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IMPROVED MATRIX INEQUALITIES USING RADICAL CONVEXITY

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Abstract. Convex functions have a key role in mathematical inequalities. In this paper, we employ radical convexity as a tool that enables us to obtain much sharper bounds than usual bounds obtained by convexity. Applications of our approach will include real functions and matrices.

1. Introduction

Convex functions have received considerable and renowned attention in the literature due to their significance in various fields of mathematics, including analysis, optimization, mathematical physics, functional analysis, and operator theory. Among the most useful applications of convexity is the way they can be used to obtain inequalities among real numbers, real functions, matrices and operators. This includes celebrated inequalities like Young, Cauchy-Schwarz, Bellman, Heinz, and many other well-established inequalities.

As a research trend in mathematical inequalities, it is of great interest to minimize the difference between the two sides of the inequality by adding a certain term to one side. Such a process is usually referred to as a refinement of inequality. For example, Young's inequality states that

$$a^{1-t}b^t \leqslant (1-t)a+tb, \ \ 0 \leqslant t \leqslant 1; \ \ a,b>0. \tag{1.1}$$

This inequality can be proved using the convexity of the function $f(t)=a^{1-t}b^t$. Then, using refinements of convex functions, we can obtain refinements of (1.1). We refer the reader to [1, 11, 12, 13, 14, 15] for a recent list of references that discuss the idea of employing convexity to obtain refinements or new proofs of known inequalities.

In [9, 10], a new treatment of convex functions was discussed, where a tool for distinguishing how much a function is convex was presented.

We recall that a convex function $f : J \subseteq \mathbb{R} \to \mathbb{R}$ is a function that satisfies

$$f((1-t)a+tb) \leqslant (1-t)f(a)+tf(b), \tag{1.2}$$

for all $0 \leqslant t \leqslant 1$ and $a,b \in J$, where J is a real interval. The inequality (1.2) has been refined in the literature, and many applications were presented for scalars and matrices. We refer the reader to [1, 11, 12, 13, 14, 15] for further discussion.

To better understand convex functions, the authors introduced the concept of p-radical convex functions in [9] as follows.

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DEFINITION 1.1. Let $f:[0,\infty)\to[0,\infty)$ be a continuous function with $f(0)=0$, and let $p\geqslant 1$ be a fixed number. If the function $g(x)=f\left(x^{\frac{1}{p}}\right)$ is convex on $[0,\infty)$, we say that f is p-radical convex.

Two examples of 2–radical convex function are listed below.

- The function $f(x):=x^2\log(x+1)$ fulfills the conditions $f(0)=0$ and $f(x)>0$ for $x>0$. And also we have $\dfrac{d^2f(\sqrt{x})}{dx^2}=\dfrac{2\sqrt{x}+3}{4\sqrt{x}(\sqrt{x}+1)^2}>0$ for $x>0$.
- The function $g(x):=(x-1)\exp(x-1)+1/e$ fulfills the conditions $g(0)=0$ and $g(x)>0$ for $x>0$. And also we have $\dfrac{d^2g(\sqrt{x})}{dx^2}=\dfrac{e^{\sqrt{x}-1}}{4\sqrt{x}}>0$ for $x>0$.

In this article, we focus on 2-radical convex functions. That is, functions $f:[0,\infty)\to[0,\infty)$ such that $f(\sqrt{x})$ is convex and $f(0)=0$. We remark that a 2-radical convex function is necessarily monotone increasing; see [9, Proposition 1.1].

In particular, we present several inequalities that considerably refine some known inequalities in the literature for convex functions, then present applications in matrix settings.

The advantage of this work will be as follows. For example, the function $f(x)=x^2$ is convex. Applying convexity implies certain bounds. However, when looking at the function $f(x)=x^2$ as a 2-radical convex function, we will be able to obtain sharper bounds. Thus, it is always wise to check first if the function is 2-radical convex when dealing with convex functions. Our results can be easily extended to p-radical convex functions.

Applications that include inequalities for convex functions and matrices will be presented, emphasizing that the obtained results are much sharper than the existing bounds in the literature. In the end, examples with visual explanations are given.

2. Inequalities for 2-radical convex functions

In this section, we further explore 2-radical convex functions by presenting several inequalities of these functions. These inequalities significantly improve the corresponding inequalities when the function is treated as a convex function. However, for organizational purposes, we split this section into two subsections.

2.1. The domain $t\in[0,1]$

In [9, Corollary 2.1], it has been shown that if f is 2-radical convex, then

$$f((1-t)a+tb)+f\left(\sqrt{t(1-t)}|a-b|\right)\leqslant(1-t)f(a)+tf(b), \tag{2.1}$$

for $0\leqslant t\leqslant 1$ and $a,b\geqslant 0$. We notice that this is a refinement of (1.2).

As we mentioned earlier, our goal is to show how treating convex functions as radical convex functions imply better bounds. For the first result, we recall that a convex function $f:[0,\infty)\to[0,\infty)$ satisfies the inequality [6, 7]

$$f((1-t)a+tb)+2r\left(\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right)\leqslant(1-t)f(a)+tf(b),\quad 0\leqslant t\leqslant 1,\ a,b\geqslant 0,\qquad(2.2)$$

where $r=\min\{t,1-t\}$. This provides a refinement of (1.2). In the following result, we show that radical convex functions can have better bounds than the one presented in (2.2).

