EXISTENCE AND ASYMPTOTIC BEHAVIOR OF SQUARE–MEAN S–ASYMPTOTICALLY PERIODIC SOLUTIONS FOR STOCHASTIC EVOLUTION EQUATION INVOLVING DELAY

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Abstract. This paper studies the stochastic evolution equations with finite delay. By means of the compact semigroup theory and Schauder fixed point theorem, the existence of square-mean S-asymptotically periodic mild solutions is obtained under certain growth conditions. In addition, using the contraction mapping principle and Gronwall integral inequality, the uniqueness and global asymptotic stability of the square-mean S-asymptotically periodic mild solutions are discussed. Finally, an example is given to illustrate our abstract results.

1. Introduction

In this paper, we assume that $\mathbb{H}$ and $\mathbb{K}$ are two real separable Hilbert spaces and $\mathcal{L}(\mathbb{K}, \mathbb{H})$ denotes the space of all bounded linear operators from $\mathbb{K}$ into $\mathbb{H}$. For convenience, we will use the same notation $(\cdot, \cdot)$ to denote the inner product of $\mathbb{H}$ and $\mathbb{K}$, and use $\|\cdot\|$ to denote the norms in $\mathbb{H}$, $\mathbb{K}$ and $\mathcal{L}(\mathbb{K}, \mathbb{H})$ without any confusion. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with some filtration $\{\mathcal{F}_t\}_{t \geqslant 0}$ satisfying the usual conditions, that is, the filtration is a right continuous increasing family and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$. Let $\{\mathbb{W}(t) : t \geqslant 0\}$ be a $\mathbb{K}$-valued Wiener process with a finite trace nuclear covariance operator $Q \geqslant 0$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Moreover, we assume that $L^2(\Omega, \mathbb{H})$ is the collection of all strong measurable square-integrable $\mathbb{H}$-valued random variables.

In this paper, we investigate the existence and global asymptotic behavior of square-mean S-asymptotically periodic mild solutions of the following delayed stochastic evolution equation

$$\begin{cases} u'(t) + Au(t) = F(t, u_t) + G(t, u_t)\mathbb{W}'(t), & t \geqslant 0, \\ u(t) = \varphi(t), & t \in [-r, 0], \end{cases} \tag{1.1}$$

where the state $u(\cdot)$ takes values in $\mathbb{H}$; $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ is a closed linear operator and $T(t)$ $(t \geqslant 0)$ is a C_0-semigroup generated by $-A$ in $\mathbb{H}$; the nonlinear functions $F : \mathbb{R}^+ \times \mathcal{B} \to L^2(\Omega, \mathbb{H})$ and $G : \mathbb{R}^+ \times \mathcal{B} \to \mathcal{L}(\mathbb{K}, \mathbb{H})$ are continuous functions which will be appointed later; $\mathcal{B}$ is a phase space specified later, $\varphi \in \mathcal{B}$ is a given initial data. For $t \geqslant 0$, $u_t \in \mathcal{B}$ denotes a stochastic process defined by $u_t(s) = u(t+s)$, $s \in [-r, 0]$, here $u : [-r, \infty) \to L^2(\Omega, \mathbb{H})$ is a random bounded continuous process.

Due to the application of stochastic differential equations in physics, chemistry, social sciences, big data, finance and other fields, more and more scholars pay attention to the study of stochastic differential equations. Stochastic evolution equation theory is an important branch of stochastic differential equations. Because many models with noise or random interference in applied disciplines can be transformed into abstract stochastic evolution equations, many researchers have studied the existence, uniqueness and asymptotic stability of solutions for stochastic evolution equations, and obtained some meaningful results(see [17, 10, 3, 5, 15, 40, 8, 1, 23, 16] and the references therein). In addition, the evolution equation with delay has a wide background and important application value in many disciplines such as chemistry, physics, biology, ecology, economy, humanities and realistic mathematical models. Hence, the theory of delayed partial differential equations, especially the existence and global asymptotic stability of solutions, has attracted extensive attention in recent years (see [41, 18, 24, 25, 2, 34, 30, 4] and the references therein).

Naturally, more and more researchers have paid their attentions to the existence, uniqueness and global asymptotic behavior of solutions for stochastic evolution equations with delays and some interesting results have brought to our view (see [36, 38, 29, 45, 14, 37, 22, 42, 43, 28, 39, 9]). Taniguchi [38] studied the existence, uniqueness, p-th moment and almost sure Lyapunov exponents of mild solutions for delayed stochastic partial functional differential equations by means of analytic semigroups. Luo [29] obtained the exponential stability of mild solutions for delayed stochastic partial differential equations by applying the principle of compressed mapping and stochastic integral techniques. Xu et al. [42] investigated square-mean exponential stability of delayed stochastic partial differential equations by establishing differential inequality with delays and utilizing the stochastic analysis technique. Shen and Ren [37] proved an existence and uniqueness result of the mild solution for a neutral stochastic partial differential equations with finite delay driven by Rosenblatt process in a real separable Hilbert space. Zhang et al. [45] established the convergence for a class of highly nonlinear stochastic differential equations with delay under the local Lipschitz condition plus Khasminskii-type condition. Gao and Li [14] obtained the existence and the mean-square exponential stability of mild solutions for impulsive stochastic partial differential equations with varying-time delays by means of the Hausdorff measure of noncompactness and some inequality technique. Hu and Huang [22] investigated the delay dependent stability of the semidiscrete and fully discrete systems for a linear stochastic delay partial differential equation by using the standard central difference scheme in space and the stochastic exponential Euler method in time. Yan and Han [43] obtained the globally exponential stability of p-mean piecewise pseudo almost periodic mild solutions of impulsive partial stochastic differential equations with infinite delay in Hilbert spaces by means of operator semigroups theory and stochastic analysis tech-

niques.

On the other hand, the periodic problem of partial differential equations is also an important research field in recent years. However, in real life, many phenomena observed by people do not meet the strict periodicity due to many interference factors. To better study these periodic phenomena and characterize these mathematical models, many researchers have some generalized periodic functions, such as almost periodic functions, asymptotic periodic functions, asymptotic almost periodic functions, pseudo almost periodic functions and S-asymptotic periodic functions and so on. It is worth noting that, S-asymptotically period function between asymptotically periodic function and asymptotically almost periodic function, is a more general approximate period function, which was first proposed and established by Henríquez et al. [20]. The existence and uniqueness of S-asymptotically periodic solutions for differential equations have been investigated in [19, 20, 32, 33, 11, 12, 13, 35, 6].

However, the existence of S-asymptotically periodic solutions for the stochastic evolution equations with delay have not received much attention, specially, most of the existing literatures assume that the nonlinear terms satisfy the Lipschitz condition to get the existence of global mild solution. Furthermore, the global asymptotic behavior of S-asymptotically ω-periodic solutions for delayed stochastic evolution equations is also an untreated topic in the literatures. Thus, inspired by the above literature, we investigate the existence and global asymptotic behavior of the S-asymptotically periodic mild solutions for the delayed stochastic evolution equation.

This article is organized as follows. Section 2 introduces some notions, definitions, and preliminary facts. In section 3, based on the Schauder fixed point theorem, the existence of the square-mean S-asymptotically periodic mild solution of the equation (1.1) is obtained in the case that the corresponding linear partial differential operator generates a compact semigroup, which is very convenient for equations with compact resolvent. Specially, we only assume that the nonlinear functions satisfy some growth conditions that are weaker than Lipschitz conditions. In Section 4, the uniqueness and global exponential stability of the square-mean S-asymptotically ω-periodic mild solution of the equation (1.1) are considered by using the principle of contraction mapping and Gronwall integral inequality. Since our condition involves the growth index of C_0-semigroup or the first eigenvalue of the infinitesimal generator of compact semigroup, our results improve and generalize the conclusions in the existing literature. Finally, an example is given to illustrate our abstract results.

2. Preliminaries

In this paper, we always assume that $\mathbb{H}$ and $\mathbb{K}$ are two real separable Hilbert spaces. For convenience, the same notations $\|\cdot\|$ and $(\cdot, \cdot)$ are used to represent the norms and the inner products in $\mathbb{H}$ and $\mathbb{K}$, respectively. Let $\mathcal{L}(\mathbb{H})$ be the Banach space of all bounded linear operators with the topology defined by operator norm on $\mathbb{H}$. Let $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ be a closed linear operator and $T(t)$ $(t \geqslant 0)$ is a C_0-semigroup generated by $-A$ in $\mathbb{H}$. From [31] it follows that there are constants $M \geqslant 1$ and $v \in \mathbb{R}$ such that

$$\|T(t)\| \leqslant M e^{vt}, \quad t \geqslant 0. \tag{2.1}$$

The growth exponent of the C_0-semigroup $T(t)$ $(t \geqslant 0)$ is given by

$$v_0 = \inf\{v \in \mathbb{R} \mid \text{There exists } M \geqslant 1 \text{ such that } \|T(t)\| \leqslant Me^{vt}, \ \forall t \geqslant 0\}. \qquad (2.2)$$

The semigroup $T(t)$ $(t \geqslant 0)$ is called exponentially stable whenever $v_0 < 0$. Next, we give the definition of compact semigroups. A C_0-semigroup $T(t)$ is said to be compact if $T(t)$ is compact for each $t > 0$.

