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POLYNOMIAL DIFFERENTIATION COMPOSITION OPERATORS FROM H^p SPACES TO WEIGHTED–TYPE SPACES ON THE UNIT BALL

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Abstract. We characterize the boundedness, compactness, and estimate essential norm of a polynomial differentiation composition operator from the Hardy space H^p to the weighted-type spaces of holomorphic functions on the unit ball.

1. Introduction

Let $\mathbb{N}_0$ be the set of nonnegative integers. If $k,l \in \mathbb{N}_0$, $k \leqslant l$, then the notation $j=\overline{k,l}$ is an abbreviation for the notation $j=k,k+1,\dots,l$. Let $\mathbb{B}=\mathbb{B}^n \subset \mathbb{C}^n$ be the open unit ball, $\mathbb{S}=\partial\mathbb{B}$ its boundary, $d\sigma$ the normalized Lebesgue measure on $\mathbb{S}$, $\mathbb{D}=\mathbb{B}^1$, $\langle z,w\rangle$ the inner product in $\mathbb{C}^n$, $|z|=\langle z,z\rangle^{1/2}$, D_j the partial derivative operator

$$D_j f(z)=\frac{\partial f}{\partial z_j}(z), \quad j\in\{1,2,\dots,n\},$$

$S(\Omega)$ the family of holomorphic self-maps of a domain Ω, $H(\Omega)$ the space of holomorphic functions on Ω ([22,23,48]), and $H^p(\mathbb{B})=H^p$, $p>0$, the Hardy space consisting of all $f\in H(\mathbb{B})$ such that

$$\|f\|_{H^p}=\sup_{0\leqslant r<1}\left(\int_{\mathbb{S}}|f(r\zeta)|^p d\sigma(\zeta)\right)^{1/p}<+\infty,$$

(see, e.g., [23,48]). For $p\geqslant 1$ it is a Banach space.

By $W(\Omega)$ we denote the family of positive and continuous functions on Ω and call them weights. Let $\mu\in W(\mathbb{B})$. The weighted-type space $H^\infty_\mu(\mathbb{B})=H^\infty_\mu$ is defined as follows

$$H^\infty_\mu(\mathbb{B}):=\left\{f\in H(\mathbb{B}):\|f\|_{H^\infty_\mu}:=\sup_{z\in\mathbb{B}}\mu(z)|f(z)|<+\infty\right\}.$$

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For $\mu(z) \equiv 1$ we get the space of bounded holomorphic functions $H^\infty(\mathbb{B}) = H^\infty$ with the supremum norm $\|\cdot\|_\infty$. The little weighted-type space $H^\infty_{\mu,0}(\mathbb{B}) = H^\infty_{\mu,0}$ is a closed subspace of H^∞_μ consisting of $f \in H(\mathbb{B})$ such that

$$\lim_{|z|\to 1} \mu(z)|f(z)| = 0.$$

The spaces and operators on them and their generalizations have been studied a lot (see, for instance, [2, 10, 16, 21, 29, 30, 33, 36, 37, 38, 39, 43, 45, 50] and the related references therein).

Beside the differentiation operator $Df = f'$, some attention to researchers attracted the composition operator $C_\varphi f = f \circ \varphi$, where $\varphi \in S(\Omega)$, the multiplication operator $M_u f = uf$, where $u \in H(\Omega)$, as well as their products. Among the products containing differentiation operators, the operators DC_φ and $C_\varphi D$ have been studied among the first ones (see, e.g., [7, 14, 15, 19] and the references therein).

The following extension of the operator $C_\varphi D$ attracted also some attention

$$D^m_{\varphi,u} := M_u C_\varphi D^m \tag{1}$$

on subspaces of $H(\mathbb{D})$ (see, e.g., [8,13,16,29,32,33,44,45,46,49,50,51,52,53,54,55]).

The following n-dimensional variant of operator (1)

$$\mathfrak{R}^m_{\varphi,u} := M_u C_\varphi \mathfrak{R}^m, \tag{2}$$

where $\mathfrak{R}$ is the radial differentiation operator was introduced in [34]. The investigation was continued in [35, 38, 39].

Investigations of sums of the operators in (1) was initiated by Stević and Sharma. The first published results can be found in [40] and [41]. An extension of the sum in [40] and [41] appeared in [42]. The investigation was continued, for instance, in [1,5,6,9,17, 47]. For some other product type operators consult, e.g., [10,11,12,20,27,28,26,31,43] and the related references therein.

Investigations of sums of the operators in (2) was suggested by Stević soon after finishing [42], but the first published results can be found in recent paper [37]. Beside the sums he also suggested studying the polynomial differentiation composition operator of the form

$$P^m_{D,\varphi} f := \sum_{j=0}^m u_j C_\varphi D_{l_j} \cdots D_{l_1} f, \quad f \in H(\mathbb{B}), \tag{3}$$

where $m \in \mathbb{N}_0$, $u_j \in H(\mathbb{B})$, $j = \overline{0,m}$, and $\varphi \in S(\mathbb{B})$ (see [36]).

Let X and Y be two normed spaces. A linear operator $T : X \to Y$ is called bounded if there is $M \geqslant 0$ such that $\|Tf\|_Y \leqslant M\|f\|_X$ for every $f \in X$. If it maps bounded sets in X into relatively compact ones, then it is called compact [4, 24]. The essential norm of the operator $T : X \to Y$ is defined as follows

$$\|T\|_{e,X\to Y} = \inf\{\|T+K\|_{X\to Y} : K \text{ is compact from } X \text{ to } Y\},$$

where $\|\cdot\|_{X\to Y}$ denote the operator norm. The operator T is compact if and only if $\|T\|_{e,X\to Y} = 0$. We denote the unit ball in X by B_X.