THEOREM 2.1. *Let $f:[0,\infty)\to[0,\infty)$ be a 2-radical convex function and let $a,b\geqslant 0$. Then for any $0\leqslant t\leqslant 1$,*

$$\begin{aligned}&f((1-t)a+tb)+f\left(\sqrt{\frac{r|1-2t|}{2}}|a-b|\right)+2r\left(\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right)\\&\leqslant(1-t)f(a)+tf(b)\end{aligned}$$

where $r=\min\{t,1-t\}$.

Proof. Assume that $0\leqslant t\leqslant 1/2$. Noting (2.1), we have

$$\begin{aligned}&(1-t)f(a)+tf(b)-2r\left(\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right)\\&=(1-2t)f(a)+2tf\left(\frac{a+b}{2}\right)\\&\geqslant f\left((1-2t)a+2t\frac{a+b}{2}\right)+f\left(\sqrt{2t(1-2t)}\frac{|a-b|}{2}\right)\\&=f((1-t)a+tb)+f\left(\sqrt{2t(1-2t)}\frac{|a-b|}{2}\right).\end{aligned}$$

This proves the desired inequality when $0\leqslant t\leqslant 1/2$. Now if $1/2\leqslant t\leqslant 1$, we have

$$\begin{aligned}&(1-t)f(a)+tf(b)-2r\left(\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right)\\&=(2t-1)f(b)+(2-2t)f\left(\frac{a+b}{2}\right)\\&\geqslant f\left((2t-1)b+(2-2t)\frac{a+b}{2}\right)+f\left(\sqrt{(2t-1)(2-2t)}\frac{|a-b|}{2}\right)\\&=f((1-t)a+tb)+f\left(\sqrt{2(1-t)(2t-1)}\frac{|a-b|}{2}\right).\end{aligned}$$

Noting that $r=\min\{t,1-t\}$, the proof is complete. $\square$

So, Theorem 2.1 adds the refining term $f\left(\sqrt{\frac{r|1-2t|}{2}}\,|a-b|\right)$ to the left side of (2.2). Of course, when f is 2-radical convex.

The following result presents a more explicit refinement of (1.2) for radical convex functions. In this result, we use the simple inequality $g(\alpha x) \leqslant \alpha g(x)$, when $g:[0,\infty)\to[0,\infty)$ is convex such that $g(0)=0$, and $0\leqslant\alpha\leqslant 1$.

THEOREM 2.2. *Let $f:[0,\infty)\to[0,\infty)$ be a 2-radical convex function. Then for any $0\leqslant t\leqslant 1$ and $a,b>0$,*

$$f\left((1-t)a+tb\right)\leqslant\frac{\left((1-t)a+tb\right)^2}{(1-t)a^2+tb^2}\left((1-t)f(a)+tf(b)\right).$$

In particular,

$$f\left(\frac{a+b}{2}\right)\leqslant\frac{(a+b)^2}{4\left(a^2+b^2\right)}\left(f(a)+f(b)\right).$$

Proof. Let $g(x)=f(\sqrt{x})$, $x\in[0,\infty)$. Then g is an increasing convex function on $[0,\infty)$ and $g(\alpha x)\leqslant\alpha g(x)$ when $0<\alpha\leqslant 1$. Thus, for $a,b>0$,

$$\begin{aligned}
g\left(\left((1-t)a+tb\right)^2\right)&=g\left(\frac{\left((1-t)a+tb\right)^2}{(1-t)a^2+tb^2}\left(1-t\right)a^2+tb^2\right)\\
&\leqslant\frac{\left((1-t)a+tb\right)^2}{(1-t)a^2+tb^2}g\left((1-t)a^2+tb^2\right)\\
&\leqslant\frac{\left((1-t)a+tb\right)^2}{(1-t)a^2+tb^2}\left((1-t)g\left(a^2\right)+tg\left(b^2\right)\right),
\end{aligned}$$

that is,

$$f\left((1-t)a+tb\right)\leqslant\frac{\left((1-t)a+tb\right)^2}{(1-t)a^2+tb^2}\left((1-t)f(a)+tf(b)\right),$$

as desired. □

The fact that Theorem 2.2 provides a refinement of (1.2) follows because $\frac{((1-t)a+tb)^2}{(1-t)a^2+tb^2}\leqslant 1$.

In the following, we discuss the behavior of the constant appearing in Theorem 2.2.