As is known to all, the compact semigroup is continuous in the uniform operator topology, hence v_0 can also be determined by $\sigma(A)$ (the spectrum of A)

$$v_0 = -\inf\{\text{Re}\lambda \mid \lambda \in \sigma(A)\}. \qquad (2.3)$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with some filtration $\{\mathcal{F}_t\}_{t \geqslant 0}$ satisfying the usual conditions, that is, the filtration is a right continuous increasing family and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$. Let $\{\mathbb{W}(t) : t \geqslant 0\}$ be a $\mathbb{K}$-valued Wiener process defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with a finite trace nuclear covariance operator $Q \geqslant 0$ satisfying $Tr(Q) = \sum_{k=1}^{\infty} \lambda_k = \lambda < \infty$ and $Qe_k = \lambda_k e_k, k \in \mathbb{N}$, where $\{e_k : k \in \mathbb{N}\}$ is a complete orthonormal basis in $\mathbb{K}$. Assume that $\{\mathbb{W}_k, k \in \mathbb{N}\}$ is a series of independent one-dimensional standard Wiener processes defined on $(\Omega, \mathcal{F}, \mathbb{P})$, then

$$\mathbb{W}(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} \mathbb{W}_k(t) e_k, \quad t \geqslant 0.$$

For $\phi, \psi \in \mathcal{L}(\mathbb{K}, \mathbb{H})$, deonte $(\phi, \psi) = Tr(\phi Q \psi^*)$, where ψ^* is the adjoint of the operator ψ. Hence, for every bounded operator $\psi \in L(\mathbb{K}, \mathbb{H})$, one can find

$$\|\psi\|_Q^2 = Tr(\psi Q \psi^*) = \sum_{k=1}^{\infty} \|\sqrt{\lambda_k} \psi e_k\|^2.$$

We use the collection of all strongly-measurable, square-integrable $\mathbb{H}$-valued random variables to represent $L^2(\Omega, \mathbb{H})$, which is a Banach space with the norm

$$\|x(\cdot)\|_{L^2} = (\mathbb{E}\|x(\cdot, \varpi)\|^2)^{\frac{1}{2}}, \ \varpi \in \Omega,$$

where $\mathbb{E}(\cdot)$ is the expectation defined by $\mathbb{E}x = \int_{\Omega} x(\varpi) d\mathbb{P}$. Let

$$L_0^2(\Omega, \mathbb{H}) := \{x \in L^2(\Omega, \mathbb{H}) \mid x \text{ is } \mathcal{F}_0 - \text{measurable}\},$$

then $L_0^2(\Omega, \mathbb{H})$ is a subspace of $L^2(\Omega, \mathbb{H})$.

Specially, according to [10, Lemma 7.7], one can obtain the following result.

LEMMA 2.1. *Let $p \geqslant 2$. For any $t > 0$, if $L(\mathbb{K}, \mathbb{H})$-valued predictable process Φ satisfies $\mathbb{E} \int_0^t \|\Phi(s)\|^2 ds < \infty$ then*

$$\sup_{0 \leqslant s \leqslant t} \mathbb{E}\left\| \int_0^s \Phi(\tau) d\mathbb{W}(\tau) \right\|^p \leqslant \left(\frac{p(p-1)}{2} \right)^{\frac{p}{2}} \left(\int_0^t (\mathbb{E}\|\Phi(s)\|^p)^{\frac{2}{p}} ds \right)^{\frac{p}{2}}. \qquad (2.4)$$

A stochastic process $u : [0, \infty) \to L^2(\Omega, \mathbb{H})$ is called randomly bounded if $\sup_{t \geqslant 0} \mathbb{E}\|u(t)\|^2 < \infty$, and randomly continuous if $\lim_{t \to s} \mathbb{E}\|u(t) - u(s)\|^2 = 0$ for all

$t,s \geqslant 0$. We denote by C_{sb} the Banach space of all stochastically bounded continuous processes from $[0,\infty)$ into $L^2(\Omega,\mathbb{H})$ with the norm $\|u\|_C=(\sup_{t\geqslant 0}\mathbb{E}\|u(t)\|^2)^{\frac{1}{2}}$, and denote by $\mathcal{B}=C([-r,0],L_0^2(\Omega,\mathbb{H}))$ the Banach space of all stochastically bounded continuous processes from $[-r,0]$ into $L_0^2(\Omega,\mathbb{H})$ with the norm $\|\varphi\|_\mathcal{B}=(\sup_{s\in[-r,0]}\mathbb{E}\|\varphi(s)\|^2)^{\frac{1}{2}}$, $r > 0$ is a constant.

DEFINITION 2.2. Let $u \in C_{sb}$, if there exists $\omega > 0$ such that $\lim_{t\to\infty}\mathbb{E}\|u(t+\omega) - u(t)\|^2 = 0$, then u is said to be a square-mean S-asymptotically ω-periodic function. Here, ω is said to be an asymptotic periodic of u.

Let $SAP_\omega(L^2(\Omega,\mathbb{H})) \subset C_{sb}$ be composed of all square mean S-asymptotic ω-periodic stochastic processes with uniform convergence norm $\|\cdot\|_C$. From the [20], it follows that $SAP_\omega(L^2(\Omega,\mathbb{H}))$ is a Banach space. If $u \in SAP_\omega(L^2(\Omega,\mathbb{H}))$, then we can easily test and verify that the function $t \to u_t \in SAP_\omega(\mathcal{B})$ (see [26]).

DEFINITION 2.3. Let $\varphi \in \mathcal{B}$. An $\mathbb{H}$-valued stochastic process $u : [-r,\infty) \to \mathbb{H}$ is said to be a mild solution of the problem (1.1) if

(1) $u(t)$ is an $\mathcal{F}_t$-adapted stochastic process for $t \geqslant 0$;

(2) $u(t) \in \mathbb{H}$ has cádlág paths on $t \in [0,\infty)$ almost surely;

(3) $u(t) = \varphi(t)$ for $t \in [-r,0]$, and

$$u(t) = T(t)\varphi(0) + \int_0^t T(t-s)F(s,u_s)ds + \int_0^t T(t-s)G(s,u_s)d\mathbb{W}(s) \quad \text{for } t \geqslant 0. \quad (2.5)$$

Moreover, if $u|_{t\geqslant 0} \in SAP_\omega(L^2(\Omega,\mathbb{H}))$, then u is said to be an S-asymptotically ω-periodic mild solution of the problem (1.1).

DEFINITION 2.4. Assume that u is a square-mean S-asymptotically ω-periodic mild solution of the problem (1.1) corresponding to the initial conditions $u(s) = \varphi(s)$ for $s \in [-r,0]$, and $v(t)$ is a mild solution of the problem (1.1) corresponding to the initial conditions $v(s) = \phi(s)$, $s \in [-r,0]$. If there exist positive constants $\overline{M}$ and α, such that
$$\mathbb{E}\|u(t) - v(t)\|^2 \leqslant \overline{M}\|\varphi - \phi\|_\mathcal{B}^2 \cdot e^{-\alpha t}, \quad t \geqslant 0,$$
then the square-mean S-asymptotically ω-periodic mild solution u is said to be globally exponentially stable.

LEMMA 2.5. ([44] Schauder fixed point theorem) *Let X be a Banach space and D be a bounded convex closed set in X. If $\mathcal{Q}: D \to D$ is completely continuous, then $\mathcal{Q}$ has a fixed point in D.*

3. Existence result

Assume that $h : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous and nondecreasing function satisfying $h(t) \geqslant 1$ for all $t \in \mathbb{R}^+$ and $\lim_{t\to\infty} h(t) = \infty$. Introduce the space

$$C_h := \left\{ u \in C([0,\infty),L^2(\Omega,\mathbb{H})) \,\Big|\, \lim_{t\to\infty} \frac{\mathbb{E}\|u(t)\|^2}{h(t)} = 0 \right\}$$

as the space of (stochastically continuous) processes with the finite squared norm

$$\|u\|_h^2 = \sup_{t \geqslant 0} \frac{\mathbb{E}\|u(t)\|^2}{h(t)}.$$

Obviously, C_h is a Banach space. Note that since $h(t) \geqslant 1$, we have that $\|u\|_h \leqslant \|u\|_C$, and therefore $C_{sb} \subset C_h$.

From [7, 21], one can easily verify the following result.

LEMMA 3.1. *A set $D \subset C_h$ is relatively compact if the following conditions hold*

(i) $\lim_{t \to \infty} \frac{1}{h(t)} \mathbb{E}\|u(t)\|^2 = 0$ *uniformly for every $u \in D$;*

(ii) for any constant $a > 0$, the functions in D are equicontinuous in $[0,a]$;

(iii) $D(t) = \{u(t) : u \in D\}$ is relatively compact in $L^2(\Omega, \mathbb{H})$ for every $t \geqslant 0$.

