x)). \tag{2.2}$$

Notice that for n fixed, $\Phi_j^{(n)}(x)$ is the distribution function of the normalized partial sum of the i.i.d. sequence $\{\bar{W}_k^{(u_{n,i})} - 1\}_{i\geqslant 1}$ under the quenched law $\mathbb{P}_\xi$. We can apply the classical Berry-Esseen inequality to $\Phi_j^{(n)}(x)$.

LEMMA 2.1. (Berry-Esseen inequality for i.i.d. sequence, [5, 22]) *Let X be a random variable with $\mathbb{E}X = \mu$ and $\mathbb{E}(X-\mu)^2 = \sigma^2$. Let X_n be independent copies of X and set $S_n = \sum_{i=1}^n (X_i - \mu)$. If $\sigma > 0$ and $\mathbb{E}|X|^a < \infty$ for some $a > 2$, then*

$$\left|\mathbb{P}\left(\frac{S_n}{\sqrt{n}\sigma} \leqslant x\right) - \Phi(x)\right| \leqslant C \max\left\{1, \mathbb{E}\left|\frac{X-\mu}{\sigma}\right|^a\right\} n^{-\gamma/2}(1+|x|)^{-a},$$

where $\gamma = \min\{a-2, 1\}$.

Put

$$A(\xi,\ a) = \mathbb{E}_\xi \left|\frac{\bar{W}_k - 1}{\sigma(\xi)}\right|^a \quad \text{and} \quad B(\xi,\ a) = \mathbb{E}_\xi \left|\frac{W_k - \bar{W}_k}{\sigma(\xi)}\right|^b. \tag{2.3}$$

For $a > 2$, set

$$\gamma = \min\{a-2,\ 1\}.$$

By Lemma 2.1, we have a.s. for all $x \in \mathbb{R}$,

$$\left|\Phi_j^{(n)}(x) - \Phi(x)\right| \leqslant C \max\{1, A(T^n\xi, a)\}\, j^{-\gamma/2}(1+|x|)^{-a} \tag{2.4}$$

provided that $A(\xi, a) < \infty$ a.s. for some $a > 2$. Noticing (2.4) and the fact that

$$G_j^{(n)}(x) - \Phi(x) = G_j^{(n)}(x) - \Phi_j^{(n)}(x) + \Phi_j^{(n)}(x) - \Phi(x), \tag{2.5}$$

we need to deal with $G_j^{(n)}(x) - \Phi_j^{(n)}(x)$.

It is easy to see that

$$\begin{aligned} G_j^{(n)}(x) - \Phi_j^{(n)}(x) =& \mathbb{P}_\xi\left(j^{-1/2}S_j^{(n)} + j^{-1/2}U^{(n)} \leqslant x,\ j^{-1/2}S_j^{(n)} > x\right) \\ &- \mathbb{P}_\xi\left(j^{-1/2}S_j^{(n)} + j^{-1/2}U^{(n)} > x,\ j^{-1/2}S_j^{(n)} \leqslant x\right). \end{aligned} \tag{2.6}$$

Take $m = [\sqrt{j}]$. Denote $\nu_\xi^{(n)}(\mathrm{d}s,\ \mathrm{d}t) = \mathbb{P}_\xi(j^{-1/2}S_m^{(n)} \in \mathrm{d}s, j^{-1/2}U^{(n)} \in \mathrm{d}t)$. We deduce that

$$\begin{aligned} &\mathbb{P}_\xi\left(j^{-1/2}S_j^{(n)} + j^{-1/2}U^{(n)} \leqslant x,\ j^{-1/2}S_j^{(n)} > x\right) \\ =& \int \mathbb{P}_\xi\left(s + t + j^{-1/2}(S_j^{(n)} - S_m^{(n)}) \leqslant x,\ s + j^{-1/2}(S_j^{(n)} - S_m^{(n)}) > x\right) \nu_\xi^{(n)}(\mathrm{d}s,\ \mathrm{d}t) \\ =& \int_{t<0} \left[\Phi_{j-m}^{(n)}((x-s-t)R_j) - \Phi_{j-m}^{(n)}((x-s)R_j)\right] \nu_\xi^{(n)}(\mathrm{d}s,\ \mathrm{d}t), \end{aligned} \tag{2.7}$$