REMARK 2.1. For $0\leqslant t\leqslant 1$ and $x>0$, define

$$f_t(x):=\frac{\left((1-t)x+t\right)^2}{(1-t)x^2+t};\ x>0,\ \ 0\leqslant t\leqslant 1.$$

Since

$$\frac{df_t(x)}{dx}=\frac{-2t(1-t)(x-1)\left\{(1-t)x+t\right\}}{\left\{(1-t)x^2+t\right\}^2},$$

one can easily verify that the function f_t is decreasing when $x \geqslant 1$, and increasing when $x \leqslant 1$. So, $f_t(x) \leqslant f_t(1) = 1$. In addition, we can find that the minimum value of $f_t(x)$ is the inverse of the Kantorovich constant $K(x) := \frac{(x+1)^2}{4x}$ in the following. Since we have

$$\frac{df_t(x)}{dt} = \frac{(x-1)^2\{(1-t)x+t\}\{(1+x)t-x\}}{\{(1-t)x+t\}^2},$$

it follows that $\frac{df_t(x)}{dt} \leqslant 0$ if $0 \leqslant t \leqslant \frac{x}{1+x}$, and $\frac{df_t(x)}{dt} \geqslant 0$ if $1 \geqslant t \geqslant \frac{x}{1+x}$. Thus we have $f_0(x) = f_1(x) = 1 \geqslant f_t(x) \geqslant f_{\frac{x}{1+x}}(x) = K^{-1}(x)$.

In the following corollary, we present another version of the inequalities in Theorem 2.2, but in a form where the refining constant is independent of a,b. This, of course, requires an additional assumption, as follows.

COROLLARY 2.1. *Let $f : [0,\infty) \to [0,\infty)$ be a 2-radical convex function and let* $0 \leqslant t \leqslant 1$.

(i) If $0 < m' \leqslant a \leqslant m \leqslant M \leqslant b \leqslant M'$, then

$$f((1-t)a+tb) \leqslant \frac{((1-t)m+tM)^2}{(1-t)m^2+tM^2}((1-t)f(a)+tf(b)).$$

(ii) If $0 < m' \leqslant b \leqslant m \leqslant M \leqslant a \leqslant M'$, then

$$f((1-t)a+tb) \leqslant \frac{((1-t)M+tm)^2}{(1-t)M^2+tm^2}((1-t)f(a)+tf(b)).$$

Proof. (i) On account of the assumption and utilizing Remark 2.1, we have

$$\frac{((1-t)a+tb)^2}{(1-t)a^2+tb^2} = f_t\left(\frac{a}{b}\right) \leqslant f_t\left(\frac{m}{M}\right) = \frac{((1-t)m+tM)^2}{(1-t)m^2+tM^2},$$

thereupon

$$f((1-t)a+tb) \leqslant \frac{((1-t)m+tM)^2}{(1-t)m^2+tM^2}((1-t)f(a)+tf(b)),$$

as desired.

The inequality in part (ii) can be obtained similarly. $\square$

Similar to Remark 2.1, we have the following lemma, which will be used to present the more general case of Theorem 2.2.

LEMMA 2.1. *The function*

$$g_{t,x}(y) = \frac{((1-t)x+t)^y}{(1-t)x^y+t}; \quad x>0, \ \ 0 \leqslant t \leqslant 1,$$

is decreasing when $y \geqslant 1$.

Proof. We calculate

$$\frac{dg_{t,x}(y)}{dy} = \frac{\{(1-t)x+t\}^y h(t,x,y)}{\{(1-t)x^y+t\}^2},$$

where

$$h(t,x,y) := -(1-t)x^y \log x + \{(1-t)x^y+t\} \log\{(1-t)x+t\}.$$

We also calculate

$$\frac{dh(t,x,y)}{dy} = -(1-t)x^y \log x \log \frac{x}{(1-t)x+t}.$$

For both cases $0 < x \leqslant 1$ and $x \geqslant 1$, we find $\frac{dh(t,x,y)}{dy} \leqslant 0$ by an elementarily calculation. Thus we have

$$h(t,x,y) \leqslant h(t,x,1) = -(1-t)x\log x + \{(1-t)x+t\}\log\{(1-t)x+t\}.$$

Since we calculate

$$\frac{dh(t,x,1)}{dx} = (1-t)\log \frac{(1-t)x+t}{x},$$

we have $\frac{dh(t,x,1)}{dx} \geqslant 0$ for $0 < x \leqslant 1$ and $\frac{dh(t,x,1)}{dx} \leqslant 0$ for $x \geqslant 1$. Thus we have $h(t,x,1) \leqslant h(t,1,1) = 0$. From $h(t,x,y) \leqslant 0$, we find that the function $g_{t,x}(y)$ is decreasing for $y \geqslant 1$. $\square$

Now we present the general form of Theorem 2.2, for p-radical convex functions when $p \geqslant 2$, thanks to Lemma 2.1.

THEOREM 2.3. *Let $f : [0,\infty) \to [0,\infty)$ be a p-radical convex function for some $p \geqslant 2$. Then for any $0 \leqslant t \leqslant 1$ and $a,b > 0$,*

$$f((1-t)a+tb) \leqslant \frac{((1-t)a+tb)^p}{(1-t)a^p+tb^p}((1-t)f(a)+tf(b)).$$

We know that a convex function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ satisfies the super-additive inequality

$$f(a+b) \geqslant f(a)+f(b); a,b \geqslant 0. \tag{2.3}$$

Again, this inequality can be sharpened if f is a radical convex function, as follows.