There has been a huge recent interest in investigating the boundedness, compactness, and estimating essential norms of concrete operators on spaces of holomorphic functions (see, e.g., [3, 5, 6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 19, 18, 20, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 40, 41, 42, 37, 38, 39, 43, 44, 45, 46, 47, 49, 50, 51, 52, 53, 54, 55] and the references therein).

In this article we characterize the boundedness and compactness of the operator $P^m_{D,\varphi} : H^p \to H^\infty_\mu$ (or $H^\infty_{\mu,0}$), for $p \geqslant 1$, and estimate the essential norm of the operator in the case $p > 1$.

Let C denote unspecified nonnegative constants. They can change from line to line. The notation $a \lesssim b$ (resp. $a \gtrsim b$) means that there is $C > 0$ such that $a \leqslant Cb$ (resp. $a \geqslant Cb$). If $a \lesssim b$ and $b \lesssim a$, then we use the notation $a \asymp b$.

2. Auxiliary results

Our first auxiliary result is a characterization for the compactness. It is proved in a standard way [25], because of which we omit the proof.

LEMMA 1. *Let $p \geqslant 1$, $u_j \in H(\mathbb{B})$, $j = \overline{0,m}$, $\varphi \in S(\mathbb{B})$, $\mu \in W(\mathbb{B})$ and $Y \in \{H^\infty_\mu(\mathbb{B}), H^\infty_{\mu,0}(\mathbb{B})\}$. Then the bounded operator $P^m_{D,\varphi} : H^p(\mathbb{B}) \to Y$ is compact if and only if for any bounded sequence $(f_k)_{k\in\mathbb{N}} \subset H^p(\mathbb{B})$ such that $f_k \to 0$ uniformly on compacts of $\mathbb{B}$ as $k \to +\infty$, we have*

$$\lim_{k\to+\infty} \|P^m_{D,\varphi} f_k\|_{H^\infty_\mu} = 0.$$

The following folklore lemma is a consequence of Cauchy's estimate for derivatives and a known estimate for the point evaluation functional on $H^p(\mathbb{B})$ ([23, 48]).

LEMMA 2. *Let $p > 0$ and $N \in \mathbb{N}_0$. Then for every multi-index $\vec{l} = (l_1, l_2, \dots, l_j)$ such that $|\vec{l}| = N$, there is $C_{\vec{l}} > 0$ such that*

$$\left| \frac{\partial^N f(z)}{\partial z_{k_1}^{l_1} \partial z_{k_2}^{l_2} \cdots \partial z_{k_j}^{l_j}} \right| \leqslant \frac{C_{\vec{l}} \|f\|_{H^p}}{(1-|z|^2)^{\frac{n}{p}+N}},$$

for every $f \in H^p(\mathbb{B})$ and $z \in \mathbb{B}$.

The following result, which is a consequence of [23, Proposition 1.4.10] and monotonicity of the integral means, gives a known family of test functions in H^p space.

LEMMA 3. *Let $p > 0$, $a \geqslant 0$ and $w \in \mathbb{B}$. Then the function*

$$f_{w,a}(z) = \frac{(1-|w|^2)^{\frac{n}{p}+a}}{(1-\langle z,w\rangle)^{\frac{2n}{p}+a}}, \tag{4}$$

belongs to $H^p(\mathbb{B})$.

Moreover, we have

$$\sup_{w\in\mathbb{B}} \|f_{w,a}\|_{H^p} \lesssim 1. \tag{5}$$

The following lemma is a known generalization of Lemma 1 in [18].

LEMMA 4. *A closed set K in $H^\infty_{\mu,0}(\mathbb{B})$ is compact if and only if it is bounded and*

$$\lim_{|z|\to 1} \sup_{f\in K} \mu(z)|f(z)| = 0.$$

The following lemma gives a useful family of test functions.

LEMMA 5. *Let $p>0$, $m\in\mathbb{N}$ and $w\in\mathbb{B}$. Then for each $s\in\{0,1,\dots,m\}$ there are $c_k^{(s)}$, $k=\overline{0,m}$, such that the function*

$$h_w^{(s)}(z) = \sum_{k=0}^m c_k^{(s)} f_{w,k}(z)$$

where $f_{w,a}$ is defined in (4), satisfies

$$D_{l_s}\cdots D_{l_1} h_w^{(s)}(w) = \frac{\overline{w}_{l_1}\overline{w}_{l_2}\cdots\overline{w}_{l_s}}{(1-|w|^2)^{\frac{n}{p}+s}} \tag{6}$$

and

$$D_{l_t}\cdots D_{l_1} h_w^{(s)}(w) = 0, \tag{7}$$

for every $t\in\{0,1,\dots,m\}\setminus\{s\}$.

We also have

$$\sup_{w\in\mathbb{B}} \|h_w^{(s)}\|_{H^p} \lesssim 1. \tag{8}$$

Proof. Let

$$h_w(z) = \sum_{k=0}^m c_k f_{w,k}(z).$$

and $d_k = \frac{2n}{p}+k$, $k\in\mathbb{N}_0$. Then

$$D_{l_t}\cdots D_{l_1} h_w(z) = \sum_{k=0}^m c_k \frac{d_k d_{k+1}\cdots d_{k+t-1}\overline{w}_{l_1}\overline{w}_{l_2}\cdots\overline{w}_{l_t}(1-|w|^2)^{\frac{n}{p}+k}}{(1-\langle z,w\rangle)^{d_k+t}},$$

for $t\in\mathbb{N}_0$, and consequently

$$D_{l_t}\cdots D_{l_1} h_w(w) = \frac{\overline{w}_{l_1}\overline{w}_{l_2}\cdots\overline{w}_{l_t}}{(1-|w|^2)^{\frac{n}{p}+t}} \sum_{k=0}^m c_k \prod_{l=0}^{t-1} d_{k+l},$$

for $t\in\mathbb{N}_0$.