where $R_j = \frac{\sqrt{j}}{\sqrt{j-m}}$. Similarly, we can obtain

$$\begin{aligned} &\mathbb{P}_\xi\left(j^{-1/2}S_j^{(n)} + j^{-1/2}U^{(n)} > x,\ j^{-1/2}S_j^{(n)} \leqslant x\right) \\ =& \int_{t>0} \left[\Phi_{j-m}^{(n)}((x-s)R_j) - \Phi_{j-m}^{(n)}((x-s-t)R_j)\right] \nu_\xi^{(n)}(\mathrm{d}s, \mathrm{d}t). \end{aligned} \tag{2.8}$$

Noticing (2.4), we calculate

$$
\begin{aligned}
&|\Phi_{j-m}^{(n)}((x-s)R_j)-\Phi_{j-m}^{(n)}((x-s-t)R_j)|\\
\leqslant&|\Phi((x-s)R_j)-\Phi((x-s-t)R_j)|+|\Phi_{j-m}^{(n)}((x-s)R_j)-\Phi((x-s)R_j)|\\
&+|\Phi_{j-m}^{(n)}((x-s-t)R_j)-\Phi((x-s-t)R_j)|\\
\leqslant&|\Phi((x-s)R_j)-\Phi((x-s-t)R_j)|\\
&+C\max\{1,A(T^n\xi,a)\}\,j^{-\gamma/2}\left[(1+|x-s|)^{-a}+(1+|x-s-t|)^{-a})\right],
\end{aligned}
\tag{2.9}
$$

where we have used the fact that $R_j \geqslant 1$. Combining (2.6)–(2.9), we get

$$
\left|G_j^{(n)}(x)-\Phi_j^{(n)}(x)\right| \leqslant I_{1j}^{(n)}+C\max\{1,A(T^n\xi,a)\}\,j^{-\gamma/2}\left(I_{2j}^{(n)}+I_{3j}^{(n)}\right), \tag{2.10}
$$

where

$$
\begin{aligned}
I_{1j}^{(n)}&=\int|\Phi((x-s)R_j)-\Phi((x-s-t)R_j)|\nu_\xi^{(n)}(\mathrm{d}s,\ \mathrm{d}t),\\
I_{2j}^{(n)}&=\int(1+|x-s|)^{-a}\nu_\xi^{(n)}(\mathrm{d}s,\ \mathrm{d}t),\\
I_{3j}^{(n)}&=\int(1+|x-s-t|)^{-a}\nu_\xi^{(n)}(\mathrm{d}s,\ \mathrm{d}t).
\end{aligned}
$$

We will give upper bounds for $I_{lj}^{(n)}$, $l=1,2,3$, in Section 3. Before that work, we present two preliminary lemmas.

LEMMA 2.2. *Let $c>0$ be a fixed constant and $\tilde{\varepsilon}\in[0,1]$. Assume that $A(\xi,a)<\infty$ a.s. for some $a>2$. Then a.s., for all $|x|\geqslant 1$,*

$$
\int_{|s|>c|x|}\nu_\xi^{(n)}(\mathrm{d}s,\ \mathrm{d}t)\leqslant C\max\left\{1,\ A(T^n\xi,\ a)^{\tilde{\varepsilon}}\right\}j^{-\gamma\tilde{\varepsilon}/2}(1+|x|)^{-a\tilde{\varepsilon}}, \tag{2.11}
$$

where $\gamma=\min\{a-2,1\}$.