THEOREM 2.4. *Let $f : [0,\infty) \to [0,\infty)$ be a 2-radical convex function. If $a,b > 0$, then*

$$f(a)+f(b) \leqslant \frac{a^2+b^2}{(a+b)^2} f(a+b).$$

Proof. One can easily check that every 2-radical convex function satisfies the following inequality

$$f(tx) \leqslant t^2 f(x);\ 0 \leqslant t \leqslant 1. \tag{2.4}$$

This implies, for $a, b > 0$,

$$\begin{aligned} f(a) + f(b) &= f\left(\frac{a}{a+b}(a+b)\right) + f\left(\frac{b}{a+b}(a+b)\right) \\ &\leqslant \frac{a^2}{(a+b)^2} f(a+b) + \frac{b^2}{(a+b)^2} f(a+b) \quad \text{(by (2.4))} \\ &= \frac{a^2+b^2}{(a+b)^2} f(a+b). \end{aligned}$$

This completes the proof. □

2.2. The domain $t \notin [0,1]$

When $f : [0,\infty) \to [0,\infty)$ is convex, the following inequality holds

$$(1+t)f(a) - tf(b) \leqslant f((1+t)a + tb), \tag{2.5}$$

provided that $t \geqslant 0$ or $t \leqslant -1$, so that $(1+t)a - tb \geqslant 0$. This latter condition is assumed to guarantee that $(1+t)a - tb \in [0,\infty)$; the domain of f. Interestingly, radical convex functions satisfy better bounds than this. We refer the reader to [4, 16, 17] for further discussion of (2.5) when f is a convex function.

THEOREM 2.5. *Let $f : [0,\infty) \to [0,\infty)$ be a 2-radical convex function and let $a, b \geqslant 0$.*

(i) If $t \geqslant 0$ and $(1+t)a - tb \geqslant 0$, then

$$(1+t) f(a) - t f(b) + (1+t) f\left(\sqrt{t}\,|a-b|\right) \leqslant f((1+t)a - tb).$$

(ii) If $t \leqslant -1$ and $(1+t)a - tb \geqslant 0$, then

$$(1+t) f(a) - t f(b) - t f\left(\sqrt{-(1+t)}\,|a-b|\right) \leqslant f((1+t)a - tb).$$

Proof. Let $t \geqslant 0$. Since

$$a = \frac{1}{1+t}((1+t)a - tb) + \frac{t}{1+t} b,$$

we have by (2.1),

$$\begin{aligned} f(a) &= f\left(\frac{1}{1+t}((1+t)a - tb) + \frac{t}{1+t} b\right) \\ &\leqslant \frac{1}{1+t} f((1+t)a - tb) + \frac{t}{1+t} f(b) - f\left(\sqrt{t}\,|a-b|\right). \end{aligned}$$

On the other hand, when $t \leqslant -1$, we have

$$b = -\frac{1}{t}((1+t)a - tb) + \frac{1+t}{t}a.$$

By (2.1), we infer that

$$\begin{aligned} f(b) &= f\left(-\frac{1}{t}((1+t)a - tb) + \frac{1+t}{t}a\right) \\ &\leqslant -\frac{1}{t}f((1+t)a - tb) + \frac{1+t}{t}f(a) - f\left(\sqrt{-(1+t)}\,|a-b|\right). \end{aligned}$$

This completes the proof. $\square$

REMARK 2.2. The inequalities in Theorem 2.5 are equivalent to

$$(1-t)f(a) + tf(b) + tf\left(\sqrt{t-1}\,|a-b|\right) \leqslant f((1-t)a + tb)\ ; t \geqslant 1,$$

and

$$(1-t)f(a) + tf(b) + (1-t)f\left(\sqrt{-t}\,|a-b|\right) \leqslant f((1-t)a + tb);\ t \leqslant 0.$$

On the other hand, the following theorem presents the reversed version of Theorem 2.2, when $t \notin [0,1]$.

THEOREM 2.6. *Let $f : [0,\infty) \to [0,\infty)$ be a 2-radical convex function. If (i) $0 < a < b$ and $t > 1$, or (ii) $0 < b < a$ and $t < 0$, then for any $t \notin [0,1]$,*

$$(1-t)f(a) + tf(b) \leqslant \frac{(1-t)a^2 + tb^2}{((1-t)a + tb)^2} f((1-t)a + tb).$$

Proof. Let $g(x) = f(\sqrt{x})$, $x \in [0,\infty)$. Then g is an increasing convex function on $[0,\infty)$ and we have $g(\alpha x) \leqslant \alpha g(x); 0 < \alpha \leqslant 1$. We note that $\frac{(1-t)a^2+tb^2}{((1-t)a+tb)^2} \leqslant 1$ for all $a,b > 0$ and $t \notin [0,1]$, and we also have $\frac{(1-t)a^2+tb^2}{((1-t)a+tb)^2} > 0$ for (i) or (ii). Thus, by Remark 2.2,

$$\begin{aligned} (1-t)g\left(a^2\right) + tg\left(b^2\right) &\leqslant g\left((1-t)a^2 + tb^2\right) \\ &= g\left(\frac{(1-t)a^2 + tb^2}{((1-t)a + tb)^2}((1-t)a + tb)^2\right) \\ &\leqslant \frac{(1-t)a^2 + tb^2}{((1-t)a + tb)^2} g\left(((1-t)a + tb)^2\right). \end{aligned}$$

This completes the proof. $\square$

In the following remark, we discuss the constant appearing in Theorem 2.6.