THEOREM 3.2. *Assume that $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ is a closed linear operator and $-A$ generates an exponentially stable compact semigroup $T(t)$ $(t \geqslant 0)$ in Hilbert space $\mathbb{H}$, whose growth exponent denotes $v_0 < 0$. Let functions $F(t, \cdot) : \mathcal{B} \to L^2(\Omega, \mathbb{H})$ and $G(t, \cdot) : \mathcal{B} \to \mathcal{L}(\mathbb{K}, \mathbb{H})$ be continuous for a.e. $t \geqslant 0$, which satisfy the following conditions*

(H1) there exists $\omega > 0$ such that

$$\lim_{t \to \infty} \mathbb{E}\|F(t+\omega, \phi) - F(t, \phi)\|^2 = 0, \quad \lim_{t \to \infty} \mathbb{E}\|G(t+\omega, \phi) - G(t, \phi)\|^2 = 0,$$

uniformly for $\phi \in \mathcal{B}$,

(H2) for all $t \geqslant 0$ and $\phi \in \mathcal{B}$, there exist nonnegative constants a_1, b_1 and positive constants a_0, b_0 such that

$$\mathbb{E}\|F(t, h^{\frac{1}{2}}(t)\phi)\|^2 \leqslant a_1\|\phi\|_{\mathcal{B}}^2 + a_0, \quad \mathbb{E}\|G(t, h^{\frac{1}{2}}(t)\phi)\|^2 \leqslant b_1\|\phi\|_{\mathcal{B}}^2 + b_0,$$

then for a given $\varphi \in \mathcal{B}$, the problem (1.1) has at least one square-mean S-asymptotically ω-periodic mild solution u satisfying $u|_{t \geqslant 0} \in SAP_\omega(L^2(\Omega, \mathbb{H}))$ and $u(t) = \varphi(t)$ for $t \in [-r, 0]$ provided that

$$\frac{3M^2 a_1}{|v_0|^2} + \frac{3M^2 b_1}{2|v_0|} < 1. \tag{3.1}$$

Proof. For a given $\varphi \in \mathcal{B}$, $u \in C_h$, we define the function $u[\varphi] : [-r, \infty) \to L^2(\Omega, \mathbb{H})$ as follows:

$$u[\varphi](t) = \begin{cases} u(t), & \text{for } t \geqslant 0 \\ \varphi(t), & \text{for } t \in [-r, 0]. \end{cases}$$

We denote

$$C_{\varphi, h} = \{u \in C_h : u(0) = \varphi(0)\}. \tag{3.2}$$

Then $C_{\varphi, h}$ is a closed subspace of C_h.

Define an operator $\mathcal{Q}$ on $C_{\varphi,h}$ by

$$\mathcal{Q}u(t) = T(t)\varphi(0) + \int_0^t T(t-s)F(s,u[\varphi]_s)ds + \int_0^t T(t-s)G(s,u[\varphi]_s)d\mathbb{W}(s), \quad (3.3)$$

for each $u \in C_{\varphi,h}$ and $t \geqslant 0$. From continuity of the functions F,G and the condition (H2), one can easily verify that $\mathcal{Q} : C_{\varphi,h} \to C_{\varphi,h}$ is well defined. In fact, for every $u \in C_{\varphi,h}$ and $t \geqslant 0$, one can see $\mathbb{E}\|u(t)\|^2 \leqslant h(t)\|u\|_h^2$ and

$$
\begin{aligned}
\|u[\varphi]_t\|_{\mathcal{B}}^2 &= \sup_{s\in[-r,0]} \mathbb{E}\|u[\varphi](t+s)\|^2 \\
&\leqslant \sup_{s\in[-r,0]} \mathbb{E}\|\varphi(s)\|^2 + \sup_{t\geqslant 0} \mathbb{E}\|u(t)\|^2 \\
&\leqslant h(t)\|\varphi\|_{\mathcal{B}}^2 + h(t)\|u\|_h^2
\end{aligned}
\quad (3.4)
$$

By the condition (H2), the Hölder inequality, (2.1) and (2.4), we obtain

$$
\begin{aligned}
&\mathbb{E}\|\mathcal{Q}u(t)\|^2 \\
&\leqslant 3E\|T(t)\varphi(0)\|^2 + 3E\left\| \int_0^t T(t-s)F(s,u[\varphi]_s)ds \right\|^2 \\
&\quad + 3E\left\| \int_0^t T(t-s)G(s,u[\varphi]_s)d\mathbb{W}(s) \right\|^2 \\
&\leqslant 3M^2\mathbb{E}\|\varphi(0)\|^2 + 3\int_0^t \|T(t-s)\|ds \int_0^t \|T(t-s)\| \cdot \mathbb{E}\|F(s,u[\varphi]_s)\|^2 ds \\
&\quad + 3\int_0^t \|T(t-s)\|^2 \cdot \mathbb{E}\|G(s,u[\varphi]_s)\|^2 ds \\
&\leqslant 3M^2\|\varphi\|_{\mathcal{B}}^2 + 3\int_0^t \|T(t-s)\|ds \int_0^t \|T(t-s)\| \left(a_1 \frac{\|u[\varphi]_s\|_{\mathcal{B}}^2}{h(s)} + a_0 \right) ds \\
&\quad + 3\int_0^t \|T(t-s)\|^2 \left(b_1 \frac{\|u[\varphi]_s\|_{\mathcal{B}}^2}{h(s)} + b_0 \right) ds \\
&\leqslant 3M^2\|\varphi\|_{\mathcal{B}}^2 + 3\left(\int_0^t Me^{v_0(t-s)}ds \right)^2 \left(a_1 \sup_{s\geqslant 0} \frac{\|u[\varphi]_s\|_{\mathcal{B}}^2}{h(s)} + a_0 \right) \\
&\quad + 3M^2 \int_0^t e^{2v_0(t-s)}ds \left(b_1 \sup_{s\geqslant 0} \frac{\|u[\varphi]_s\|_{\mathcal{B}}^2}{h(s)} + b_0 \right) \\
&\leqslant 3M^2 \left(1 + \frac{a_1}{|v_0|^2} + \frac{b_1}{2|v_0|} \right) \|\varphi\|_{\mathcal{B}}^2 + \frac{3M^2}{|v_0|^2}(a_1\|u\|_h^2 + a_0) + \frac{3M^2}{2|v_0|}(b_1\|u\|_h^2 + b_0).
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&\frac{1}{h(t)}\mathbb{E}\|\mathcal{Q}u(t)\|^2 \\
&\leqslant \frac{3M^2}{h(t)} \left(\left(1 + \frac{a_1}{|v_0|^2} + \frac{b_1}{2|v_0|} \right) \|\varphi\|_{\mathcal{B}}^2 + \frac{1}{|v_0|^2}(a_1\|u\|_h^2 + a_0) + \frac{1}{2|v_0|}(b_1\|u\|_h^2 + b_0) \right) \\
&\to 0, \quad \text{as } t \to \infty,
\end{aligned}
\quad (3.5)
$$

which implies that $\mathcal{Q}: C_{\varphi,h} \to C_{\varphi,h}$ is well defined. Therefore, according to (3.2) and Definition 2.3, we can assert u is the fixed point of operator $\mathcal{Q}$, then $u[\varphi]$ is the mild solution for the problem (1.1) on $[-r, \infty)$. Moreover, if $u \in SAP_\omega(L^2(\Omega, \mathbb{H}))$, then $u[\varphi]$ is the square-mean S-asymptotically ω-periodic mild solution of the problem (1.1).

Next, we complete the proof in five steps.

Step 1. $\mathcal{Q}$ is continuous on $C_{\varphi,h}$.

Let the sequence $\{u^{(n)}\} \subset C_{\varphi,h}$ and $u^{(n)} \to u$ in $C_{\varphi,h}$ as $n \to \infty$, then $u^{(n)}[\varphi]_t \to u[\varphi]_t (n \to \infty)$ for any $t \in [0, \infty)$. By the continuity of F and G, for any $\varepsilon > 0$ and large enough n, one can see

$$\mathbb{E}\|F(t, u^{(n)}[\varphi]_t) - F(t, u[\varphi]_t)\|^2 \leqslant \frac{|v_0|^2 \varepsilon}{M^2}, \quad a.e. \ t \geqslant 0. \tag{3.6}$$

$$\mathbb{E}\|G(t, u^{(n)}[\varphi]_t) - G(t, u[\varphi]_t)\|^2 \leqslant \frac{2|v_0|\varepsilon}{M^2}, \quad a.e. \ t \geqslant 0. \tag{3.7}$$

Hence, by the dominated convergence theorem and (3.6), (3.7), one can find

$$\mathbb{E}\|\mathcal{Q}u^{(n)}(t) - \mathcal{Q}u(t)\|^2$$

$$\leqslant 2\mathbb{E}\left\|\int_0^t T(t-s) \cdot (F(s, u^{(n)}[\varphi]_s) - F(s, u[\varphi]_s))ds\right\|^2$$

$$+ 2\mathbb{E}\left\|\int_0^t T(t-s) \cdot (G(s, u^{(n)}[\varphi]_s) - G(s, u[\varphi]_s))d\mathbb{W}(s)\right\|^2$$

$$\leqslant 2\int_0^t \|T(t-s)\|ds \int_0^t \|T(t-s)\| \cdot \mathbb{E}\|F(s, u^{(n)}[\varphi]_s) - F(s, u[\varphi]_s)\|^2 ds$$

$$+ 2\int_0^t \|T(t-s)\|^2 \cdot \mathbb{E}\|G(s, u^{(n)}[\varphi]_s) - G(s, u[\varphi]_s)\|^2 ds$$

$$\leqslant 2\left(\int_0^t M e^{v_0(t-s)}ds\right)^2 \cdot \frac{|v_0|^2 \varepsilon}{M^2} + 2M^2 \int_0^t e^{2v_0(t-s)}ds \cdot \frac{2|v_0|\varepsilon}{M^2}$$

$$\leqslant 2 \cdot \frac{M^2}{|v_0|^2} \cdot \frac{|v_0|^2 \varepsilon}{M^2} + 2 \cdot \frac{M^2}{2|v_0|} \cdot \frac{2|v_0|\varepsilon}{M^2}$$

$$\leqslant 4\varepsilon.$$

Thus

$$\|\mathcal{Q}u^{(n)} - \mathcal{Q}u\|_h = \left(\sup_{t \geqslant 0} \frac{1}{h(t)}\mathbb{E}\|\mathcal{Q}u^{(n)}(t) - \mathcal{Q}u(t)\|^2\right)^{\frac{1}{2}} \to 0, \ \text{as } n \to \infty,$$

which implies that the operator $\mathcal{Q}: C_{\varphi,h} \to C_{\varphi,h}$ is continuous.