Proof. By (2.4), we derive that

$$
\begin{aligned}
&\int_{|s|>c|x|}\nu_\xi^{(n)}(\mathrm{d}s,\ \mathrm{d}t)\\
=&\mathbb{P}_\xi(|j^{-1/2}S_m^{(n)}|>c|x|)\\
=&1-\Phi_m^{(n)}\left(c\frac{\sqrt{j}}{\sqrt{m}}|x|\right)+\Phi_m^{(n)}\left(-c\frac{\sqrt{j}}{\sqrt{m}}|x|\right)\\
\leqslant&2\left[1-\Phi\left(c\frac{\sqrt{j}}{\sqrt{m}}|x|\right)\right]+C\max\{1,A(T^n\xi,a)\}\,m^{-\gamma/2}\left(1+c\frac{\sqrt{j}}{\sqrt{m}}|x|\right)^{-a}.
\end{aligned}
\tag{2.12}
$$

In addition,

$$
1-\Phi\left(c\frac{\sqrt{j}}{\sqrt{m}}|x|\right)\leqslant C\left(\frac{\sqrt{j}}{\sqrt{m}}|x|\right)^{-2a}\leqslant Cj^{-a/2}(1+|x|)^{-a}, \tag{2.13}
$$

and

$$m^{-\gamma/2}\left(1+c\frac{\sqrt{j}}{\sqrt{m}}|x|\right)^{-a}\leqslant Cj^{-\gamma/2}(1+|x|)^{-a}, \tag{2.14}$$

since $a>\gamma$. Combining (2.12) with (2.13) and (2.14) yields

$$\int_{|s|>c|x|}\nu_{\xi}^{(n)}(\mathrm{d}s,\ \mathrm{d}t)\leqslant C\max\{1,\ A(T^n\xi,\ a)\}\,j^{-\gamma/2}(1+|x|)^{-a}. \tag{2.15}$$

Since $\tilde{\varepsilon}\in[0,1]$ and

$$\int_{|s|>c|x|}\nu_{\xi}^{(n)}(\mathrm{d}s,\ \mathrm{d}t)\leqslant 1, \tag{2.16}$$

we obtain (2.11) by (2.15) and (2.16). $\square$

LEMMA 2.3. *Let $c>0$ be a fixed constant and $\tilde{\varepsilon}\in[0,1]$. Assume that $B(\xi,\ b)<\infty$ for some $b>0$. Then a.s., for all $|x|\geqslant 1$,*

$$\int_{|t|>c|x|}\nu_{\xi}^{(n)}(\mathrm{d}s,\ \mathrm{d}t)\leqslant CB(T^n\xi,\ b)^{\tilde{\varepsilon}}j^{-b\tilde{\varepsilon}/2}(1+|x|)^{-b\tilde{\varepsilon}}. \tag{2.17}$$

Proof. By Markov's inequality, we get

$$\begin{aligned}\int_{|t|>c|x|}\nu_{\xi}^{(n)}(\mathrm{d}s,\ \mathrm{d}t)&=\mathbb{P}_{\xi}\left(\left|j^{-1/2}U^{(n)}\right|>c|x|\right)\\&\leqslant Cj^{-b/2}\mathbb{E}_{\xi}\left|U^{(n)}\right|^{b}|x|^{-b}\\&\leqslant CB(T^n\xi,\ b)j^{-b/2}(1+|x|)^{-b},\end{aligned}$$

which leads to (2.17), since $\tilde{\varepsilon}\in[0,1]$ and $\int_{|t|>c|x|}\nu_{\xi}^{(n)}(\mathrm{d}s,\ \mathrm{d}t)\leqslant 1$. $\square$

3. Proofs of Theorems 1.2 and 1.3

In this section, we will go to proofs of the main theorems. Recall (2.10). From it we see that in order to get the rate of convergence of $G_j^{(n)}(x)-\Phi_j^{(n)}(x)$, we need to analyse I_{lj}^{n}, $l=1,2,3$. Recall $A(\xi,\ a)$ and $B(\xi,\ b)$ defined in (2.3).