REMARK 2.3. We consider the function

$$f(t,x) := \frac{(1-t)x^2+t}{\{(1-t)x+t\}^2}, \quad x>0, \quad t \notin [0,1].$$

We see $f(t,1)=1$ trivially. Simple calculations show that

$$\frac{df(t,x)}{dt} = \frac{(x-1)^2\{(1+x)t-x\}}{\{(x-1)t-x\}^3}.$$

Now we can easily verify the following.

(i) If $0<x<1$ and $t>1$, then $f(t,x)$ is monotone decreasing when $t>1$.

(ii) If $x>1$ and $t<0$, then $f(t,x)$ is monotone increasing when $t<0$.

We also have

$$\frac{df(t,x)}{dx} = \frac{2t(1-t)(1-x)}{\{(t-1)x-t\}^3}.$$

Thus we also have the following:

(i) If $0<x<1$ and $t>1$, then $f(t,x)$ is monotone increasing in $0<x<1$.

(ii) If $x>1$ and $t<0$, then $f(t,x)$ is monotone decreasing in $x>1$.

3. Matrix inequalities via 2-radical convex functions

In this section, we present improved matrix inequalities using 2-radical convex functions. For this, we need to remind the reader of some terminologies. Let $\mathscr{M}_n$ denote the algebra of all $n\times n$ complex matrices. A matrix $A\in\mathscr{M}_n$ is said to be Hermitian if $A^*=A$, where A^* denotes the conjugate transpose of A. If $A\in\mathscr{M}_n$ is such that $\langle Ax,x\rangle \geqslant 0$ for all $x\in\mathbb{C}^n$, then A is said to be positive semi-definite, and we write $A\geqslant 0$. If $A\geqslant 0$ is invertible, then A is said to be positive definite, and we write $A>0$. When $A,B\in\mathscr{M}_n$ are Hermitian such that $A-B\geqslant 0$, then we write $A\geqslant B$. In particular, we write $A\leqslant M$ for the scalar M if $A\leqslant MI$, where I is the identity matrix. When $A\in\mathscr{M}_n$ is Hermitian, we use the notation $\lambda_k(A)$ to denote the k-th largest eigenvalue of A. That is, $\lambda_1(A)\geqslant\lambda_2(A)\geqslant\cdots\geqslant\lambda_n(A)$.

If $A,B\in\mathscr{M}_n$ are such that $\sum_{i=1}^k\lambda_i(A)\leqslant\sum_{i=1}^k\lambda_i(B)$ for all $1\leqslant k\leqslant n$, then we write $\lambda(A)\prec_w\lambda(B)$. This is usually referred to as A being majorized by B.

While convex functions satisfy (1.2), they do not satisfy the matrix inequality $f((1-t)A+tB)\leqslant(1-t)f(A)+tf(B)$. Rather, they satisfy the weaker majorization inequality [2]

$$\lambda(f((1-t)A+tB))\prec_w\lambda((1-t)f(A)+tf(B)),\ 0\leqslant t\leqslant 1 \tag{3.1}$$

where A,B are Hermitian, with eigenvalues in the domain of f. When f is monotone and convex, (3.1) was strengthened in [2, Theorem 2.9] as follows

$$\lambda(f((1-t)A+tB))\leqslant\lambda((1-t)f(A)+tf(B)),\ 0\leqslant t\leqslant 1, \tag{3.2}$$

where we say that $\lambda(X) \leqslant \lambda(Y)$ for two Hermitian matrices $X, Y \in \mathscr{M}_n$ if $\lambda_j(X) \leqslant \lambda_j(Y)$ for each $j = 1, \cdots, n$.

The following theorem shows that radical convex functions satisfy better bounds than (3.1) and (3.2). In the proof of this theorem, we need to recall that when $f : J \to \mathbb{R}$ is convex, and $A \in \mathscr{M}_n$ is Hermitian such that the spectrum of A is in J, then [5, p. 281]

$$f(\langle Ax, x\rangle) \leqslant \langle f(A)x, x\rangle, \tag{3.3}$$

for any unit vector $x \in \mathbb{C}^n$. We also recall that if $X \in \mathscr{M}_n$ is Hermitian, then [5, p. 58]

$$\lambda_j(X) = \max_{\dim \mathfrak{M} = j} \min\{\langle Xx, x\rangle, \|x\| = 1, x \in \mathfrak{M}\}, 1 \leqslant j \leqslant n, \tag{3.4}$$

where the maximum is taken over all possible subspaces $\mathfrak{M}$ of $\mathbb{C}^n$ with dimension j.

Finding possible relations among the eigenvalues of certain matrices has received renowned attention in the literature. We refer the reader to [8] and the related references therein. Now we improve (3.1) and (3.2).