For any $R > \|\varphi\|_{\mathcal{B}}^2$, let

$$\overline{\Theta}_R := \{u \in C_{\varphi,h} | \ \|u\|_h^2 \leqslant R - \|\varphi\|_{\mathcal{B}}^2\}. \tag{3.8}$$

Clearly, $\overline{\Theta}_R$ is a closed ball in $C_{\varphi,h}$.

Step 2. There is a constant $R_0 > 0$ such that $\mathcal{Q}(\overline{\Theta}_{R_0}) \subset \overline{\Theta}_{R_0}$.

If it was invalid, then there exist $u \in \overline{\Theta}_R$ such that $\|\mathcal{Q}u\|_h^2 > R - \|\varphi\|_{\mathcal{B}}^2$ for any $R > \|\varphi\|_{\mathcal{B}}^2$. By the Hölder inequality, the condition (H2), one can see for $t \geqslant 0$,

$$
\begin{aligned}
\frac{\mathbb{E}\|(\mathcal{Q}u)(t)\|^2}{h(t)} &\leqslant \mathbb{E}\|(\mathcal{Q}u)(t)\|^2 \\
&\leqslant 3\mathbb{E}\|T(t)\varphi(0)\|^2 + 3\mathbb{E}\left\| \int_0^t T(t-s)F(s,u[\varphi]_s)ds \right\|^2 \\
&\quad + 3\mathbb{E}\left\| \int_0^t T(t-s)G(s,u[\varphi]_s)d\mathbb{W}(s) \right\|^2 \\
&\leqslant 3M^2\mathbb{E}\|\varphi(0)\|^2 + 3\int_0^t \|T(t-s)\|ds \int_0^t \|T(t-s)\| \cdot \mathbb{E}\|F(s,u[\varphi]_s)\|^2 ds \\
&\quad + 3\int_0^t \|T(t-s)\|^2 \cdot \mathbb{E}\|G(s,u[\varphi]_s)\|^2 ds \\
&\leqslant 3M^2\|\varphi\|_{\mathcal{B}}^2 + 3\left(\int_0^t Me^{v_0(t-s)}ds \right)^2 \left(a_1 \sup_{s \geqslant 0} \frac{\|u[\varphi]_s\|_{\mathcal{B}}^2}{h(s)} + a_0 \right) \\
&\quad + 3M^2 \int_0^t e^{2v_0(t-s)}ds \left(b_1 \sup_{s \geqslant 0} \frac{\|u[\varphi]_s\|_{\mathcal{B}}^2}{h(s)} + b_0 \right) \\
&\leqslant 3M^2\|\varphi\|_{\mathcal{B}}^2 + \frac{3M^2}{|v_0|^2}(a_1 R + a_0) + \frac{3M^2}{2|v_0|}(b_1 R + b_0),
\end{aligned}
$$

thus,

$$
R - \|\varphi\|_{\mathcal{B}}^2 \leqslant 3M^2\|\varphi\|_{\mathcal{B}}^2 + \frac{3M^2}{|v_0|^2}(a_1 R + a_0) + \frac{3M^2}{2|v_0|}(b_1 R + b_0). \tag{3.9}
$$

Dividing both sides of (3.9) by R and taking the lower limit as $R \to \infty$, and combining with (3.1), one can get that

$$
1 \leqslant \frac{3M^2 a_1}{|v_0|^2} + \frac{3M^2 b_1}{2|v_0|} < 1,
$$

which is a contradiction. Therefore, there is a constant $R_0 > 0$ such that $\mathcal{Q}(\overline{\Theta}_{R_0}) \subset \overline{\Theta}_{R_0}$.

Moreover, according to the property of the function $h(t)$, it is easy to see that for any $u \in \overline{\Theta}_{R_0}$

$$
\lim_{t \to \infty} \frac{1}{h(t)} E\|\mathcal{Q}u(t)\|^2 = 0. \tag{3.10}
$$

Step 3. The set $\Lambda(t) := \{\mathcal{Q}u(t) | u \in \overline{\Theta}_{R_0}, t \in [0,a]\}$ is relatively compact in $L^2(\Omega, \mathbb{H})$ for every $a \in (0, \infty)$.

Define a set

$$
\Lambda_\varepsilon(t) := \{\mathcal{Q}_\varepsilon u(t) | u \in \overline{\Theta}_{R_0}, t \in [0,a], \varepsilon \in (0,t)\},
$$

where

$$\mathcal{Q}_\varepsilon u(t) = T(t)\varphi(0) + \int_0^{t-\varepsilon} T(t-s)F(s, u[\varphi]_s)ds$$

$$+ \int_0^{t-\varepsilon} T(t-s)G(s, u[\varphi]_s)d\mathbb{W}(s)$$

$$= T(t)\varphi(0) + T(\varepsilon)\int_0^{t-\varepsilon} T(t-\varepsilon-s)F(s, u[\varphi]_s)ds$$

$$+ T(\varepsilon)\int_0^{t-\varepsilon} T(t-\varepsilon-s)G(s, u[\varphi]_s)d\mathbb{W}(s).$$

Since the semigroup $T(t)$ $(t \geqslant 0)$ is compact, the set $\Lambda_\varepsilon(t)$ is relatively compact in $L^2(\Omega, \mathbb{H})$ for every $\varepsilon \in (0,t)$. Futhermore, for every $u \in \overline{\Theta}_{R_0}$, $t \in [0,a]$, by the Hölder inequality, (2.4) and the condition (H2), one can obtain

$$\mathbb{E}\|\mathcal{Q}u(t) - \mathcal{Q}_\varepsilon u(t)\|^2$$

$$= \mathbb{E}\left\| \int_{t-\varepsilon}^t T(t-s)F(s, u[\varphi]_s)ds + \int_{t-\varepsilon}^t T(t-s)G(s, u[\varphi]_s)d\mathbb{W}(s) \right\|^2$$

$$\leqslant 2\mathbb{E}\left\| \int_{t-\varepsilon}^t T(t-s)F(s, u[\varphi]_s)ds \right\|^2 + 2\mathbb{E}\left\| \int_{t-\varepsilon}^t T(t-s)G(s, u[\varphi]_s)d\mathbb{W}(s) \right\|^2$$

$$\leqslant 2M^2 \left(\int_{t-\varepsilon}^t e^{v_0(t-s)}ds \right)^2 \left(a_1 \sup_{s\geqslant 0} \frac{\|u[\varphi]_s\|_{\mathcal{B}}^2}{h(s)} + a_0 \right)$$

$$+ 2M^2 \int_{t-\varepsilon}^t e^{2v_0(t-s)}ds \left(b_1 \sup_{s\geqslant 0} \frac{\|u[\varphi]_s\|_{\mathcal{B}}^2}{h(s)} + b_0 \right)$$

$$\leqslant 2M^2 (a_1(\|\varphi\|_{\mathcal{B}}^2 + \|u\|_h^2) + a_0) \left(\int_{t-\varepsilon}^t e^{v_0(t-s)}ds \right)^2$$

$$+ 2M^2 (b_1(\|\varphi\|_{\mathcal{B}}^2 + \|u\|_h^2) + b_0) \int_{t-\varepsilon}^t e^{2v_0(t-s)}ds$$

$$\leqslant 2M^2 (a_1 R_0 + a_0) \left(\int_{t-\varepsilon}^t e^{v_0(t-s)}ds \right)^2$$

$$+ 2M^2 (b_1 R_0 + b_0) \int_{t-\varepsilon}^t e^{2v_0(t-s)}ds$$

$$\to 0 \text{ as } \varepsilon \to 0.$$

Therefore, the set $\Lambda(t) \subset L^2(\Omega, \mathbb{H})$ is relatively compact for any $t \in [0,a]$.

Step 4. We show that $\mathcal{Q}(\overline{\Theta}_{R_0})$ is a family of locally equicontinuous functions in $C_{\varphi,h}$.