LEMMA 3.1. *Let $\tilde{\varepsilon}\in[0,1]$. If $A(\xi,\ a)<\infty$ a.s. for some $a>2$, then a.s., for all $x\in\mathbb{R}$,*

$$I_{2j}^{(n)}\leqslant C\max\left\{1,\ A(T^n\xi,\ a)^{\tilde{\varepsilon}}\right\}(1+|x|)^{-a\tilde{\varepsilon}}. \tag{3.1}$$

Proof. For $|x|<1$, noticing that $(1+|x|)^{-a\tilde{\varepsilon}}\geqslant 2^{-a\tilde{\varepsilon}}$, we have

$$I_{2j}^{(n)}\leqslant 1\leqslant C(1+|x|)^{-a\tilde{\varepsilon}}. \tag{3.2}$$

For $|x| \geqslant 1$, by Lemma 2.2,

$$\begin{aligned} I_{2j}^{(n)} &\leqslant \int_{|s| \leqslant \frac{1}{2}|x|} (1+|x-s|)^{-a} \nu_{\xi}^{(n)}(\mathrm{d}s,\ \mathrm{d}t) + \int_{|s| > \frac{1}{2}|x|} \nu_{\xi}^{(n)}(\mathrm{d}s,\ \mathrm{d}t) \\ &\leqslant \left(1+\frac{1}{2}|x|\right)^{-a\tilde{\varepsilon}} + C \max\left\{1,\ A(T^n\xi,\ a)^{\tilde{\varepsilon}}\right\} j^{-\gamma\tilde{\varepsilon}/2}(1+|x|)^{-a\tilde{\varepsilon}} \\ &\leqslant C \max\left\{1,\ A(T^n\xi,\ a)^{\tilde{\varepsilon}}\right\}(1+|x|)^{-a\tilde{\varepsilon}}. \end{aligned} \tag{3.3}$$

Combining (3.2) and (3.3), we get (3.1). $\square$

LEMMA 3.2. *Let $\tilde{\varepsilon} \in [0,1]$. If $A(\xi,\ a) < \infty$ a.s. for some $a > 2$ and $B(\xi,\ b) < \infty$ for some $b > 0$, then a.s., for all $x \in \mathbb{R}$,*

$$I_{3j}^{(n)} \leqslant C \max\left\{1,\ A(T^n\xi,\ a)^{\tilde{\varepsilon}},\ B(T^n\xi,\ b)^{\tilde{\varepsilon}}\right\}(1+|x|)^{-(a\wedge b)\tilde{\varepsilon}}. \tag{3.4}$$

Proof. For $|x| < 1$, it is obvious that

$$I_{3j}^{(n)} \leqslant 1 \leqslant C(1+|x|)^{-(a\wedge b)\tilde{\varepsilon}}. \tag{3.5}$$

For $|x| \geqslant 1$, applying Lemmas 2.2 and 2.3, we obtain

$$\begin{aligned} I_{3j}^{(n)} &\leqslant \int_{|s+t| \leqslant \frac{1}{2}|x|} (1+|x-s-t|)^{-a} \nu_{\xi}^{(n)}(\mathrm{d}s,\ \mathrm{d}t) \\ &\quad + \int_{|s| > \frac{1}{4}|x|} \nu_{\xi}^{(n)}(\mathrm{d}s,\ \mathrm{d}t) + \int_{|t| > \frac{1}{4}|x|} \nu_{\xi}^{(n)}(\mathrm{d}s,\ \mathrm{d}t) \\ &\leqslant \left(1+\frac{1}{2}|x|\right)^{-a\tilde{\varepsilon}} + \max\left\{1,\ A(T^n\xi,\ a)^{\tilde{\varepsilon}}\right\}(1+|x|)^{-a\tilde{\varepsilon}} \\ &\quad + B(T^n\xi,\ b)^{\tilde{\varepsilon}}(1+|x|)^{-b\tilde{\varepsilon}} \\ &\leqslant \max\left\{1,\ A(T^n\xi,\ a)^{\tilde{\varepsilon}},\ B(T^n\xi,\ b)^{\tilde{\varepsilon}}\right\}(1+|x|)^{-(a\wedge b)\tilde{\varepsilon}}. \end{aligned} \tag{3.6}$$