THEOREM 3.1. *Let $f : [0, \infty) \to [0, \infty)$ be a 2-radical convex function, $A, B \in \mathscr{M}_n$ be positive definite, and let $0 \leqslant t \leqslant 1$.*

(i) If $0 < m' \leqslant A \leqslant m \leqslant M \leqslant B \leqslant M'$, then

$$\lambda(f((1-t)A + tB)) \leqslant \frac{((1-t)m + tM)^2}{(1-t)m^2 + tM^2} \lambda(((1-t)f(A) + tf(B))).$$

(ii) If $0 < m' \leqslant B \leqslant m \leqslant M \leqslant A \leqslant M'$, then

$$\lambda(f((1-t)A + tB)) \leqslant \frac{((1-t)M + tm)^2}{(1-t)M^2 + tm^2} \lambda(((1-t)f(A) + tf(B))).$$

Proof. We prove the case $0 < m' \leqslant A \leqslant m \leqslant M \leqslant B \leqslant M'$. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $(1-t)A + tB$, for any $0 \leqslant t \leqslant 1$. Then, for each $1 \leqslant j \leqslant n$,

$$\begin{aligned}
&\lambda_j(f((1-t)A + tB)) \\
&= \max_{\dim \mathfrak{M} = j} \min\{\langle f((1-t)A + tB)x, x\rangle, \|x\| = 1, x \in \mathfrak{M}\} \quad \text{(by (3.4))} \\
&= f\left(\max_{\dim \mathfrak{M} = j} \min\{\langle ((1-t)A + tB)x, x\rangle, \|x\| = 1, x \in \mathfrak{M}\}\right) \quad \text{(since } f \text{ is increasing)} \\
&= f\left(\max_{\dim \mathfrak{M} = j} \min\{(1-t)\langle Ax, x\rangle + t\langle Bx, x\rangle, \|x\| = 1, x \in \mathfrak{M}\}\right) \\
&= \max_{\dim \mathfrak{M} = j} \min\{f((1-t)\langle Ax, x\rangle + t\langle Bx, x\rangle), \|x\| = 1, x \in \mathfrak{M}\}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(since } f \text{ is increasing)} \\
&\leqslant \frac{((1-t)m + tM)^2}{(1-t)m^2 + tM^2} \max_{\dim \mathfrak{M} = j} \min\{(1-t)f(\langle Ax, x\rangle) + tf(\langle Bx, x\rangle), \|x\| = 1, x \in \mathfrak{M}\}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{((1-t)m+tM)^2}{(1-t)m^2+tM^2} \max_{\dim \mathfrak{M}=j} \min\{(1-t)\langle f(A)x,x\rangle + t\langle f(B)x,x\rangle, \|x\|=1, x\in\mathfrak{M}\} \\
&= \frac{((1-t)m+tM)^2}{(1-t)m^2+tM^2} \max_{\dim \mathfrak{M}=j} \min\{\langle((1-t)f(A)+tf(B))x,x\rangle, \|x\|=1, x\in\mathfrak{M}\} \\
&= \frac{((1-t)m+tM)^2}{(1-t)m^2+tM^2} \lambda_j((1-t)f(A)+tf(B))
\end{aligned}
$$

where we obtained the first inequality using Corollary 2.1, the second inequality using (3.3) and the last equality using (3.4). This completes the proof. □

Again, in [2], it has been shown that when $f:[0,\infty)\to[0,\infty)$ is convex such that $f(0)=0$, then

$$
\lambda(f(X^*AX)) \prec_w \lambda(X^*f(A)X), \tag{3.5}
$$

for positive semi-definite matrix A, and any contraction $X\in\mathscr{M}_n$, that is $\|X\|\leqslant 1$. When f is monotone, the sign $\prec_w$ was replaced by $\leqslant$. We present a sharper bound satisfied by the radical convex functions in the following.

THEOREM 3.2. *Let $f:[0,\infty)\to[0,\infty)$ be a 2-radical convex function, $A\geqslant 0$ and let $X\in\mathscr{M}_n$ be a contraction. Then*

$$
\lambda(f(X^*AX)) \leqslant \|X\|^2\lambda(X^*f(A)X).
$$

Proof. Let $\lambda_1,\lambda_2,\ldots,\lambda_n$ be the eigenvalues of X^*AX. Since f is increasing, we have, for $1\leqslant j\leqslant n$,

$$
\begin{aligned}
\lambda_j(f(X^*AX)) &= f(\lambda_j(X^*AX)) \\
&= f\left(\max_{\dim \mathfrak{M}=j} \min\{\langle X^*AXx,x\rangle, \|x\|=1, x\in\mathfrak{M}\}\right) \\
&= \max_{\dim \mathfrak{M}=j} \min\{f(\langle X^*AXx,x\rangle), \|x\|=1, x\in\mathfrak{M}\}.
\end{aligned}
$$

