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ESTIMATING THE REMAINDER OF AN ALTERNATING p–SERIES USING HYPERGEOMETRIC FUNCTIONS

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Abstract. In this paper, using hypergeometric functions, we provide sharp estimates of the remainder of the alternating p-series, $\sum_{n\geqslant 1}\frac{(-1)^{n-1}}{n^p}$, where $p\geqslant 2$ is an integer. We show that the largest ρ and the largest σ such that the inequalities

$$\frac{1}{2(n+1)^p-\rho}\leqslant\left|\sum_{k=n+1}^{\infty}\frac{(-1)^{k-1}}{k^p}\right|\leqslant\frac{1}{2n^p+\sigma},$$

hold for any integer $n\geqslant 1$ are

$$\rho(p)=2^{p+1}-\frac{1}{1-(1-2^{1-p})\zeta(p)}\text{ and }\sigma(p)=\frac{1}{1-(1-2^{1-p})\zeta(p)}-2,$$

where $\zeta(p)=\sum_{k=1}^{\infty}\frac{1}{k^p}$, the Riemann zeta function.

1. Introduction

Let $f:[1,\infty)\longrightarrow(0,\infty)$ be a function, satisfying the following properties:

$$f(n+1)<f(n),\ \text{for all } n\in\mathbb{N}, \tag{1.1a}$$

$$\lim_{n\to\infty}f(n)=0, \tag{1.1b}$$

$$\Delta f(n)<\Delta f(n+1),\ \text{for all } n\in\mathbb{N}, \tag{1.1c}$$

where

$$\Delta f(n):=f(n+1)-f(n). \tag{1.2}$$

Throughout this paper, we denote by

$$g(n):=\frac{1}{f(n)}. \tag{1.3}$$

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Consider the Leibniz series $\sum_{n=1}^{\infty}(-1)^{n-1}f(n)$, we denote by

$$R_n := \sum_{k=n+1}^{\infty}(-1)^{k-1}f(k), \tag{1.4}$$

its remainder of order n. We have

$$|R_n| + |R_{n+1}| = f(n+1), \text{ for } n \geqslant 1, \tag{1.5}$$

and according to [2, 3] $(|R_n|)_n$ is decreasing. Therefore the following inequalities hold:

$$\frac{f(n+1)}{2} < |R_n| < \frac{f(n)}{2}. \tag{1.6}$$

The above inequalities can be rewritten as follows:

$$\frac{1}{2g(n+1)} < |R_n| < \frac{1}{2g(n)}. \tag{1.7}$$

For more information about estimates of the remainder of some alternating series, see for instance [5, 7, 8].

It is natural to ask the following question: which are the best constants ρ and σ (the largest ρ and the largest σ) such that the inequalities

$$\frac{1}{2g(n+1)-\rho} < |R_n| < \frac{1}{2g(n)+\sigma} \tag{1.8}$$

hold, for every $n \geqslant 1$?

Similar questions have been stated (cf. [6]) for the alternating harmonic series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ and the Gregory-Leibniz series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}$.

The aim of this paper is to give a positive answer for the previous question for $g(n) = n^p$, where $p \geqslant 2$ is an integer. Indeed the best constants are

$$\rho(p) = 2^{p+1} - \frac{1}{1-(1-2^{1-p})\zeta(p)} \text{ and } \sigma(p) = \frac{1}{1-(1-2^{1-p})\zeta(p)} - 2,$$

where $\zeta(p) = \sum_{k=1}^{\infty}\frac{1}{k^p}$ is the Riemann zeta function.