For each $u \in \overline{\Theta}_{R_0}$ and $0 \leqslant t_1 < t_2 \leqslant a$, from (3.3), it follows that

$$\mathbb{E}\|\mathcal{Q}u(t_2) - \mathcal{Q}u(t_1)\|^2$$

$$\leqslant 5\mathbb{E}\|T(t_2)\varphi(0) - T(t_1)\varphi(0)\|^2$$

$$+ 5\mathbb{E}\left\| \int_0^{t_1} (T(t_2-s) - T(t_1-s))F(s, u[\varphi]_s)ds \right\|^2$$

$$+5\mathbb{E}\left\|\int_{t_1}^{t_2} T(t_2-s)F(s,u[\varphi]_s)ds\right\|^2$$

$$+5\mathbb{E}\left\|\int_{0}^{t_1} (T(t_2-s)-T(t_1-s))G(s,u[\varphi]_s)d\mathbb{W}(s)\right\|^2$$

$$+5\mathbb{E}\left\|\int_{t_1}^{t_2} T(t_2-s)G(s,u[\varphi]_s)d\mathbb{W}(s)\right\|^2$$

$$:= J_1+J_2+J_3+J_4+J_5.$$

Next, we check that $J_i \to 0$ independently of $u \in \overline{\Theta}_{R_0}$ as $t_2-t_1 \to 0$, $i=1,2,3,4,5$. By the equicontinuity of semigroup $\|T(t)\|(t \geqslant 0)$, one can obtain

$$\begin{aligned} J_1 &= 5\mathbb{E}\|T(t_2)\varphi(0)-T(t_1)\varphi(0)\|^2 \\ &\leqslant 5\|T(t_2)-T(t_1)\|^2\|\varphi\|_{\mathcal{B}}^2 \\ &\leqslant 5\|T(t_2-t_1)-I\|^2\|T(t_1)\|^2\|\varphi\|_{\mathcal{B}}^2 \\ &\to 0 \text{ as } t_2-t_1 \to 0. \end{aligned}$$

From the condition (H2), there exist positive constants M_f, M_g such that for all $u \in \overline{\Theta}_{R_0}$

$$\sup_{t \geqslant 0}\mathbb{E}\|F(t,u[\varphi]_t)\|^2 \leqslant M_f, \quad \sup_{t \geqslant 0}\mathbb{E}\|G(t,u[\varphi]_t)\|^2 \leqslant M_g. \tag{3.11}$$

For $t_1=0$ and $t_2>0$, it is easy to see that $J_2=0$. For $t_1>0$ and taking $\varepsilon>0$ small enough which is independent of t_1 and t_2, by (3.11), and the Hölder inequality, one can get that

$$\begin{aligned} J_2 &= 5\mathbb{E}\left\|\int_{0}^{t_1}(T(t_2-s)-T(t_1-s))F(s,u[\varphi]_s)ds\right\|^2 \\ &\leqslant 5\int_{0}^{t_1}\|T(t_2-s)-T(t_1-s)\|ds \\ &\quad \times \int_{0}^{t_1}\|T(t_2-s)-T(t_1-s)\|\mathbb{E}\|F(s,u[\varphi]_s)\|^2 ds \\ &\leqslant \frac{10MM_f}{|\nu_0|}\left(\int_{0}^{t_1-\varepsilon}\|T(t_2-s)-T(t_1-s)\|ds\right. \\ &\quad \left.+\int_{t_1-\varepsilon}^{t_1}\|T(t_2-s)-T(t_1-s)\|ds\right) \\ &\leqslant \frac{10MM_f}{|\nu_0|}\left(\|T(t_2-t_1+\varepsilon)-T(\varepsilon)\|\int_{0}^{t_1-\varepsilon}\|T(t_1-s-\varepsilon)\|ds\right. \\ &\quad \left.+\int_{t_1-\varepsilon}^{t_1}(\|T(t_2-s)\|+\|T(t_1-s)\|)ds\right) \\ &\leqslant \frac{10MM_f}{|\nu_0|}\left(\frac{M}{|\nu_0|}\|T(t_2-t_1+\varepsilon)-T(\varepsilon)\|+2M\varepsilon\right) \\ &\to 0 \text{ as } t_2-t_1 \to 0, \ \varepsilon \to 0. \end{aligned}$$

By (3.11), and the Hölder inequality, one can obtain

$$
\begin{aligned}
J_3 &= 5\mathbb{E}\left\| \int_{t_1}^{t_2} T(t_2 - s)F(s, u[\varphi]_s)ds \right\|^2 \\
&\leqslant 5 \int_{t_1}^{t_2} \|T(t_2 - s)\|ds \int_{t_1}^{t_2} \|T(t_2 - s)\|\mathbb{E}\|F(s, u[\varphi]_s)\|^2 ds \\
&\leqslant 5M_f \left(\int_{t_1}^{t_2} \|T(t_2 - s)\|ds \right)^2 \\
&\leqslant 5M^2 M_f (t_2 - t_1)^2 \\
&\to 0 \ \text{ as } t_2 - t_1 \to 0.
\end{aligned}
$$

On the other hand, for $t_1 = 0$ and $t_2 > 0$, it is easy to see that $J_4 = 0$. For $t_1 > 0$ and $\varepsilon > 0$ small enough, by (2.4) and (3.11), one can see that

$$
\begin{aligned}
J_4 &= 5\mathbb{E}\left\| \int_0^{t_1} (T(t_2 - s) - T(t_1 - s))G(s, u[\varphi]_s)d\mathbb{W}(s) \right\|^2 \\
&\leqslant 5 \int_0^{t_1} \|T(t_2 - s) - T(t_1 - s)\|^2 \mathbb{E}\|G(s, u[\varphi]_s)\|^2 ds \\
&\leqslant 5M_g \left(\int_0^{t_1 - \varepsilon} \|T(t_2 - s) - T(t_1 - s)\|^2 ds \right. \\
&\quad \left. + \int_{t_1 - \varepsilon}^{t_1} \|T(t_2 - s) - T(t_1 - s)\|^2 ds \right) \\
&\leqslant 5M_g \left(\|T(t_2 - t_1 + \varepsilon) - T(\varepsilon)\|^2 \int_0^{t_1 - \varepsilon} \|T(t_1 - s - \varepsilon)\|^2 ds \right. \\
&\quad \left. + \int_{t_1 - \varepsilon}^{t_1} (\|T(t_2 - s)\|^2 + \|T(t_1 - s)\|^2)ds \right) \\
&\leqslant 5M^2 M_g \left(\frac{1}{2|v_0|} \|T(t_2 - t_1 + \varepsilon) - T(\varepsilon)\|^2 + 2\varepsilon \right) \\
&\to 0 \ \text{ as } t_2 - t_1 \to 0, \ \varepsilon \to 0.
\end{aligned}
$$

And from (2.4), (3.11), one can obtain

$$
\begin{aligned}
J_5 &= 5\mathbb{E}\left\| \int_{t_1}^{t_2} T(t_2 - s)G(s, u[\varphi]_s)d\mathbb{W}(s) \right\|^2 \\
&\leqslant 5 \int_{t_1}^{t_2} \|T(t_2 - s)\|^2 \mathbb{E}\|G(s, u[\varphi]_s)\|^2 ds \\
&\leqslant 5M^2 M_g (t_2 - t_1) \\
&\to 0 \ \text{ as } t_2 - t_1 \to 0.
\end{aligned}
$$

Thus, it can be deduced that $\mathbb{E}\|\mathcal{Q}u(t_2) - \mathcal{Q}u(t_1)\|^2 \to 0$ independently of $u \in \overline{\Theta}_{R_0}$ as $t_2 - t_1 \to 0$, which means that $\mathcal{Q} : \overline{\Theta}_{R_0} \to \overline{\Theta}_{R_0}$ is equicontinuous in $[0, a]$. Hence, the operator $\mathcal{Q} : \overline{\Theta}_{R_0} \to \overline{\Theta}_{R_0}$ is locally equicontinuous.

Now, according to Lemma 3.1, we can deduce that $\mathcal{Q}\overline{\Theta}_{R_0}$ is relatively compact in $C_{\varphi, h}$, which implies that $\mathcal{Q} : \overline{\Theta}_{R_0} \to \overline{\Theta}_{R_0}$ is completely continuous.

Step 5. We show that $\mathcal{Q}$ is $SAP_{\omega}(L^2(\Omega, \mathbb{H}))$-valued.

Obviously, $SAP_{\omega}(L^2(\Omega, \mathbb{H}))$ is a closed subspace of C_{sb} and $u \in SAP_{\omega}(L^2(\Omega, \mathbb{H}))$ implies that the function $t \mapsto u[\varphi]_t$ belongs to $SAP_{\omega}(\mathcal{B})$.

For a given $\varphi \in \mathcal{B}$, and $u \in SAP_{\omega}(L^2(\Omega, \mathbb{H}))$, we consider

$$
\begin{aligned}
&(\mathcal{Q}u)(t+\omega) - (\mathcal{Q}u)(t) \\
&= T(t+\omega)\varphi(0) + \int_0^{t+\omega} T(t+\omega-s)F(s, u[\varphi]_s)ds \\
&\quad + \int_0^{t+\omega} T(t+\omega-s)G(s, u[\varphi]_s)d\mathbb{W}(s) - T(t)\varphi(0) \\
&\quad - \int_0^t T(t-s)F(s, u[\varphi]_s)ds - \int_0^t T(t-s)G(s, u[\varphi]_s)d\mathbb{W}(s) \\
&= T(t+\omega)\varphi(0) - T(t)\varphi(0) + \int_0^{\omega} T(t+\omega-s)F(s, u[\varphi]_s)ds \\
&\quad + \int_0^t T(t-s)(F(s+\omega, u[\varphi]_{s+\omega}) - F(s, u[\varphi]_s))ds \\
&\quad + \int_0^{\omega} T(t+\omega-s)G(s, u[\varphi]_s)d\mathbb{W}(s) \\
&\quad + \int_0^t T(t-s)(G(s+\omega, u[\varphi]_{s+\omega}) - G(s, u[\varphi]_s))d\mathbb{W}(s) \\
&= T(t+\omega)\varphi(0) - T(t)\varphi(0) + \int_0^{\omega} T(t+\omega-s)F(s, u[\varphi]_s)ds \\
&\quad + \int_0^t T(t-s)(F(s+\omega, u[\varphi]_{s+\omega}) - F(s, u[\varphi]_{s+\omega}))ds \\
&\quad + \int_0^t T(t-s)(F(s, u[\varphi]_{s+\omega}) - F(s, u[\varphi]_s))ds \\
&\quad + \int_0^{\omega} T(t+\omega-s)G(s, u[\varphi]_s)d\mathbb{W}(s) \\
&\quad + \int_0^t T(t-s)(G(s+\omega, u[\varphi]_{s+\omega}) - G(s, u[\varphi]_{s+\omega}))d\mathbb{W}(s) \\
&\quad + \int_0^t T(t-s)(G(s, u[\varphi]_{s+\omega}) - G(s, u[\varphi]_s))d\mathbb{W}(s) \\
&:= I_1(t) + I_2(t) + I_3(t) + I_4(t) + I_5(t) + I_6(t) + I_7(t).
\end{aligned}
$$