Now, we have

$$
\begin{aligned}
f(\langle X^*AXx,x\rangle) &= f(\langle AXx,Xx\rangle) \\
&= f\left(\|Xx\|^2\left\langle A\frac{Xx}{\|Xx\|},\frac{Xx}{\|Xx\|}\right\rangle\right) \\
&\leqslant \|Xx\|^4 f\left(\left\langle A\frac{Xx}{\|Xx\|},\frac{Xx}{\|Xx\|}\right\rangle\right) \quad \text{(by (2.4))} \\
&\leqslant \|Xx\|^4\left\langle f(A)\frac{Xx}{\|Xx\|},\frac{Xx}{\|Xx\|}\right\rangle \quad \text{(by (3.3))} \\
&= \|Xx\|^2\langle X^*f(A)Xx,x\rangle \\
&\leqslant \|X\|^2\langle X^*f(A)Xx,x\rangle.
\end{aligned}
$$

Consequently,

$$\begin{aligned}\lambda_j\left(f(X^*AX)\right) &= \max_{\dim \mathfrak{M}=j} \min\{f\left(\langle X^*AXx,x\rangle\right), \|x\|=1, x\in\mathfrak{M}\}\\ &\leqslant \|X\|^2 \max_{\dim \mathfrak{M}=j} \min\{\langle X^*f(A)Xx,x\rangle : \|x\|=1, x\in\mathfrak{M}\}\\ &= \|X\|^2\lambda_j\left(X^*f(A)X\right) \quad \text{(by (3.4))}.\end{aligned}$$

This completes the proof. $\square$

Now we recall that if $A,B\in\mathscr{M}_n$ are Hermitian, then we have the following equivalence for some unitary matrix $U\in\mathscr{M}_n$

$$\lambda(A)\prec_w\lambda(B)\Leftrightarrow A\leqslant U^*BU.$$

Consequently, the inequality in Theorem 3.2 implies

$$f\left(X^*AX\right)\leqslant\|X\|^2U^*\left(X^*f(A)X\right)U, \tag{3.6}$$

for some unitary matrix $U\in\mathscr{M}_n$.

In the following, we present another interesting consequence of radical convexity applied to matrix inequalities. In [3, (1)], it has been shown that if $f:[0,\infty)\to[0,\infty)$ is convex monotone such that $f(0)=0$, and if $A,B\geqslant 0$, then two unitary matrices U,V exist such that

$$Uf(A)U^*+Vf(B)V^*\leqslant f(A+B). \tag{3.7}$$

In the following, a better estimate can be derived using radical convexity. We notice here that a 2-radical convex function is necessarily monotone [9, Proposition 1.1].

THEOREM 3.3. *Let $A,B\in\mathscr{M}_n$ be positive semi-definite, and let $f:[0,\infty)\to[0,\infty)$ be a 2-radical convex function. Then there exist two unitary matrices U,V and two contractions X,Y such that*

$$\frac{1}{\|X\|^2}U^*f(A)U+\frac{1}{\|Y\|^2}V^*f(B)V\leqslant f(A+B).$$

Proof. Following [3], we can assume that $A+B$ is invertible. Then

$$A=X(A+B)X^* \text{ and } B=Y(A+B)Y^*$$

where $X=A^{\frac12}(A+B)^{-\frac12}$ and $Y=B^{\frac12}(A+B)^{-\frac12}$ are contractions. Hence, using (3.6) there exist a unitary matrix U_0 such that

$$\begin{aligned}f(A) &= f\left(X(A+B)X^*\right)\\ &\leqslant \|X\|^2U_0^*Xf(A+B)X^*U_0\\ &= \|X\|^2U_0^*X(f(A+B))^{\frac12}(f(A+B))^{\frac12}X^*U_0.\end{aligned}$$

If we let $T = X(f(A+B))^{\frac{1}{2}}$, the above inequality can be written as

$$\begin{aligned} f(A) &\leqslant \|X\|^2 U_0^*(TT^*)U_0 \\ &= \|X\|^2 U_0^*(U_1^*T^*TU_1)U_0 \\ &= \|X\|^2 (U_1U_0)^*(f(A+B))^{\frac{1}{2}} X^*X(f(A+B))^{\frac{1}{2}}(U_1U_0) \end{aligned}$$

where we have used the fact that TT^* and T^*T are unitary congruent. Here U_1 is unitary. Letting $U = U_0^*U_1^*$, the above inequality is equivalent to

$$U^*f(A)U \leqslant \|X\|^2 f^{\frac{1}{2}}(A+B)X^*Xf^{\frac{1}{2}}(A+B). \tag{3.8}$$

Similarly, there exists a unitary V such that

$$V^*f(B)V \leqslant \|Y\|^2 f^{\frac{1}{2}}(A+B)Y^*Yf^{\frac{1}{2}}(A+B). \tag{3.9}$$

Adding (3.8) and (3.9), we get

$$\frac{1}{\|X\|^2}U^*f(A)U + \frac{1}{\|Y\|^2}V^*f(B)V \leqslant f(A+B),$$

due to $XX^* + YY^* = I$. This completes the proof. $\square$

The fact that Theorem 3.3 refines (3.7) follows from the observation $\|X\|, \|Y\| \leqslant 1$.