Since the determinant of the system

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ d_0 & d_1 & \cdots & d_m \\ \vdots & \vdots & \ddots & \vdots \\ \prod_{k=0}^{s-1} d_k & \prod_{k=0}^{s-1} d_{k+1} & \cdots & \prod_{k=0}^{s-1} d_{k+m} \\ \vdots & \vdots & \ddots & \vdots \\ \prod_{k=0}^{m-1} d_k & \prod_{k=0}^{m-1} d_{k+1} & \cdots & \prod_{k=0}^{m-1} d_{k+m} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_s \\ \vdots \\ c_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \tag{9}$$

is not equal to zero ([30, Lemma 3]), we have that for any $s \in \{0,1,\dots,m\}$, it has a unique solution $c_k := c_k^{(s)}$, $k = \overline{0,m}$. It is easy to see that the function satisfying (6) and (7) is given by $h_w^{(s)}(z) := \sum_{k=0}^m c_k^{(s)} f_{w,k}(z)$, and that (5) implies (8). $\square$

3. Main results

The main results in the paper are presented in this section.

THEOREM 1. *Let* $p \geqslant 1$, $m \in \mathbb{N}$, $\mu \in W(\mathbb{B})$, $u_j \in H(\mathbb{B})$, $j = \overline{0,m}$, $\varphi = (\varphi_1, \dots, \varphi_n) \in S(\mathbb{B})$,

$$\min_{j=\overline{1,n}} \inf_{z \in \mathbb{B}} |\varphi_j(z)| \geqslant \delta > 0. \tag{10}$$

Then $P_{D,\varphi}^m : H^p(\mathbb{B}) \to H_\mu^\infty(\mathbb{B})$ *is bounded if and only if*

$$L_j := \sup_{z \in \mathbb{B}} \frac{\mu(z)\big|u_j(z)\big|}{(1-|\varphi(z)|^2)^{\frac{n}{p}+j}} < +\infty, \quad j = \overline{0,m}. \tag{11}$$

Moreover, if the operator is bounded, then we have

$$\|P_{D,\varphi}^m\|_{H^p \to H_\mu^\infty} \asymp \sum_{j=0}^m L_j. \tag{12}$$

Proof. Suppose that $P_{D,\varphi}^m : H^p(\mathbb{B}) \to H_\mu^\infty(\mathbb{B})$ is bounded. Lemma 5 implies that for each $s \in \{0,1,\dots,m\}$ and $\varphi(w) \in \mathbb{B}$, there is $h_{\varphi(w)}^{(s)} \in H^p(\mathbb{B})$ such that

$$D_{l_s} \cdots D_{l_1} h_{\varphi(w)}^{(s)}(\varphi(w)) = \frac{\overline{\varphi_{l_1}(w)\varphi_{l_2}(w)} \cdots \overline{\varphi_{l_s}(w)}}{(1-|\varphi(w)|^2)^{\frac{n}{p}+s}}, \tag{13}$$

$$D_{l_t} \cdots D_{l_1} h_{\varphi(w)}^{(s)}(\varphi(w)) = 0, \tag{14}$$

for every $t \in \{0,1,\dots,m\} \setminus \{s\}$. We also have $\sup_{w\in\mathbb{B}} \|h^{(s)}_{\varphi(w)}\|_{H^p} < +\infty$.

This together with the boundedness, (13), (14), as well as (10), implies

$$
\begin{aligned}
\|P^m_{D,\varphi}\|_{H^p\to H^\infty_\mu} \gtrsim &\|P^m_{D,\varphi} h^{(s)}_{\varphi(w)}\|_{H^\infty_\mu}\\
&= \sup_{z\in\mathbb{B}} \mu(z)\left|\sum_{j=0}^m u_j(z) D_{l_j}\cdots D_{l_1} h^{(s)}_{\varphi(w)}(\varphi(z))\right|\\
&\geqslant \mu(w)\left|\sum_{j=0}^m u_j(w) D_{l_j}\cdots D_{l_1} h^{(s)}_{\varphi(w)}(\varphi(w))\right|\\
&= \mu(w)|u_s(w)|\frac{\overline{|\varphi_{l_1}(w)|}\cdots\overline{|\varphi_{l_s}(w)|}}{(1-|\varphi(w)|^2)^{\frac{n}{p}+s}}\\
&\geqslant \delta^s \frac{\mu(w)|u_s(w)|}{(1-|\varphi(w)|^2)^{\frac{n}{p}+s}},
\end{aligned}
\tag{15}
$$

for every $w \in \mathbb{B}$, from which it easily follows that $L_s < +\infty$, $s \in \{0,1,\dots,m\}$, and

$$
L_s \lesssim \|P^m_{D,\varphi}\|_{H^p\to H^\infty_\mu}, \quad s = \overline{0,m},
$$

and consequently

$$
\sum_{j=0}^m L_j \lesssim \|P^m_{D,\varphi}\|_{H^p\to H^\infty_\mu}. \tag{16}
$$