In order to achieve this goal, we introduce the following sequences (x_n) and (y_n) defined by:

$$|R_n| = \frac{1}{2g(n)+x_n} = \frac{1}{2g(n+1)-y_n}. \tag{1.9}$$

In [9], the author has introduced the sequence (θ_n) by the implicit relation

$$|R_n| = \frac{1}{2g(n+\theta_n)}, \tag{1.10}$$

and proved that

$$0 < \theta_n < 1. \tag{1.11}$$

Immediately, we can derive the equalities

$$x_n = 2\left(g(n+\theta_n) - g(n)\right), \tag{1.12a}$$

$$y_n = 2\left(g(n+1) - g(n+\theta_n)\right). \tag{1.12b}$$

In section 2, we give some preliminary results regarding the monotonicity of the sequences (x_n) and (y_n) in the general setting. In section 3, we focus on the particular case of p-series.

2. Preliminary results

Thanks to Equations (1.11), (1.12a) and (1.12b), we have the following lemma.

LEMMA 2.1. *The sequences (x_n) and (y_n) satisfy the following properties.*

(i) $x_n + y_n = 2\Delta g(n)$ *for all* $n \geqslant 1$.

(ii) $0 < x_n, y_n < 2\Delta g(n)$, *for all* $n \geqslant 1$.

(iii) The best constants in (1.8) are

$$\rho = \inf_{n\geqslant 1}(y_n),$$

$$\sigma = \inf_{n\geqslant 1}(x_n).$$

In order to find σ, we discuss the monotonicity of the sequence (x_n). First, we introduce the sequence:

$$t_n := \sqrt{(\Delta g(n))^2 + g(n+1)^2} - g(n), \tag{2.1}$$

for $n \geqslant 1$.

PROPOSITION 2.2. *Let n be a positive integer. Then the following statements are equivalent.*

(i) $x_{n+1} > x_n$;

(ii) $x_n < t_n$;

(iii) $x_{n+1} > t_n$.

Proof. The equality (1.5) means

$$\frac{1}{2g(n)+x_n}+\frac{1}{2g(n+1)+x_{n+1}}=\frac{1}{g(n+1)}. \tag{2.2}$$

Hence $x_{n+1}>x_n$ is equivalent to each of the following inequalities.

$$\frac{1}{2g(n)+x_n}+\frac{1}{2g(n+1)+x_n}>\frac{1}{g(n+1)}, \tag{2.3}$$

$$\frac{1}{2g(n)+x_{n+1}}+\frac{1}{2g(n+1)+x_{n+1}}<\frac{1}{g(n+1)}. \tag{2.4}$$

Direct computations show that Inequalities (2.3) and (2.4) are equivalent to

$$(x_n+g(n))^2<2g^2(n+1)-2g(n+1)g(n)+g^2(n)=(t_n+g(n))^2$$

and

$$(x_{n+1}+g(n))^2>2g^2(n+1)-2g(n+1)g(n)+g^2(n)=(t_n+g(n))^2,$$

respectively. This completes the proof, as x_n, x_{n+1}, t_n and $g(n)$ are positive real numbers. □

We denote by

$$\delta_n := 2\Delta g(n)-t_n, \tag{2.5a}$$
$$\beta_n := 2\Delta g(n+1)-t_n. \tag{2.5b}$$

Then, combining Lemma 2.1 and Proposition 2.2, we get the following corollary.

COROLLARY 2.3. *For any positive integer n, the following statements are equivalent.*

(i) $x_{n+1}>x_n$;

(ii) $y_n>\delta_n$;

(iii) $y_{n+1}<\beta_n$.

In order to discuss the monotonicity of the sequence (y_n), we introduce the sequence:

$$\lambda_n := g(n+2)-\sqrt{(\Delta g(n+1))^2+g(n+1)^2}, \tag{2.6}$$

for $n \geqslant 1$.

PROPOSITION 2.4. *Let n be a positive integer. Then the following statements are equivalent:*

(i) $y_{n+1}>y_n$;

(ii) $y_n<\lambda_n$;

(iii) $y_{n+1} > \lambda_n$.