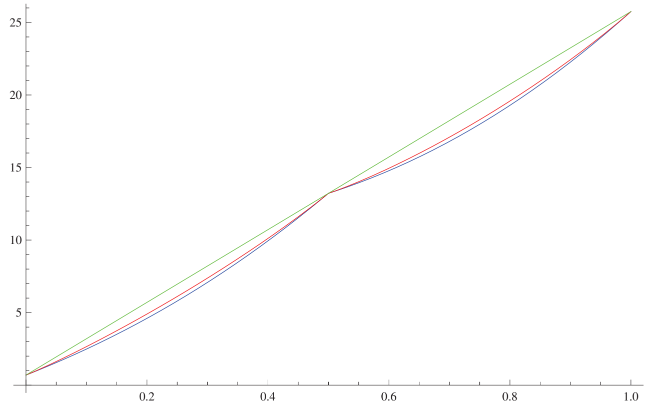
4. Examples and comments

In this short section, we present the significance of the above results by comparing the convex version with the radical convex version of the same inequality. This will enable the reader to appreciate radical convexity better.

EXAMPLE 4.1. The function $f(x) = x^2\log(x+1)$ is 2-convex, hence it is convex. Treating f as a convex function, we may apply (2.2) while treating it as a 2-radical convex function allows us to use Theorem 2.1. It is clear that the inequality in this theorem implies a better bound. In the following figure, we used $a = 1, b = 4$ and plotted the left side in (2.2) in blue, the left side of the inequality in Theorem 2.1 in red, and the common right side of both inequalities in green.

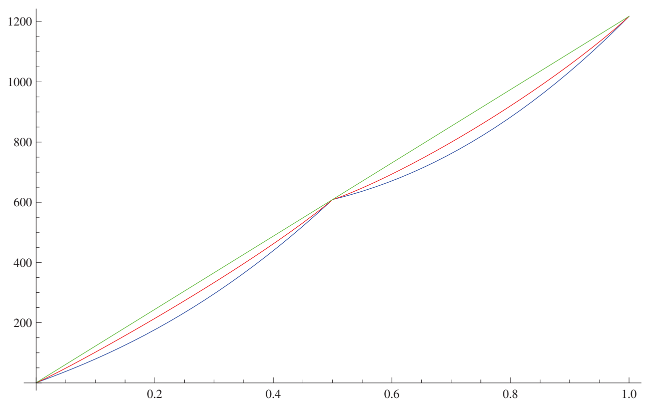
The graph shows how Theorem 2.1 provides a visual refinement of (2.2).

At this point, it is worthwhile mentioning that the inequality in Theorem 2.1 and the inequality (2.2) become identical for the function $f(x) = x^2$.



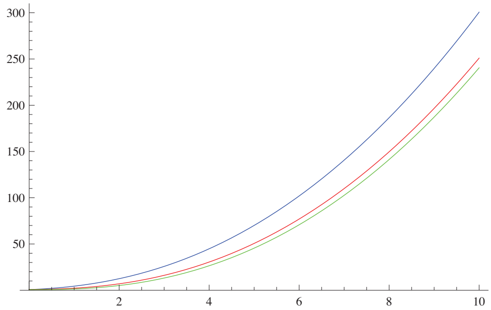
Another visual explanation is given in the following example.

EXAMPLE 4.2. In this example, we plot the (1.2) together with the inequality in Theorem 2.2. While it is clear that the latter inequality is better, here we give a visual clarification for the reader. We also notice that the inequalities become identical when $f(x) = x^2$. In the following figure, we plot the left side of (1.2) in blue, the left side of the inequality in Theorem 2.2, and the common right side of both inequalities in green for the function $f(x) = x^2 \log(x+1)$, $a = 1$, $b = 20$.



Now we look at (2.3) and Theorem 2.4.

EXAMPLE 4.3. Again, for the function $f(x) = x^2 \log(x+1)$, we let $a = 1$. Then, as functions of b, we plot the left side of (2.3) in blue, the left side of the inequality in Theorem 2.4 in red and the common right side of both inequalities in green.



As an upper bound of the green curve, the figure shows how much the red curve is better than the blue one, thanks to radical convexity.

As a conclusion of the above three examples, we emphasize the following. When dealing with a convex function and its inequalities, it is always better to test whether the function is 2-radical convex or not. It is much better to employ inequalities governing radical convex functions if it is radical convex. This will always lead to better and sharper bounds!

Now moving to the matrix inequalities, the term $\frac{((1-t)m+tM)^2}{(1-t)m^2+tM^2}$ in Theorem 3.1 provides a refinement of (3.1). Direct calculus computations show how this quantity can be small for larger values of $\frac{M}{m}$.

Now looking at (3.5) and Theorem 3.2, one can easily comprehend the way this theorem sharpens (3.5). By making $\|X\|$ so small, the inequality in the theorem provides a significant improvement of (3.5).

A similar argument can be said about the relation between (3.7) and Theorem 3.3.

Conclusion

The authors have recently defined radical convex functions. This paper employed radical convexity to obtain several significant improvements in known inequalities. As a conclusion of our discussion, we emphasize testing a given convex function for radical convexity. If the given function is radical convex, then treating it this way provides much sharper bounds than just convexity. This approach helps obtain stronger bounds in many scenarios.

Declarations

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