If (11) holds, then Lemma 2 implies

$$
\begin{aligned}
\mu(z)|P^m_{D,\varphi} f(z)| = &\mu(z)\left|\sum_{j=0}^m u_j(z) D_{l_j}\cdots D_{l_1} f(\varphi(z))\right|\\
&\leqslant C\sum_{j=0}^m \frac{\mu(z)|u_j(z)|}{(1-|\varphi(z)|^2)^{\frac{n}{p}+j}}\|f\|_{H^p},
\end{aligned}
\tag{17}
$$

from which along with (11), the boundedness of $P^m_{D,\varphi} : H^p(\mathbb{B}) \to H^\infty_\mu(\mathbb{B})$ follows, as well as the asymptotic estimate

$$
\|P^m_{D,\varphi}\|_{H^p\to H^\infty_\mu} \lesssim \sum_{j=0}^m L_j. \tag{18}
$$

Asymptotic estimates (16) and (18) imply (12). □

THEOREM 2. *Let $p \geqslant 1$, $m \in \mathbb{N}$, $u_j \in H(\mathbb{B})$, $j = \overline{0,m}$, $\varphi \in S(\mathbb{B})$, and $\mu \in W(\mathbb{B})$. Then $P^m_{D,\varphi} : H^p(\mathbb{B}) \to H^\infty_{\mu,0}(\mathbb{B})$ is bounded if and only if $P^m_{D,\varphi} : H^p(\mathbb{B}) \to H^\infty_\mu(\mathbb{B})$ is bounded and*

$$
\lim_{|z|\to 1} \mu(z)|u_j(z)| = 0, \quad j = \overline{0,m}. \tag{19}
$$

Proof. If $P^m_{D,\varphi}: H^p(\mathbb{B}) \to H^\infty_\mu(\mathbb{B})$ is bounded and (19) holds, then for any polynomial p we have

$$\mu(z)\left|\sum_{j=0}^m u_j(z)D_{l_j}\cdots D_{l_1}p(\varphi(z))\right| \leqslant \sum_{j=0}^m \mu(z)|u_j(z)||D_{l_j}\cdots D_{l_1}p(\varphi(z))|$$
$$\leqslant \sum_{j=0}^m \mu(z)|u_j(z)|\|D_{l_j}\cdots D_{l_1}p\|_\infty,$$

from which together with (19) it easily follows that $P^m_{D,\varphi}p \in H^\infty_{\mu,0}(\mathbb{B})$.

Since for every $f \in H^p(\mathbb{B})$ there is a sequence of polynomials $(p_k)_{k\in\mathbb{N}}$ such that

$$\lim_{k\to+\infty}\|f-p_k\|_{H^p}=0,$$

and the following inequality holds

$$\|P^m_{D,\varphi}f - P^m_{D,\varphi}p_k\|_{H^\infty_\mu} \leqslant \|P^m_{D,\varphi}\|_{H^p\to H^\infty_\mu}\|f-p_k\|_{H^p},$$

by letting $k\to+\infty$, and using the fact that $\overline{H^\infty_{\mu,0}(\mathbb{B})} = H^\infty_\mu(\mathbb{B})$, we have $P^m_{D,\varphi}f \in H^\infty_{\mu,0}(\mathbb{B})$, from which the boundedness of $P^m_{D,\varphi}: H^p(\mathbb{B}) \to H^\infty_{\mu,0}(\mathbb{B})$ follows.

Suppose that $P^m_{D,\varphi}: H^p(\mathbb{B}) \to H^\infty_{\mu,0}(\mathbb{B})$ is bounded. Then $P^m_{D,\varphi}: H^p(\mathbb{B}) \to H^\infty_\mu(\mathbb{B})$ is also such. Since $f_0(z)\equiv 1 \in H^p(\mathbb{B})$, we have $P^m_{D,\varphi}(f_0) \in H^\infty_{\mu,0}(\mathbb{B})$, that is

$$\lim_{|z|\to 1}\mu(z)|P^m_{D,\varphi}(f_0)(z)| = \lim_{|z|\to 1}\mu(z)|u_0(z)| = 0. \tag{20}$$

Hence (19) holds for $j=0$.

Suppose that for some $s\in\{1,2,\dots,m-1\}$, (19) holds for $0\leqslant j\leqslant s$. Let

$$f_{s+1}(z) = z_{l_1}z_{l_2}\cdots z_{l_{s+1}}.$$

Since $f_{s+1}\in H^p(\mathbb{B})$, we have $P^m_{D,\varphi}(f_{s+1}) \in H^\infty_{\mu,0}(\mathbb{B})$. Note that

$$f_{s+1}(z) = z_1^{\alpha_1}\cdots z_n^{\alpha_n},$$

where $\alpha_j\in\mathbb{N}_0$, $j=\overline{1,n}$, are such that $\sum_{j=1}^n\alpha_j = s+1$. It is easy to see that for each $t\in\mathbb{N}_0$, $0\leqslant t\leqslant s+1$

$$D_{j_t}\cdots D_{j_1}f_{s+1}(z) = \gamma_t z_1^{\alpha_1-k_1(t)}\cdots z_n^{\alpha_n-k_n(t)},$$

for some $\gamma_t\in\mathbb{N}$, where $k_i(t)$ is the number of operators D_i in the product $D_{j_t}\cdots D_{j_1}$. Note that $\sum_{j=1}^n k_i(t)=t$ and

$$D_{j_{s+1}}\cdots D_{j_1}f_{s+1}(z) = \gamma_{s+1}, \tag{21}$$

for some $\gamma_{s+1}\in\mathbb{N}$. Hence

$$\lim_{|z|\to 1}\mu(z)|P^m_{D,\varphi}f_{s+1}(z)| = \lim_{|z|\to 1}\mu(z)\left|\sum_{j=0}^{s+1}u_j(z)\gamma_j\prod_{i=1}^n(\varphi_i(z))^{\alpha_i-k_i(j)}\right| = 0,$$