Proof. Here, considering (1.5) and (1.9), we have

$$\frac{1}{2g(n+1)-y_{n+1}}+\frac{1}{2g(n+2)-y_{n+1}}=\frac{1}{g(n+1)}.$$

So $y_n < y_{n+1}$ is equivalent to each of the following inequalities:

$$\frac{1}{2g(n+1)-y_n}+\frac{1}{2g(n+2)-y_n}>\frac{1}{g(n+1)} \tag{2.7}$$

and

$$\frac{1}{2g(n+1)-y_{n+1}}+\frac{1}{2g(n+2)-y_{n+1}}<\frac{1}{g(n+1)}. \tag{2.8}$$

Again, direct computations show that Inequalities 2.7 and 2.8 are equivalent to

$$(g(n+2)-y_n)^2 > 2g^2(n+1)-2g(n+1)g(n+2)+g^2(n+2)=(\lambda_n+g(n+2))^2$$

and

$$(g(n+2)-y_{n+1})^2 < 2g^2(n+1)-2g(n+1)g(n+2)+g^2(n+2)=(\lambda_n+g(n+2))^2,$$

respectively. As, in addition, $g(n+2)-y_n>0$ and $g(n+2)-y_{n+1}>0$, these inequalities are equivalent to $y_n<\lambda_n$ and $\lambda_n<y_{n+1}$, respectively, completing the proof. □

If we denote by

$$\mu_n := 2\Delta g(n+1)-\lambda_n \tag{2.9a}$$

$$\alpha_n := 2\Delta g(n)-\lambda_n, \tag{2.9b}$$

then combining Lemma 2.1 and Proposition 2.4, we obtain the following corollary.

COROLLARY 2.5. *For any positive integer n, the following conditions are equivalent:*

(i) $y_{n+1} < y_n$;

(ii) $x_n < \alpha_n$;

(iii) $x_{n+1} > \mu_n$.

REMARK 2.6. The equivalences between the reversed inequalities in Proposition 2.2, Corollary 2.3, Proposition 2.4 and Corollary 2.5, remain true.

Now, we will discuss how the monotonicity of the sequence (x_n) influences that of (y_n) and vice versa.

PROPOSITION 2.7.

(i) If (x_n) is increasing, then so is (y_n).

(ii) If (y_n) is decreasing, then so is (x_n).

For the proof, we need a straightforward lemma.

LEMMA 2.8. *For each $n \geqslant 2$, the following inequalities hold.*

(i) $\lambda_{n-1} < \delta_n$.

(ii) $t_n < \mu_{n-1}$.

Proof of Proposition 2.7. Assume that (x_n) is increasing. Then, according to Corollary 2.3, $y_n > \delta_n$ for every positive integer n. Hence, by Lemma 2.8, $y_n > \lambda_{n-1}$ for every integer $n \geqslant 2$. Thus, using Proposition 2.4, we conclude that (y_n) is increasing.

Now, suppose that (y_n) is decreasing. Then by Lemma 2.4, $y_n < \lambda_{n-1}$ for all $n \geqslant 2$. So, again, by Lemma 2.8, we obtain $y_n < \delta_n$. Consequently, combining Corollary 2.3 and Remark 2.6, we get (x_n) is decreasing. □

REMARK 2.9. The converse of each statement in Proposition 2.7 does not hold. It suffices to consider $g(n) = n$; then $x_n = 2\theta_n$ and $y_n = 2(1-\theta_n)$. As (θ_n) is decreasing (see [10]), the sequence (x_n) is decreasing and (y_n) is increasing.

In the next result, we use the convexity of the function g and the monotonicity of the sequence (θ_n) to derive the monotonicity of (x_n) or (y_n).

PROPOSITION 2.10. *The following properties hold.*

(i) If g is strictly concave and (θ_n) is decreasing, then (x_n) is decreasing.

(ii) If g is strictly convex and (θ_n) is decreasing, then (y_n) is increasing.

Proof.

(i) As $x_n = 2(g(n+\theta_n) - g(n))$, it suffices to show that the sequence $\Big(g(n+\theta_n) - g(n)\Big)$ is decreasing.