Combining (3.5) and (3.6), we get (3.4). $\square$

LEMMA 3.3. *Assume that $A(\xi,\ a) < \infty$ a.s. for some $a > 2$ and $B(\xi,\ b) < \infty$ a.s. for some $b > 0$. Let $\varepsilon \in (0,\ 1]$ satisfying $\varepsilon < b$. Then a.s., for all $x \in \mathbb{R}$,*

$$I_{1j}^{(n)} \leqslant C \max\{1, A(T^n\xi, a), B(T^n\xi, b)\}\, j^{-\delta/2}(1+|x|)^{-\lambda}, \tag{3.7}$$

where $\lambda = (a\wedge b)(1-\frac{\varepsilon}{b})$ and $\delta = \min\{a-2,\ \varepsilon\}$.

Proof. We first consider $|x| < 1$. It is obvious that

$$|\Phi((x-s)R_j) - \Phi((x-s-t)R_j)| \leqslant \min\{1,\ C|t|\} \leqslant C|t|^{\varepsilon}.$$

Therefore, by Hölder's inequality,

$$
\begin{aligned}
I_{1j}^{(n)} &\leqslant C\int |t|^{\varepsilon}\nu_{\xi}^{(n)}(\mathrm{d}s,\ \mathrm{d}t) \leqslant C\left(\int |t|^{b}\nu_{\xi}^{(n)}(\mathrm{d}s,\ \mathrm{d}t)\right)^{\varepsilon/b} \\
&= CB(T^n\xi,\ b)^{\varepsilon/b} j^{-\varepsilon/2} \leqslant C\max\{1,\ B(T^n\xi,\ b)\} j^{-\delta/2}(1+|x|)^{-\lambda}.
\end{aligned}
$$

Next we consider $|x| \geqslant 1$. Decompose

$$
I_{1j}^{(n)} = I_{Aj}^{(n)} + I_{Bj}^{(n)},
$$

where

$$
\begin{aligned}
I_{Aj}^{(n)} &= \int_{(x-s)(x-s-t)<0} |\Phi((x-s)R_j) - \Phi((x-s-t)R_j)|\nu_{\xi}^{(n)}(\mathrm{d}s,\ \mathrm{d}t), \\
I_{Bj}^{(n)} &= \int_{(x-s)(x-s-t)\geqslant 0} |\Phi((x-s)R_j) - \Phi((x-s-t)R_j)|\nu_{\xi}^{(n)}(\mathrm{d}s,\ \mathrm{d}t).
\end{aligned}
$$

For $I_{Aj}^{(n)}$, noticing that

$$
\{(x-s)(x-s-t)<0\} \subset \left\{|s| > \frac{1}{2}|x|\right\} \cup \left\{|t| > \frac{1}{2}|x|\right\},
$$

by Lemmas 2.2 and 2.3, we get a.s., for $|x| \geqslant 1$,

$$
\begin{aligned}
I_{Aj}^{(n)} &\leqslant \int_{|s|>\frac{1}{2}|x|} \nu_{\xi}^{(n)}(\mathrm{d}s,\ \mathrm{d}t) + \int_{|t|>\frac{1}{2}|x|} \nu_{\xi}^{(n)}(\mathrm{d}s,\ \mathrm{d}t) \\
&\leqslant C\max\{1,\ A(T^n\xi,\ a), B(T^n\xi, b)\}\, j^{-\frac{1}{2}\min\{a-2,\ 1,\ b\}}(1+|x|)^{-(a\wedge b)}.
\end{aligned} \tag{3.8}
$$