from which, along with $|\varphi_i(z)| < 1$, $i = \overline{1,n}$, $\alpha_i \geqslant k_i(j)$, for $i = \overline{1,n}$, $j = \overline{0,s+1}$, the hypothesis $u_j \in H^\infty_{\mu,0}(\mathbb{B})$, $j = \overline{0,s}$, (21) and $\gamma_{s+1} \neq 0$, we obtain

$$\lim_{|z|\to 1} \mu(z)|u_{s+1}(z)| = 0.$$

Hence (19) holds for $j = \overline{0,m}$. $\square$

THEOREM 3. *Let $p \geqslant 1$, $m \in \mathbb{N}$, $u_j \in H(\mathbb{B})$, $j = \overline{0,m}$, $\varphi \in S(\mathbb{B})$, $\mu \in W(\mathbb{B})$, and (10) holds. Then the operator $P^m_{D,\varphi} : H^p(\mathbb{B}) \to H^\infty_\mu(\mathbb{B})$ is compact if and only if the operator is bounded and the following condition holds*

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)\big|u_j(z)\big|}{(1-|\varphi(z)|^2)^{\frac{n}{p}+j}} = 0, \tag{22}$$

for $j \in \{0,1,\ldots,m\}$.

Proof. Suppose $P^m_{D,\varphi} : H^p(\mathbb{B}) \to H^\infty_\mu(\mathbb{B})$ is bounded and (22) holds. Then for every $\varepsilon > 0$ there is $\delta \in (0,1)$ such that for $|\varphi(z)| > \delta$

$$\frac{\mu(z)\big|u_j(z)\big|}{(1-|\varphi(z)|^2)^{\frac{n}{p}+j}} < \varepsilon, \quad j = \overline{0,m}. \tag{23}$$

Suppose that $\sup_{k\in\mathbb{N}} \|f_k\|_{H^p} \leqslant M$ and

$$f_k \to 0 \tag{24}$$

uniformly on compacts of $\mathbb{B}$. Let $K_\delta = \{z \in \mathbb{B} : |\varphi(z)| > \delta\}$. Then Lemma 2 and (23) imply

$$\begin{aligned}
\|P^m_{D,\varphi} f_k\|_{H^\infty_\mu} &= \sup_{z\in\mathbb{B}} \mu(z)\Big|\sum_{j=0}^m u_j(z) D_{l_j}\cdots D_{l_1} f_k(\varphi(z))\Big| \\
&\leqslant \sup_{z\in K_\delta} \mu(z)\Big|\sum_{j=0}^m u_j(z) D_{l_j}\cdots D_{l_1} f_k(\varphi(z))\Big| \\
&\quad + \sup_{z\in\mathbb{B}\setminus K_\delta} \mu(z)\Big|\sum_{j=0}^m u_j(z) D_{l_j}\cdots D_{l_1} f_k(\varphi(z))\Big| \\
&\leqslant C\sum_{j=0}^m \sup_{z\in K_\delta} \frac{\mu(z)\big|u_j(z)\big|}{(1-|\varphi(z)|^2)^{\frac{n}{p}+j}} \|f_k\|_{H^p} \\
&\quad + C\sum_{j=0}^m \sup_{z\in\mathbb{B}\setminus K_\delta} \mu(z)|u_j(z)||D_{l_j}\cdots D_{l_1} f_k(\varphi(z))| \\
&\leqslant (m+1)MC\varepsilon + C\sum_{j=0}^m \sup_{z\in\mathbb{B}\setminus K_\delta} \mu(z)|u_j(z)| \sup_{|\varphi(z)|\leqslant\delta} \big|D_{l_j}\cdots D_{l_1} f_k(\varphi(z))\big| \\
&\leqslant (m+1)MC\varepsilon + C\sum_{j=0}^m \|u_j\|_{H^\infty_\mu} \sup_{|w|\leqslant\delta} \big|D_{l_j}\cdots D_{l_1} f_k(w)\big|.
\end{aligned} \tag{25}$$

Condition (24) together with Cauchy's estimate imply

$$D_{l_j}\cdots D_{l_1} f_k \to 0, \tag{26}$$

uniformly on compacts of $\mathbb{B}$ as $k \to +\infty$, for $j = \overline{0,m}$.

Let

$$f_s(z) = \prod_{j=1}^{s} z_{l_j}, \quad s = \overline{0,m}.$$

Arguing as in the proof of Theorem 2 we get $u_j \in H_\mu^\infty$, $j = \overline{0,m}$, from which along with (26), the compactness of $|w| \leqslant \delta$, and (25), we easily obtain

$$\lim_{k\to+\infty} \|P_{D,\varphi}^m f_k\|_{H_\mu^\infty} = 0.$$

This fact with Lemma 1 implies the compactness of $P_{D,\varphi}^m M_u : H^p(\mathbb{B}) \to H_\mu^\infty(\mathbb{B})$.

If $P_{D,\varphi}^m : H^p(\mathbb{B}) \to H_\mu^\infty(\mathbb{B})$ is compact, then it is bounded. If $\|\varphi\|_\infty < 1$, then (22) holds.