First, let us recall the Chordal Slope Lemma for a strictly convex function α: if $x < y < z$, then

$$\frac{\alpha(y)-\alpha(x)}{y-x} < \frac{\alpha(z)-\alpha(x)}{z-x} < \frac{\alpha(z)-\alpha(y)}{z-y}.$$

Now, as $n < n+\theta_n < n+1 < n+1+\theta_{n+1}$ and $-g$ is convex, we get

$$\frac{g(n+\theta_n)-g(n)}{\theta_n} > \frac{g(n+1+\theta_{n+1})-g(n+1)}{\theta_{n+1}}.$$

As a result, we obtain

$$\frac{g(n+\theta_n)-g(n)}{g(n+1+\theta_{n+1})-g(n+1)} > \frac{\theta_n}{\theta_{n+1}} > 1,$$

showing that $\big(g(n+\theta_n)-g(n)\big)$ is decreasing. Therefore (x_n) is decreasing.

(ii) As $y_n = 2\big(g(n+1)-g(n+\theta_n)\big)$, it is sufficient to show that the sequence $\big(g(n+1)-g(n+\theta_n)\big)$ is increasing. Applying the Chordal Slope Lemma to the inequalities: $n+\theta_n < n+1 < n+1+\theta_{n+1} < n+2$, we obtain

$$\frac{g(n+1)-g(n+\theta_n)}{1-\theta_n} < \frac{g(n+2)-g(n+1+\theta_{n+1})}{1-\theta_{n+1}}.$$

Thus $g(n+1)-g(n+\theta_n) < \big(g(n+2)-g(n+1+\theta_{n+1})\big)\dfrac{1-\theta_n}{1-\theta_{n+1}}$. As the sequence (θ_n) is decreasing, we get $g(n+1)-g(n+\theta_n) < g(n+2)-g(n+1+\theta_{n+1})$, as desired. □

For the p-series, using the fact that (θ_n) is decreasing (see [10]) and that $g(x)=x^p$ is strictly convex for $p>1$ and strictly concave for $p<1$, we have the following corollary.

COROLLARY 2.11. *Let p be a positive real number and $g(n)=n^p$.*

(i) If $p>1$, then (x_n) is decreasing.

(ii) If $p<1$, then (y_n) is increasing.

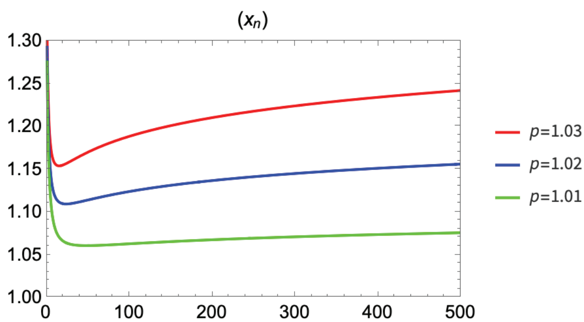

Figure 1: *According to Corollary 2.11, (y_n) is decreasing for $p>1$. However, there is no similar conclusion for the sequence (x_n). This figure illustrates some particular values of p. For $p=1.01$, 1.02 or 1.03, plotting the exact values of the sequence (x_n) shows that it is neither increasing nor decreasing.*

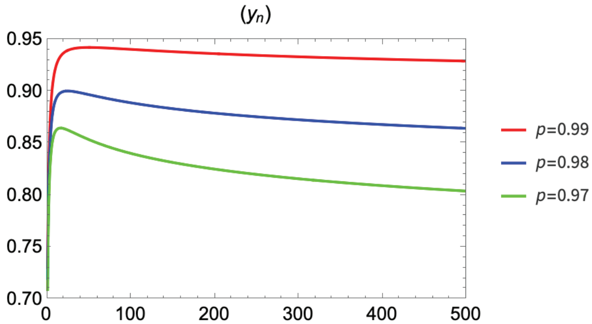

Figure 2: *According to Corollary 2.11, (x_n) is decreasing for $p<1$. However, there is no similar conclusion for the sequence (y_n). This figure illustrates some particular values of p. For $p=0.97,\ 0.98$ or 0.99, plotting the exact values of the sequence (y_n) shows that it is neither increasing nor decreasing.*

3. Alternating p-series

In this section, we focus on the alternating p-series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^p}$, for any integer $p\geqslant 2$. The main result of this paper is the following.