Now we deal with $I_{Bj}^{(n)}$. By differential mean value theorem, we get

$$
\Phi((x-s)R_j) - \Phi((x-s-t)R_j) = \varphi(\zeta)tR_j,
$$

where $\varphi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ is the density function of the standard normal distribution, and ζ is between $(x-s)R_j$ and $(x-s-t)R_j$. When $(x-s)(x-s-t) \geqslant 0$, we have

$$
\begin{aligned}
\varphi(\zeta) &\leqslant \max\{\varphi((x-s)R_j),\ \varphi((x-s-t)R_j)\} \\
&\leqslant C\max\left\{e^{-\frac{1}{2}(x-s)^2},\ e^{-\frac{1}{2}(x-s-t)^2}\right\},
\end{aligned}
$$

so that

$$
|\Phi((x-s)R_j) - \Phi((x-s-t)R_j)| \leqslant C\max\left\{e^{-\frac{1}{2}(x-s)^2},\ e^{-\frac{1}{2}(x-s-t)^2}\right\}|t|. \tag{3.9}
$$

Besides, it is clear that

$$
|\Phi((x-s)R_j) - \Phi((x-s-t)R_j)| \leqslant 1. \tag{3.10}
$$

Noticing (3.9) and (3.10), we deduce that

$$\begin{aligned} I_{Bj}^{(n)} &\leqslant C\int \min\left\{1,\ \max\left\{e^{-\frac{1}{2}(x-s)^2},\ e^{-\frac{1}{2}(x-s-t)^2}\right\}|t|\right\}\nu_\xi^{(n)}(\mathrm{d}s,\ \mathrm{d}t) \\ &\leqslant C\int \max\left\{e^{-\frac{\varepsilon}{2}(x-s)^2},\ e^{-\frac{\varepsilon}{2}(x-s-t)^2}\right\}|t|^\varepsilon \nu_\xi^{(n)}(\mathrm{d}s,\ \mathrm{d}t). \end{aligned}$$

Denote $q=\frac{b}{b-\varepsilon}$. By Hölder's inequality,

$$\begin{aligned} I_{Bj}^{(n)} \leqslant& C\left(\int \max\left\{e^{-\frac{\varepsilon}{2}q(x-s)^2},\ \ e^{-\frac{\varepsilon}{2}q(x-s-t)^2}\right\}\nu_\xi^{(n)}(\mathrm{d}s,\ \mathrm{d}t)\right)^{\frac{1}{q}} \\ &\cdot\left(\int |t|^b\nu_\xi^{(n)}(\mathrm{d}s,\ \mathrm{d}t)\right)^{\frac{\varepsilon}{b}}. \end{aligned} \tag{3.11}$$

Obviously,

$$\int |t|^b\nu_\xi^{(n)}(\mathrm{d}s,\ \mathrm{d}t)=j^{-b/2}B(T^n\xi,\ b). \tag{3.12}$$

Let $c>0$ be a constant. Using Lemma 2.2, we derive that a.s., for $|x|\geqslant 1$,

$$\begin{aligned} &\int e^{-c(x-s)^2}\nu_\xi^{(n)}(\mathrm{d}s,\ \mathrm{d}t) \\ \leqslant&\int_{|s|\leqslant\frac{1}{2}|x|} e^{-c(x-s)^2}\nu_\xi^{(n)}(\mathrm{d}s,\ \mathrm{d}t)+\int_{|s|>\frac{1}{2}|x|}\nu_\xi^{(n)}(\mathrm{d}s,\ \mathrm{d}t) \\ \leqslant& e^{-\frac{c}{4}x^2}+\int_{|s|>\frac{1}{2}|x|}\nu_\xi^{(n)}(\mathrm{d}s,\ \mathrm{d}t) \\ \leqslant& C\max\{1,\ A(T^n\xi,\ a)\}(1+|x|)^{-a}. \end{aligned} \tag{3.13}$$