Firstly, by the exponential stability of semigroup $T(t)$ $(t \geqslant 0)$,

$$
\begin{aligned}
\mathbb{E}\|I_1(t)\|^2 &\leqslant 2\mathbb{E}\|T(t+\omega)\varphi(0)\|^2 + 2\mathbb{E}\|T(t)\varphi(0)\|^2 \\
&\leqslant 2(M^2 e^{2v_0(t+\omega)} + M^2 e^{2v_0 t})\|\varphi\|_{\mathcal{B}}^2 \\
&\leqslant 4M^2 e^{2v_0 t}\|\varphi\|_{\mathcal{B}}^2 \\
&\to 0, \quad \text{as } t \to \infty.
\end{aligned}
$$

Since $u \in SAP_{\omega}(L^2(\Omega, \mathbb{H}))$, there is a constant $t_{\varepsilon,1} > 0$ such that $\mathbb{E}\|u[\varphi](t+\omega) -$

$u[\varphi](t)\|^2 \leqslant \varepsilon$ for any $t \geqslant t_{\varepsilon,1}$. Thus, $u[\varphi]_t \in SAP_\omega(\mathcal{B})$ for any $t \geqslant 0$ and $\mathbb{E}\|u[\varphi]_{t+\omega} - u[\varphi]_t\|_\mathcal{B}^2 \leqslant \varepsilon$ for any $t \geqslant t_{\varepsilon,1}$.

By the continuity of F and G, one can find that for $t \geqslant t_{\varepsilon,1}$

$$\mathbb{E}\|F(t,u[\varphi]_{t+\omega}) - F(t,u[\varphi]_t)\|^2 \leqslant \frac{|v_0|}{M}\varepsilon, \tag{3.12}$$

$$\mathbb{E}\|G(t,u[\varphi]_{t+\omega}) - G(t,u[\varphi]_t)\|^2 \leqslant \frac{2|v_0|}{M^2}\varepsilon. \tag{3.13}$$

From the condition (H1), it is easy to test that there exists a constant $t_{\varepsilon,2}$ large enough such that for $t \geqslant t_{\varepsilon,2}$,

$$\mathbb{E}\|F(t+\omega,u[\varphi]_{t+\omega}) - F(t,u[\varphi]_{t+\omega})\|^2 \leqslant \frac{|v_0|}{M}\varepsilon, \tag{3.14}$$

$$\mathbb{E}\|G(t+\omega,u[\varphi]_{t+\omega}) - G(t,u[\varphi]_{t+\omega})\|^2 \leqslant \frac{2|v_0|}{M^2}\varepsilon. \tag{3.15}$$

According to the Hölder inequality, exponential stability of semigroup $T(t)$ $(t \geqslant 0)$ and (3.11), one can see

$$\begin{aligned}
\mathbb{E}\|I_2(t)\|^2 &\leqslant \int_0^\omega \|T(t+\omega-s)\|ds \int_0^\omega \|T(t+\omega-s)\|\mathbb{E}\|F(s,u[\varphi]_s)\|^2 ds \\
&\leqslant M_f\left(\int_0^\omega \|T(t+\omega-s)\|ds\right)^2 \\
&\leqslant M^2 M_f\left(e^{v_0 t}\int_0^\omega e^{v_0(\omega-s)}ds\right)^2 \\
&\leqslant \frac{M^2 M_f e^{2v_0 t}}{|v_0|^2} \\
&\to 0, \quad \text{as } t \to \infty.
\end{aligned}$$

For $t > t_\varepsilon := \max\{t_{\varepsilon,1}, t_{\varepsilon,2}\}$, by the Hölder inequality, (3.11) and (3.14), one can see that

$$\begin{aligned}
\mathbb{E}\|I_3(t)\|^2 &\leqslant \int_0^t \|T(t-s)\|ds \int_0^t \|T(t-s)\|\mathbb{E}\|F(s+\omega,u[\varphi]_{s+\omega}) - F(s,u[\varphi]_{s+\omega})\|^2 ds \\
&\leqslant \frac{M}{|v_0|} \times \left(\int_0^{t_\varepsilon} \|T(t-s)\|\mathbb{E}\|F(s+\omega,u[\varphi]_{s+\omega}) - F(s,u[\varphi]_{s+\omega})\|^2 ds \right. \\
&\quad \left. + \int_{t_\varepsilon}^t \|T(t-s)\|\mathbb{E}\|F(s+\omega,u[\varphi]_{s+\omega}) - F(s,u[\varphi]_{s+\omega})\|^2 ds\right) \\
&\leqslant \frac{M}{|v_0|}\left(2M_f \int_0^{t_\varepsilon} \|T(t-s)\|ds + \frac{|v_0|}{M}\varepsilon \int_{t_\varepsilon}^t \|T(t-s)\|ds\right) \\
&\leqslant \frac{M}{|v_0|}\left(2MM_f \int_0^{t_\varepsilon} e^{v_0(t-s)}ds + \varepsilon\right) \\
&\leqslant \frac{M}{|v_0|}\left(2MM_f \frac{e^{v_0(t-t_\varepsilon)}}{|v_0|} + \varepsilon\right),
\end{aligned}$$

which means that $\mathbb{E}\|I_3(t)\|^2 \to 0$ as $t \to \infty$. Similarly, from (3.12), one can find that $\mathbb{E}\|I_4(t)\|^2 \to 0$ as $t \to \infty$. According to the exponential stability of semigroup $T(t)$ $(t \geqslant 0)$, (2.4) and (3.11), one can find

$$
\begin{aligned}
\mathbb{E}\|I_5(t)\|^2 &\leqslant \int_0^\omega \|T(t+\omega-s)\|^2 \mathbb{E}\|G(s,u[\varphi]_s)\|^2 ds \\
&\leqslant M_g \int_0^\omega M^2 e^{2v_0(t+\omega-s)} ds \\
&\leqslant \frac{M^2 M_g e^{2v_0 t}}{2|v_0|} \\
&\to 0, \text{ as } t \to \infty.
\end{aligned}
$$

For $t > t_\varepsilon := \max\{t_{\varepsilon,1}, t_{\varepsilon,2}\}$, by (2.4), (3.11) and (3.15), one can obtain that

$$
\begin{aligned}
\mathbb{E}\|I_6(t)\|^2 &\leqslant \int_0^t \|T(t-s)\|^2 \mathbb{E}\|G(s+\omega, u[\varphi]_{s+\omega}) - G(s, u[\varphi]_{s+\omega})\|^2 ds \\
&\leqslant \int_0^{t_\varepsilon} \|T(t-s)\|^2 \mathbb{E}\|G(s+\omega, u[\varphi]_{s+\omega}) - G(s, u[\varphi]_{s+\omega})\|^2 ds \\
&\quad + \int_{t_\varepsilon}^t \|T(t-s)\|^2 \mathbb{E}\|G(s+\omega, u[\varphi]_{s+\omega}) - G(s, u[\varphi]_{s+\omega})\|^2 ds \\
&\leqslant 2M^2 M_g \int_0^{t_\varepsilon} e^{2v_0(t-s)} ds + \frac{2|v_0|}{M^2} \varepsilon \int_{t_\varepsilon}^t M^2 e^{2v_0(t-s)} ds \\
&\leqslant \frac{M^2 M_g e^{2v_0(t-t_\varepsilon)}}{|v_0|} + \varepsilon,
\end{aligned}
$$

which implies that $\mathbb{E}\|I_6(t)\|^2 \to 0$ as $t \to \infty$. Similarly, from (3.13), we can get $\mathbb{E}\|I_7(t)\|^2$ tends to 0 as $t \to \infty$.

Thus, we can conclude that

$$
\lim_{t \to \infty} \mathbb{E}\|Qu(t+\omega) - Qu(t)\|^2 = 0,
$$

namely, $\mathcal{Q}(SAP_\omega(L^2(\Omega, \mathbb{H}))) \subset SAP_\omega(L^2(\Omega, \mathbb{H}))$.