THEOREM 3.1. *The best constants ρ and σ such that the inequalities*

$$\frac{1}{2(n+1)^p-\rho}\leqslant\left|\sum_{k=n+1}^{\infty}\frac{(-1)^{k-1}}{k^p}\right|\leqslant\frac{1}{2n^p+\sigma},$$

hold for any integer $n\geqslant 1$ are

$$\rho(p)=2^{p+1}-\frac{1}{1-(1-2^{1-p})\zeta(p)}$$

and

$$\sigma(p)=\frac{1}{1-(1-2^{1-p})\zeta(p)}-2,$$

where $\zeta(p)=\sum_{k=1}^{\infty}\frac{1}{k^p}$ is the zeta Riemann function.

We break the proof of this theorem into a sequence of lemmas.

First, let us recall the hypergeometric function ${}_q F_p$ defined as

$${}_qF_p\left((a_k)_{k=1}^{q};(b_k)_{k=1}^{p};x\right)=\sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_q)_n}{(b_1)_n\cdots(b_p)_n}\frac{x^n}{n!},\qquad(3.1)$$

where $(a)_n$ is the Pochhammer's symbol defined by $(a)_n:=a(a+1)\dots(a+n-1)$, for any $n\geqslant 1$ and $(a)_0=1$, see [1].

LEMMA 3.2. *Let $p \geqslant 1$ be an integer and $R_n = \sum_{k=n+1}^{\infty} \frac{(-1)^{k-1}}{k^p}$. Then, we have*

$$|R_n| = \frac{1}{(n+1)^p} \, {}_{\mathrm{p+1}}F_{\mathrm{p}} \Big(1, n+1, \ldots, n+1; n+2, \ldots, n+2; -1\Big). \tag{3.2}$$

Proof. The series $|R_n|$ is given by

$$\begin{aligned}
|R_n| &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(n+k)^p} \\
&= \frac{1}{(n+1)^p} \sum_{k=1}^{\infty} \frac{(n+1)^p}{(n+k)^p} (-1)^{k-1} \\
&= \frac{1}{(n+1)^p} \sum_{l=0}^{\infty} \frac{(1)_l (n+1)_l \cdots (n+1)_l}{(n+2)_l \cdots (n+2)_l} \frac{(-1)^l}{l!} \\
&= \frac{1}{(n+1)^p} \, {}_{\mathrm{p+1}}F_{\mathrm{p}} \Big(1, n+1, \ldots, n+1; n+2, \ldots, n+2; -1\Big). \quad \square
\end{aligned}$$

In [4], the authors gave an estimation of ${}_{\mathrm{p+1}}F_{\mathrm{p}}$. Indeed, for $b_k > a_k > 1$, with $k = 1, \ldots, p$ and $x > 0$, we have

$$\frac{1}{1 + x \prod_{i=1}^{p} \frac{a_i}{b_i}} < {}_{\mathrm{p+1}}F_{\mathrm{p}} \Big(1, (a_k)_{k=1}^{p}; (b_k)_{k=1}^{p}; -x\Big) < \frac{1}{1 + x \prod_{i=1}^{p} \frac{a_i - 1}{b_i - 1}}.$$

In particular, for $a_k = n+1$ and $b_k = n+2$ for $k = 1, \ldots, p$, Lemma 3.2 yields the following.