Assume that $\|\varphi\|_\infty = 1$. Let $(z_k)_{k\in\mathbb{N}} \subset \mathbb{B}$ be a sequence such that $|\varphi(z_k)| \to 1$ as $k \to +\infty$, and

$$h_k^{(s)} := h_{\varphi(z_k)}^{(s)}, \quad s = \overline{0,m},$$

where $h_w^{(s)}$, $s = \overline{0,m}$, are as in Lemma 5. Then

$$\sup_{k\in\mathbb{N}} \|h_k^{(s)}\|_{H^p} < +\infty, \quad s = \overline{0,m}, \tag{27}$$

and $h_k^{(s)} \to 0$ uniformly on compacts of $\mathbb{B}$ as $k \to +\infty$, for $s \in \{0,1,\ldots,m\}$. This along with Lemma 1 implies

$$\lim_{k\to+\infty} \|P_{D,\varphi}^m h_k^{(s)}\|_{H_\mu^\infty} = 0, \quad s = \overline{0,m}. \tag{28}$$

From (15) we have

$$\frac{\mu(z_k)|u_s(z_k)|}{(1-|\varphi(z_k)|^2)^{\frac{n}{p}+s}} \lesssim \|P_{D,\varphi}^m h_k^{(s)}\|_{H_\mu^\infty}, \quad s = \overline{0,m}. \tag{29}$$

From (28) and (29), (22) easily follows. $\square$

When $p > 1$, we can estimate the essential norm of the bounded operator $P_{D,\varphi}^m : H^p(\mathbb{B}) \to H_\mu^\infty(\mathbb{B})$ as follows.

THEOREM 4. *Let $p > 1$, $m \in \mathbb{N}$, $u_j \in H(\mathbb{B})$, $j = \overline{0,m}$, $\varphi \in S(\mathbb{B})$, $\mu \in W(\mathbb{B})$, and (10) holds. If the operator $P_{D,\varphi}^m : H^p(\mathbb{B}) \to H_\mu^\infty(\mathbb{B})$ is bounded then*

$$\|P_{D,\varphi}^m\|_{e,H^p\to H_\mu^\infty} \asymp \max_{j=\overline{1,m}} \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|u_j(z)|}{(1-|\varphi(z)|^2)^{\frac{n}{p}+j}}. \tag{30}$$

Proof. Let take a sequence $(z_k)_{k\in\mathbb{N}} \subset \mathbb{B}$ such that $|\varphi(z_k)| \to 1$ as $k \to +\infty$, and

$$h_k^{(s)} := h_{\varphi(z_k)}^{(s)}, \quad s = \overline{0,m},$$

where $h_w^{(s)}$, $s = \overline{0,m}$, are as in Lemma 5. Then (27) holds, and we have that $h_k^{(s)} \to 0$ uniformly on compacts of $\mathbb{B}$ as $k \to +\infty$, for each $s \in \{0,1,\dots,m\}$. Since the dual of $H^p(\mathbb{B})$ is known [48], it is easily verified that $h_k^{(s)} \to 0$ weakly in $H^p(\mathbb{B})$. Hence, we have

$$\lim_{k\to+\infty} \|Kh_k^{(s)}\|_{H_\mu^\infty} = 0$$

for any compact operator $K : H^p(\mathbb{B}) \to H_\mu^\infty(\mathbb{B})$.

From this, (13), (14) and (10) we have

$$\begin{aligned}\|P_{D,\varphi}^m\|_{e,H^p\to H_\mu^\infty} &\gtrsim \limsup_{k\to\infty}\left(\|P_{D,\varphi}^m h_k^{(s)}\|_{H_\mu^\infty} - \|Kh_k^{(s)}\|_{H_\mu^\infty}\right)\\ &\geqslant \limsup_{k\to\infty} \mu(z_k)|u_s(z_k)|\frac{|\varphi_{l_1}(z_k)|\cdots|\varphi_{l_s}(z_k)|}{(1-|\varphi(z_k)|^2)^{\frac{n}{p}+s}}\\ &\geqslant \delta^s \limsup_{k\to\infty}\frac{\mu(z_k)|u_s(z_k)|}{(1-|\varphi(z_k)|^2)^{\frac{n}{p}+s}}\end{aligned}$$

for each $s \in \{0,1,\dots,m\}$. This implies that the lower estimate in (30) holds.

Next we prove the upper estimate in (30). For fixed t, $0<t<1$, put $C_t f(z) = f(tz)$. Since C_t is a compact operator on $H^p(\mathbb{B})$, $P_{D,\varphi}^m C_t$ is also compact from $H^p(\mathbb{B})$ into $H_\mu^\infty(\mathbb{B})$. Thus we have

$$\|P_{D,\varphi}^m\|_{e,H^p\to H_\mu^\infty} \leqslant \sup_{\|f\|_{H^p}\leqslant 1} \|P_{D,\varphi}^m f - P_{D,\varphi}^m C_t f\|_{H_\mu^\infty}. \tag{31}$$

Now we fix $f \in H^p(\mathbb{B})$ with $\|f\|_{H^p} \leqslant 1$ and R, $0<R<1$. Note that for each $z \in \mathbb{B}$ it holds that

$$\begin{aligned}&\left|P_{D,\varphi}^m f(z) - P_{D,\varphi}^m C_t f(z)\right|\\ &= \left|\sum_{j=0}^m u_j(z)\left\{D_{l_j}\cdots D_{l_1} f(\varphi(z)) - t^j D_{l_j}\cdots D_{l_1} f(t\varphi(z))\right\}\right|.\end{aligned}$$

By combining this with Lemma 2, we have

$$\begin{aligned}&\sup_{\|f\|_{H^p}\leqslant 1}\sup_{|\varphi(z)|>R} \mu(z)\left|P_{D,\varphi}^m f(z) - P_{D,\varphi}^m C_t f(z)\right|\\ &\lesssim \sum_{j=0}^m \sup_{|\varphi(z)|>R}\frac{\mu(z)|u_j(z)|}{(1-|\varphi(z)|^2)^{\frac{n}{p}+j}}\\ &\lesssim \max_{j=\overline{0,m}}\sup_{|\varphi(z)|>R}\frac{\mu(z)|u_j(z)|}{(1-|\varphi(z)|^2)^{\frac{n}{p}+j}}.\end{aligned} \tag{32}$$