Similarly, by Lemmas 2.2 and 2.3, we have a.s., for $|x|\geqslant 1$,

$$\begin{aligned} &\int e^{-c(x-s-t)^2}\nu_\xi^{(n)}(\mathrm{d}s,\ \mathrm{d}t) \\ \leqslant&\int_{|s+t|\leqslant\frac{1}{2}|x|} e^{-c(x-s-t)^2}\nu_\xi^{(n)}(\mathrm{d}s,\ \mathrm{d}t)+\int_{|s|>\frac{1}{4}|x|}\nu_\xi^{(n)}(\mathrm{d}s,\ \mathrm{d}t) \\ &+\int_{|t|>\frac{1}{4}|x|}\nu_\xi^{(n)}(\mathrm{d}s,\ \mathrm{d}t) \\ \leqslant& C\max\{1,\ A(T^n\xi,\ a),\ B(T^n\xi,\ b)\}(1+|x|)^{-(a\wedge b)}. \end{aligned} \tag{3.14}$$

Combining (3.11)–(3.14), we get a.s., for $|x|\geqslant 1$,

$$I_{Bj}^{(n)}\leqslant C\max\{1,\ A(T^n\xi,\ a),\ B(T^n\xi,\ b)\}j^{-\varepsilon/2}(1+|x|)^{-(a\wedge b)/q}. \tag{3.15}$$

Combining (3.8) and (3.15), we assert that (3.7) is hold for $|x|\geqslant 1$. □

Proof of Theorem 1.2. Let $\tilde{\varepsilon}\in[0,1]$. Noticing (2.10), by Lemmas 3.1–3.3, we get a.s., for all $x\in\mathbb{R}$,

$$|G_j^{(n)}(x)-\Phi_j^{(n)}(x)|\leqslant C_{\tilde{\varepsilon}}(T^n\xi)j^{-\delta/2}(1+|x|)^{-(a\wedge b)\min\{\tilde{\varepsilon},\ 1-\frac{\varepsilon}{b}\}}, \tag{3.16}$$

where $\delta = \min\{a-2, \varepsilon\}$ and

$$C_{\tilde{\varepsilon}}(\xi) = C \max\left\{1,\ A(\xi,\ a)^{1+\tilde{\varepsilon}},\ B(\xi,\ b)^{1+\tilde{\varepsilon}}\right\}.$$

Combining (2.2) with (2.4), (2.5) and (3.16), we get

$$|G_n(x;\xi) - \Phi(x)| \leqslant C_{\tilde{\varepsilon}}(T^n\xi)\mathbb{E}_\xi Z_n^{-\delta/2}(1+|x|)^{-(a\wedge b)\min\{\tilde{\varepsilon},\ 1-\frac{\varepsilon}{b}\}}. \tag{3.17}$$

Since $\mathbb{E}\log^+\mathbb{E}_\xi|\frac{\bar{W}_k - 1}{\sigma(\xi)}|^a < \infty$ and $\mathbb{E}\log^+\mathbb{E}_\xi|\frac{W_k - \bar{W}_k}{\sigma(\xi)}|^b < \infty$, it can be verified that

$$\mathbb{E}\log^+ C_{\tilde{\varepsilon}}(\xi) < \infty.$$

The proof is finished by taking $\tilde{\varepsilon} = 1$ and $C(\xi) = C_1(\xi)$. $\square$

Proof of Theorem 1.3 Taking the expectation of (3.17), we obtain

$$\begin{aligned}
|G_n(x) - \Phi(x)| &= |\mathbb{E}(G_n(x;\xi) - \Phi(x))| \\
&\leqslant \mathbb{E}|G_n(x;\xi) - \Phi(x)| \\
&\leqslant \mathbb{E}C_{\tilde{\varepsilon}}(\xi)\mathbb{E}Z_n^{-\delta/2}(1+|x|)^{-(a\wedge b)\min\{\tilde{\varepsilon},\ 1-\frac{\varepsilon}{b}\}}.
\end{aligned}$$

The proof is finished by noticing that $\mathbb{E}C_{\tilde{\varepsilon}}(\xi) < \infty$. $\square$

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