From the above results, we have that

$$
\mathcal{Q} : \overline{\Theta_{R_0} \cap SAP_\omega(L^2(\Omega, \mathbb{H}))} \to \overline{\Theta_{R_0} \cap SAP_\omega(L^2(\Omega, \mathbb{H}))}
$$

is a completely continuous operator. According to the Schauder fixed point theorem, the operator $\mathcal{Q}$ has at least one fixed point $u \in \overline{\Theta_{R_0} \cap SAP_\omega(L^2(\Omega, \mathbb{H}))}$. Let $\{u^{(n)}\} \subset \overline{\Theta_{R_0} \cap SAP_\omega(L^2(\Omega, \mathbb{H}))}$ converge to u, then $\{\mathcal{Q}u^{(n)}\}$ converges to $\mathcal{Q}u = u$ uniformly in $[0, \infty)$. Therefore, we can easily deduce that $u[\varphi]$ is the square-mean S-asymptotically ω-periodic mild solution of the problem (1.1). $\square$

4. Uniqueness and global asymptotic behavior

Now, we prove the uniqueness and globally asymptotically stable property of the square-mean S-asymptotically ω-periodic mild solution of the equation (1.1).

THEOREM 4.1. *Assume that $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ is a closed linear operator and $-A$ generate an exponentially stable semigroup $T(t)$ $(t \geqslant 0)$ in Hilbert space $\mathbb{H}$, whose growth exponent denotes $v_0 < 0$. Let $F : \mathbb{R}^+ \times \mathcal{B} \to L^2(\Omega, \mathbb{H})$, $G : \mathbb{R}^+ \times \mathcal{B} \to \mathcal{L}(\mathbb{K}, \mathbb{H})$ be continuous functions and $\sup_{t \geqslant 0} \mathbb{E}\|F(t, \theta)\|^2 < \infty$, $\sup_{t \geqslant 0} \mathbb{E}\|G(t, \theta)\|^2 < \infty$. If the conditions (H1) and (H3) for all $t \geqslant 0$ and $\phi, \psi \in \mathcal{B}$, there exist positive constants a_1, b_1 such that*

$$\mathbb{E}\|F(t, \phi) - F(t, \psi)\|^2 \leqslant a_1 \|\phi - \psi\|_{\mathcal{B}}^2, \ \mathbb{E}\|G(t, \phi) - G(t, \psi)\|^2 \leqslant b_1 \|\phi - \psi\|_{\mathcal{B}}^2,$$

hold, then for a given $\varphi \in \mathcal{B}$, the problem (1.1) has a unique square-mean S-asymptotically ω-periodic mild solution provided that

$$\frac{M^2 a_1}{|v_0|^2} + \frac{M^2 b_1}{|v_0|} < \frac{1}{2}. \tag{4.1}$$

Moreover, if the constants a_1, b_1 in the condition (H3) satisfy

$$\frac{M^2 a_1 e^{|v_0|r}}{|v_0|^2} + \frac{M^2 b_1 e^{|v_0|r}}{|v_0|} < \frac{1}{3}, \tag{4.2}$$

then the unique S-asymptotically ω-periodic mild solution is globally exponentially stable in square-mean sense.

Proof. Given $\varphi \in \mathcal{B}$, and $u \in C_{sb}$, we define the function $u[\varphi] : [-r, \infty) \to L^2(\Omega, \mathbb{H})$ as follows:

$$u[\varphi](t) = \begin{cases} u(t), & \text{for } t \geqslant 0 \\ \varphi(t), & \text{for } t \in [-r, 0]. \end{cases}$$

We denote

$$C_\varphi = \{u \in C_{sb} : u(0) = \varphi(0)\}.$$

Then C_φ is a closed subspace of C_{sb}.

For $u \in C_\varphi$ and $t \geqslant 0$, let $\mathcal{Q}u(t)$ defined by (3.3). By the condition (H3), one can find

$$\mathbb{E}\|\mathcal{Q}u(t)\|^2 \leqslant 3\mathbb{E}\|T(t)\varphi(0)\|^2 + 3\mathbb{E}\left\| \int_0^t T(t-s)F(s, u[\varphi]_s)ds \right\|^2$$

$$+ 3\mathbb{E}\left\| \int_0^t T(t-s)G(s, u[\varphi]_s)d\mathbb{W}(s) \right\|^2$$

$$\leqslant 3M^2 \|\varphi\|_{\mathcal{B}}^2 + \frac{3M^2}{|v_0|^2}(2a_1(\|\varphi\|_{\mathcal{B}}^2 + \|u\|_C^2) + 2\mathbb{E}\|F(t, \theta)\|^2)$$

$$+ \frac{3M^2}{|v_0|}(2b_1(\|\varphi\|_{\mathcal{B}}^2 + \|u\|_C^2) + 2\mathbb{E}\|G(t, \theta)\|^2),$$

which implies that $\mathcal{Q}: C_\varphi \to C_\varphi$ is well defined. According to the proof of Theorem 3.2, one can see that $\mathcal{Q}(SAP_\omega(L^2(\Omega, \mathbb{H}))) \subset SAP_\omega(L^2(\Omega, \mathbb{H}))$. Thus, the fixed point u of the operator $\mathcal{Q}$ in $SAP_\omega(L^2(\Omega, \mathbb{H}))$ implies that $u[\varphi]$ is the square-mean S-asymptotically ω-periodic mild solution of the problem (1.1).

For all $t \geqslant 0$, and $u, v \in SAP_\omega(L^2(\Omega, \mathbb{H}))$, by the condition (H3), the Hölder inequality, (2.4), and (3.3), we get that

$$
\begin{aligned}
&\mathbb{E}\|\mathcal{Q}u(t) - \mathcal{Q}v(t)\|^2 \\
&= \mathbb{E}\left\| \int_0^t T(t-s)(F(s, u[\varphi]_s) - F(s, v[\varphi]_s))ds \right. \\
&\quad \left. + \int_0^t T(t-s)(G(s, u[\varphi]_s) - G(s, v[\varphi]_s))d\mathbb{W}(s) \right\|^2 \\
&\leqslant 2\mathbb{E}\left\| \int_0^t T(t-s)(F(s, u[\varphi]_s) - F(s, v[\varphi]_s))ds \right\|^2 \\
&\quad + 2\mathbb{E}\left\| \int_0^t T(t-s)(G(s, u[\varphi]_s) - G(s, v[\varphi]_s))d\mathbb{W}(s) \right\|^2 \\
&\leqslant 2 \int_0^t \|T(t-s)\|ds \int_0^t \|T(t-s)\|\mathbb{E}\|F(s, u[\varphi]_s) - F(s, v[\varphi]_s)\|^2 ds \\
&\quad + 2 \int_0^t \|T(t-s)\|^2 \mathbb{E}\|G(s, u[\varphi]_s) - G(s, v[\varphi]_s)\|^2 ds \\
&\leqslant 2 \int_0^t \|T(t-s)\|ds \int_0^t \|T(t-s)\|(a_1\|u[\varphi]_s - v[\varphi]_s\|_{\mathcal{B}}^2)ds \\
&\quad + 2 \int_0^t \|T(t-s)\|^2 (b_1\|u[\varphi]_s - v[\varphi]_s\|_{\mathcal{B}}^2)ds \\
&\leqslant \frac{2M^2 a_1}{|v_0|^2}\|u-v\|_C^2 + \frac{2M^2 b_1}{|v_0|}\|u-v\|_C^2 \\
&\leqslant 2\left(\frac{M^2 a_1}{|v_0|^2} + \frac{M^2 b_1}{|v_0|}\right)\|u-v\|_C^2,
\end{aligned}
$$

which implies that

$$
\|\mathcal{Q}u - \mathcal{Q}v\|_C^2 \leqslant 2\left(\frac{M^2 a_1}{|v_0|^2} + \frac{M^2 b_1}{|v_0|}\right)\|u-v\|_C^2, \tag{4.3}
$$

thus, by (4.1), we can conclude that $\mathcal{Q}$ is a contraction mapping. Hence, the problem (1.1) has a unique square-mean S-asymptotic ω-periodic mild solution $u^*[\varphi]$.

Next, we prove the global exponentially stability of the unique square-mean S-asymptotically ω-periodic mild solution. Assume that $u^*[\varphi]$ is the unique square-mean S-asymptotically ω-periodic mild solution of the problem (1.1). Moreover, for any $\phi \in \mathcal{B}$, it is easy to test that the problem (1.1) has a unique mild solution $v[\phi]$ with $v \in C_{sb}$.