On the other hand, by using the mean value theorem and the Cauchy inequality, we obtain

$$\begin{aligned}
&\sup_{|\varphi(z)|\leqslant R}\Big|D_{l_j}\cdots D_{l_1}f(\varphi(z))-D_{l_j}\cdots D_{l_1}f(t\varphi(z))\Big|\\
&\leqslant \sup_{|\varphi(z)|\leqslant R}(1-t)|\varphi(z)|\sup_{|w|\leqslant R}\Big|\nabla(D_{l_j}\cdots D_{l_1}f)(w)\Big|\\
&\lesssim \frac{R}{1-R}(1-t)\sup_{|w|\leqslant\frac{1+R}{2}}\Big|D_{l_j}\cdots D_{l_1}f(w)\Big|.
\end{aligned}$$

Hence, Lemma 2 also shows that

$$\sup_{|\varphi(z)|\leqslant R}\Big|D_{l_j}\cdots D_{l_1}f(\varphi(z))-D_{l_j}\cdots D_{l_1}f(t\varphi(z))\Big|\lesssim\frac{R(1-t)}{(1-R)(1-(\frac{1+R}{2})^2)^{\frac{n}{p}+j}} \tag{33}$$

for each $j\in\overline{0,m}$. Furthermore it follows from Lemma 2 that

$$\sup_{|\varphi(z)|\leqslant R}\Big|D_{l_j}\cdots D_{l_1}f(t\varphi(z))-t^jD_{l_j}\cdots D_{l_1}f(t\varphi(z))\Big|\lesssim\frac{1-t^j}{(1-R^2)^{\frac{n}{p}+j}} \tag{34}$$

for each $j\in\{0,1,\ldots,m\}$. Inequalities (33) and (34) give

$$\begin{aligned}
&\sup_{\|f\|_{H^p}\leqslant1}\sup_{|\varphi(z)|\leqslant R}\mu(z)\left|P^m_{D,\varphi}f(z)-P^m_{D,\varphi}C_tf(z)\right|\\
&\lesssim\sum_{j=0}^m\left\{\frac{R(1-t)}{(1-R)(1-(\frac{1+R}{2})^2)^{\frac{n}{p}+j}}+\frac{1-t^j}{(1-R^2)^{\frac{n}{p}+j}}\right\}\sup_{|\varphi(z)|\leqslant R}\mu(z)|u_j(z)|\\
&\to 0,
\end{aligned}\tag{35}$$

as $t\to1$. From (31), (32) and (35), we obtain

$$\|P^m_{D,\varphi}\|_{e,H^p\to H^\infty_\mu}\lesssim\max_{j\in\overline{0,m}}\sup_{|\varphi(z)|>R}\frac{\mu(z)|u_j(z)|}{(1-|\varphi(z)|^2)^{\frac{n}{p}+j}}. \tag{36}$$

Letting $R\to1^-$ in (36), we also obtain the upper estimate in (30). □

THEOREM 5. *Let $p\geqslant1$, $m\in\mathbb{N}$, $u_j\in H(\mathbb{B})$, $j=\overline{0,m}$, $\varphi\in S(\mathbb{B})$, $\mu\in W(\mathbb{B})$, and condition (10) holds. Then the operator $P^m_{D,\varphi}:H^p(\mathbb{B})\to H^\infty_{\mu,0}(\mathbb{B})$ is compact if and only if the operator is bounded and*

$$\lim_{|z|\to1}\frac{\mu(z)|u_j(z)|}{(1-|\varphi(z)|^2)^{\frac{n}{p}+j}}=0,\quad j=\overline{0,m}. \tag{37}$$

Proof. Assume (37) holds. Then (11) holds. From this and Theorem 1 the boundedness of $P^m_{D,\varphi}:H^p(\mathbb{B})\to H^\infty_\mu(\mathbb{B})$ easily follows. Letting $|z|\to1$ in (17) and using (37), we have $P^m_{D,\varphi}f\in H^\infty_{\mu,0}(\mathbb{B})$ for any $f\in H^p(\mathbb{B})$, from which the boundedness of

$P_{D,\varphi}^m : H^p(\mathbb{B}) \to H_{\mu,0}^\infty(\mathbb{B})$ follows. Taking the supremum in (17) over $\mathbb{B}$ and $B_{H^p(\mathbb{B})}$, and using (11), we obtain

$$\sup_{f\in B_{H^p(\mathbb{B})}} \sup_{z\in\mathbb{B}} \mu(z)|P_{D,\varphi}^m f(z)| \leqslant C\sum_{j=0}^m L_j < +\infty, \tag{38}$$

where L_j, $j=\overline{0,m}$, are the quantities in (11). So $\{P_{D,\varphi}^m f : f\in B_{H^p(\mathbb{B})}\}$ is a bounded subset of $H_{\mu,0}^\infty(\mathbb{B})$. Taking the supremum in (17) over $B_{H^p(\mathbb{B})}$ and letting $|z|\to 1$ we have

$$\lim_{|z|\to 1} \sup_{f\in B_{H^p(\mathbb{B})}} \mu(z)\big|P_{D,\varphi}^m f(z)\big| = 0,$$

from which along with Lemma 4 the compactness of $P_{D,\varphi}^m : H^p(\mathbb{B}) \to H_{\mu,0}^\infty(\mathbb{B})$ follows.

If $P_{D,\varphi}^m : H^p(\mathbb{B}) \to H_{\mu,0}^\infty(\mathbb{B})$ is compact, then $P_{D,\varphi}^m : H^p(\mathbb{B}) \to H_{\mu}^\infty(\mathbb{B})$ is compact, from which and Theorem 3 we get (23). From Theorem 2 we get (19), so that there is $\eta\in(0,1)$ such that

$$\mu(z)|u_j(z)| < \varepsilon(1-\delta^2)^{\frac{n}{p}+j}, \quad j=\overline{0,m},$$

when $\eta<|z|<1$, for ε chosen such that (23) holds, and consequently

$$\frac{\mu(z)|u_j(z)|}{(1-|\varphi(z)|^2)^{\frac{n}{p}+j}} \leqslant \frac{\mu(z)|u_j(z)|}{(1-\delta^2)^{\frac{n}{p}+j}} < \varepsilon, \quad j=\overline{0,m},$$

when $|\varphi(z)|\leqslant\delta$ and $\eta<|z|<1$. This along with (23) imply (37). $\square$

In addition to Theorem 5, we also obtain the estimate for the essential norm of the operator $P_{D,\varphi}^m : H^p(\mathbb{B}) \to H_{\mu,0}^\infty(\mathbb{B})$, in the case $p>1$.

THEOREM 6. *Let $p>1$, $m\in\mathbb{N}$, $u_j\in H(\mathbb{B})$, $j=\overline{0,m}$, $\varphi\in S(\mathbb{B})$, $\mu\in W(\mathbb{B})$, and condition (10) holds. If the operator $P_{D,\varphi}^m : H^p(\mathbb{B}) \to H_{\mu,0}^\infty(\mathbb{B})$ is bounded then*

$$\|P_{D,\varphi}^m\|_{e,H^p\to H_{\mu,0}^\infty} \asymp \max_{j=\overline{1,m}} \limsup_{|z|\to 1} \frac{\mu(z)\big|u_j(z)\big|}{(1-|\varphi(z)|^2)^{\frac{n}{p}+j}}. \tag{39}$$

Proof. By Theorem 2, the boundedness of $P_{D,\varphi}^m : H^p(\mathbb{B}) \to H_{\mu,0}^\infty(\mathbb{B})$ implies $u_j\in H_{\mu,0}^\infty(\mathbb{B})$ for $j=\overline{0,m}$. There are two cases to be considered.

Case $\|\varphi\|_\infty<1$. In this case we see that $P_{D,\varphi}^m : H^p(\mathbb{B}) \to H_{\mu,0}^\infty(\mathbb{B})$ is compact, so $\|P_{D,\varphi}^m\|_{e,H^p\to H_{\mu,0}^\infty}=0$. On the other hand, from $\|\varphi\|_\infty<1$ and $u_j\in H_{\mu,0}^\infty(\mathbb{B})$ we have that the limit on the right-hand side of (39) equals to zero. Hence (39) holds in this case.

Case $\|\varphi\|_\infty=1$. By Theorem 4, it is enough to prove that

$$\limsup_{|z|\to 1} \frac{\mu(z)\big|u_j(z)\big|}{(1-|\varphi(z)|^2)^{\frac{n}{p}+j}} = \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)\big|u_j(z)\big|}{(1-|\varphi(z)|^2)^{\frac{n}{p}+j}}, \tag{40}$$

for $j=\overline{0,m}$.

Note that

$$\limsup_{|z|\to 1}\frac{\mu(z)|u_j(z)|}{(1-|\varphi(z)|^2)^{\frac{n}{p}+j}}\geqslant \limsup_{|\varphi(z)|\to 1}\frac{\mu(z)|u_j(z)|}{(1-|\varphi(z)|^2)^{\frac{n}{p}+j}},\quad j=\overline{0,m}. \tag{41}$$

Assume that a sequence $(z_k)_{k\in\mathbb{N}}\subset\mathbb{B}$ satisfies

$$\limsup_{|z|\to 1}\frac{\mu(z)|u_j(z)|}{(1-|\varphi(z)|^2)^{\frac{n}{p}+j}}=\lim_{k\to\infty}\frac{\mu(z_k)|u_j(z_k)|}{(1-|\varphi(z_k)|^2)^{\frac{n}{p}+j}},\quad j=\overline{0,m}.$$

If $\sup_{k\in\mathbb{N}}|\varphi(z_k)|<1$, then since $u_j\in H^\infty_{\mu,0}(\mathbb{B})$, $j=\overline{0,m}$, we have that the first limit in (41) is zero and consequently the second one.

If $\sup_{k\in\mathbb{N}}|\varphi(z_k)|=1$, then there is a subsequence $(\varphi(z_{k_l}))_{l\in\mathbb{N}}$ such that $|\varphi(z_{k_l})|\to 1$ as $l\to\infty$. Hence we obtain

$$\begin{aligned}\limsup_{|z|\to 1}\frac{\mu(z)|u_j(z)|}{(1-|\varphi(z)|^2)^{\frac{n}{p}+j}}&=\lim_{l\to\infty}\frac{\mu(z_{k_l})|u_j(z_{k_l})|}{(1-|\varphi(z_{k_l})|^2)^{\frac{n}{p}+j}}\\&\leqslant\limsup_{|\varphi(z)|\to 1}\frac{\mu(z)|u_j(z)|}{(1-|\varphi(z)|^2)^{\frac{n}{p}+j}}.\end{aligned} \tag{42}$$

From (41) and (42), (40) follows, finishing the proof of the theorem. □

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