For every $t \geqslant 0$, from the definition of $\mathcal{Q}$, one can find

$$\mathbb{E}\|u^*[\varphi](t) - v[\phi](t)\|^2 = \mathbb{E}\|u^*(t) - v(t)\|^2$$
$$\leqslant 3\mathbb{E}\|T(t)(\varphi(0) - \phi(0))\|^2$$
$$+ 3\mathbb{E}\left\|\int_0^t T(t-s)(F(s, u^*[\varphi]_s) - F(s, v[\phi]_s))ds\right\|^2$$
$$+ 3\mathbb{E}\left\|\int_0^t T(t-s)(G(s, u^*[\varphi]_s) - G(s, v[\phi]_s))d\mathbb{W}(s)\right\|^2$$
$$\leqslant 3M^2 e^{v_0 t}\mathbb{E}\|\varphi(0) - \phi(0)\|^2 + 3\int_0^t \|T(t-s)\|ds$$
$$\times \int_0^t \|T(t-s)\|\mathbb{E}\|F(s, u^*[\varphi]_s) - F(s, v[\phi]_s)\|^2 ds$$
$$+ 3\int_0^t \|T(t-s)\|^2 \mathbb{E}\|G(s, u^*[\varphi]_s) - G(s, v[\phi]_s)\|^2 ds$$
$$\leqslant 3M^2 e^{v_0 t}\mathbb{E}\|\varphi(0) - \phi(0)\|^2 + \frac{3M}{|v_0|}\int_0^t \|T(t-s)\|(a_1\|u^*[\varphi]_s - v[\phi]_s\|_{\mathcal{B}}^2)ds$$
$$+ 3\int_0^t \|T(t-s)\|^2 (b_1\|u^*[\varphi]_s - v[\phi]_s\|_{\mathcal{B}}^2)ds$$
$$\leqslant 3M^2 e^{v_0 t}\mathbb{E}\|\varphi(0) - \phi(0)\|^2$$
$$+ \frac{3M^2}{|v_0|}\int_0^t e^{v_0(t-s)}\left(a_1 \sup_{\tau \in [-r,0]} \mathbb{E}\|u^*[\varphi](s+\tau) - v[\phi](s+\tau)\|^2\right)ds$$
$$+ 3M^2 \int_0^t e^{2v_0(t-s)}\left(b_1 \sup_{\tau \in [-r,0]} \mathbb{E}\|u^*[\varphi](s+\tau) - v[\phi](s+\tau)\|^2\right)ds$$
$$\leqslant 3M^2 e^{v_0 t}\mathbb{E}\|\varphi(0) - \phi(0)\|^2 + \left(\frac{3M^2 a_1}{|v_0|} + 3M^2 b_1\right)$$
$$\times \int_0^t e^{v_0(t-s)}\left(\sup_{\tau \in [-r,0]} \mathbb{E}\|u^*[\varphi](s+\tau) - v[\phi](s+\tau)\|^2\right)ds.$$

For any $t \geqslant 0$, let $\Psi(t) = e^{|v_0|t}\mathbb{E}\|u^*[\varphi](t) - v[\phi](t)\|^2$, one can see

$$\Psi(t) \leqslant \Lambda_1 \Psi(0) + \int_0^t \Lambda_2 \sup_{\tau \in [-r,0]} \Psi(s+\tau)ds, \tag{4.4}$$

where $\Lambda_1 = 3M^2$, $\Lambda_2 = \left(\frac{3M^2 a_1}{|v_0|} + 3M^2 b_1\right)e^{|v_0|r}$.

From the Gronwall integral inequality ([24, 27]), it follows that for all $t \geqslant 0$,

$$e^{|v_0|t} \cdot \mathbb{E}\|u^*[\varphi](t) - v[\phi](t)\|^2 = \Psi(t) \leqslant \Lambda_1\|\varphi - \phi\|_{\mathcal{B}}^2 \cdot e^{\Lambda_2 t}. \tag{4.5}$$

By (4.2), one can find that

$$
\begin{aligned}
\alpha &:= |v_0| - \Lambda_2 \\
&= |v_0| - \left(\frac{3M^2 a_1}{|v_0|} + 3M^2 b_1\right)e^{|v_0|r} \\
&= \frac{|v_0|^2 - 3M^2 a_1 e^{|v_0|r} - 3M^2 |v_0| b_1 e^{|v_0|r}}{|v_0|} \\
&= \frac{|v_0|^2 \left(1 - \frac{3M^2 a_1 e^{|v_0|r}}{|v_0|^2} - \frac{3M^2 b_1 e^{|v_0|r}}{|v_0|}\right)}{|v_0|} > 0,
\end{aligned}
$$

from (4.5), one can get

$$
\mathbb{E}\|u[\varphi](t) - v[\phi](t)\|^2 \leqslant \Lambda_1 \|\varphi - \phi\|_{\mathcal{B}}^2 \cdot e^{-\alpha t}.
$$

Therefore, the unique square-mean S-asymptotic ω-periodic mild solution of the problem (1.1) is globally exponentially stable. $\square$

5. Example

Consider the following delayed stochastic partial differential equation

$$
\begin{cases}
\dfrac{\partial}{\partial t}u(t,x) - \dfrac{\partial^2}{\partial x^2}u(t,x) = \dfrac{\sin 2\pi t}{4e^{\frac{t}{2}}}u(t+\tau,x) \\
\qquad\qquad\qquad\qquad + \dfrac{\cos 2\pi t}{8e^{\frac{t}{2}}}u(t+\tau,x)\mathbb{W}'(t), \ t \in \mathbb{R}^+, \ x \in [0,\pi], \quad (5.1) \\
u(t,0) = u(t,\pi) = 0, \quad t \in \mathbb{R}^+, \\
u(\tau,x) = \varphi(\tau,x), \quad \tau \in [-r,0], \ x \in [0,\pi],
\end{cases}
$$

where $\mathbb{W}(t)$ is a standard Wiener process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

In order to write the problem (5.1) into the abstract form (1.1), we assume that $\mathbb{H} = L^2[0,\pi]$ is a Hilbert space with the L^2-norm $\|\cdot\|_2$. Denote the operator $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ by

$$
D(A) := \{u \in \mathbb{H}| \ u', u'' \in \mathbb{H}, u(0) = u(\pi) = 0\}, \quad Au = -\frac{\partial^2 u}{\partial x^2}. \qquad (5.2)
$$

From [31], it can be known that $-A$ generates an exponentially stable compact analytic semigroup $T(t)$ $(t \geqslant 0)$ in $\mathbb{H}$ and $\|T(t)\| \leqslant e^{-t}$ for any $t \geqslant 0$, then $M = 1$, $v_0 = -1$.

For every $t \in \mathbb{R}^+$, $x \in [0,\pi]$, denote $u(t)(x) = u(t,x)$, $u_t(\tau)(x) = u(t+\tau,x)$ for every $t \in \mathbb{R}^+, x \in [0,\pi]$, $\tau \in [-r,0]$, and

$$
F(t,u_t)(x) = \frac{\sin 2\pi t}{4e^{\frac{t}{2}}}u(t+\tau,x), \qquad (5.3)
$$

$$
G(t,u_t)(x) = \frac{\cos 2\pi t}{8e^{\frac{t}{2}}}u(t+\tau,x). \qquad (5.4)
$$

For any $\omega > 0$, we have

$$\lim_{t \to \infty} \left| \frac{\sin 2\pi(t+\omega)}{4e^{\frac{(t+\omega)}{2}}} - \frac{\sin 2\pi t}{4e^{\frac{t}{2}}} \right| = 0, \quad \lim_{t \to \infty} \left| \frac{\cos 2\pi(t+\omega)}{8e^{\frac{(t+\omega)}{2}}} - \frac{\cos 2\pi t}{8e^{\frac{t}{2}}} \right| = 0.$$

Thus, F, G are continuous and satisfy the condition (H1) of Theorem 3.2. Let $h(t) = e^t$, then

$$\mathbb{E}\|F(t, e^{t/2}\eta)\|^2 \leqslant \frac{\sin^2 2\pi t}{16e^t} \mathbb{E}\|e^{t/2}\eta\|^2 \leqslant \frac{1}{16}\|\eta\|_{\mathcal{B}}^2, \tag{5.5}$$

$$\mathbb{E}\|G(t, e^{t/2}\eta)\|^2 \leqslant \frac{\cos^2 2\pi t}{64e^t} \mathbb{E}\|e^{t/2}\eta\|^2 \leqslant \frac{1}{64}\|\eta\|_{\mathcal{B}}^2, \tag{5.6}$$

for all $t \geqslant 0$, $\eta \in \mathcal{B}$. From (5.5) and (5.6), it can be seen that the condition (H2) is satisfied. By $M = 1, v_0 = -1$, one can see

$$\frac{3M^2 a_1}{|v_0|^2} + \frac{3M^2 b_1}{2|v_0|} = \frac{3}{16} + \frac{3}{128} = \frac{27}{128} < 1. \tag{5.7}$$

Hence, from Theorem 3.2, the equation (5.1) has at least one square-mean S-asymptotically 1-periodic mild solution.

From the definition of F, G, one can get

$$\mathbb{E}\|F(t, \eta_1) - F(t, \eta_2)\|^2 \leqslant \frac{1}{16}\|\eta_1 - \eta_2\|_{\mathcal{B}}^2, \tag{5.8}$$

$$\mathbb{E}\|G(t, \eta_1) - G(t, \eta_2)\|^2 \leqslant \frac{1}{64}\|\eta_1 - \eta_2\|_{\mathcal{B}}^2, \tag{5.9}$$

for every $t \geqslant 0$, $\eta_1, \eta_2 \in \mathcal{B}$, this means that the condition (H3) is satisfied. Since $M = 1$, $v_0 = -1$, one has

$$\frac{M^2 a_1}{|v_0|^2} + \frac{M^2 b_1}{|v_0|} = \frac{1}{16} + \frac{1}{64} = \frac{5}{64} < \frac{1}{2}, \tag{5.10}$$

which implies that the (4.1) holds. Thus, according to Theorem 4.1, the equation (5.1) has a unique square-mean S-asymptotically 1-periodic mild solution. For $0 < r < \ln 3$, we can find that

$$\frac{M^2 a_1 e^{|v_0|r}}{|v_0|^2} + \frac{M^2 b_1 e^{|v_0|r}}{|v_0|} = \frac{1}{16} \times 3 + \frac{1}{64} \times 3 = \frac{15}{64} < \frac{1}{3}, \tag{5.11}$$

it means that the (4.2) is holds. Therefore, from Theorem 4.1, it can be seen that the S-asymptotically 1-periodic mild solution of equation (5.1) is globally exponentially stable.

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