LEMMA 3.3. *For any integers $p \geqslant 2$ and $n \geqslant 1$, we have*

$$\frac{(n+1)^{-p}}{1 + \left(\frac{n+1}{n+2}\right)^p} < |R_n| < \frac{(n+1)^{-p}}{1 + \left(\frac{n}{n+1}\right)^p}. \tag{3.3}$$

The first inequality can be rewritten as

$$\frac{1}{g(n+1)\left(1 + \frac{g(n+1)}{g(n+2)}\right)} < |R_n|,$$

where $g(n) = n^p$. As $|R_n| = \frac{1}{2g(n) + x_n}$, the above inequality is equivalent to

$$x_n < \Delta g(n) - g(n) + \frac{g^2(n+1)}{g(n+2)}. \tag{3.4}$$

PROPOSITION 3.4. *The sequence (x_n) is increasing.*

The proof follows immediately from Proposition 2.2, Inequality (3.4) and the next lemma.

LEMMA 3.5. *For any two real integers $p \geqslant 2$ and $n \geqslant 1$, we have*

$$\Delta g(n) - g(n) + \frac{g^2(n+1)}{g(n+2)} \leqslant t_n.$$

Proof. Using the expression of t_n, we will show the inequality

$$\Delta g(n) + \frac{g^2(n+1)}{g(n+2)} \leqslant \sqrt{(\Delta g(n))^2 + g^2(n+1)},$$

which is equivalent to

$$\frac{2\Delta g(n)}{g(n+2)} + \frac{g^2(n+1)}{g^2(n+2)} \leqslant 1$$

Now, letting

$$H(n) = g^2(n+2) - 2\Delta g(n) g(n+2) - g^2(n+1),$$

the above inequality is equivalent to $H(n) \geqslant 0$, for any two integers $p \geqslant 2$ and $n \geqslant 1$.

Using Bernoulli inequality which states

$$(1+x)^r \geqslant 1 + rx$$

for any two real numbers $r \geqslant 1$ and $x \geqslant -1$, we have two useful inequalities

$$\left(1 + \frac{1}{n+1}\right)^p \geqslant 1 + \frac{p}{n+1},$$
$$\left(1 - \frac{1}{(n+1)^2}\right)^p \geqslant 1 - \frac{p}{(n+1)^2}.$$

Then, we get

$$\begin{aligned}
\frac{H(n)}{(n+1)^{2p}} &= \left(\frac{n+2}{n+1}\right)^{2p} - 2\left(\frac{n+2}{n+1}\right)^p - 1 + 2\left(\frac{n(n+2)}{(n+1)^2}\right)^p \\
&= \left(1 + \frac{1}{n+1}\right)^{2p} - 2\left(1 + \frac{1}{n+1}\right)^p - 1 + 2\left[1 - \frac{1}{(n+1)^2}\right]^p \\
&= \left[\left(1 + \frac{1}{n+1}\right)^p - 1\right]^2 - 2 + 2\left[1 - \frac{1}{(n+1)^2}\right]^p \\
&\geqslant \left[\frac{p}{n+1}\right]^2 - 2 + 2\left[1 - \frac{p}{(n+1)^2}\right] = \frac{p(p-2)}{(n+1)^2} \geqslant 0
\end{aligned}$$

as long as $p \geqslant 2$. This achieves the proof. □

Proof of Theorem 3.1. Note that (x_n) and (y_n) are increasing (see Corollary 2.11 and Proposition 3.4). By Lemma 2.1, we have $\rho(p) = \inf_{n\geqslant 1}(y_n) = y_1$ and $\sigma(p) = \inf_{n\geqslant 1}(x_n) = x_1$. □

PROPOSITION 3.6. *We have*

$$\lim_{n\to+\infty} x_n = \lim_{n\to\infty} y_n = +\infty.$$

Proof. By the mean value theorem, there exists c_n between n and $n+\theta_n$ such that

$$x_n = 2\left(g(n+\theta_n) - g(n)\right) = 2g'(c_n)\theta_n = 2pc_n^{p-1}\theta_n.$$

As $\lim_{n\to+\infty}\theta_n = \frac{1}{2}$ (see [10]), we get $\lim_{n\to+\infty} x_n = +\infty$. With a similar argument, we obtain $\lim_{n\to+\infty} y_n = +\infty$. □

EXAMPLE 3.7. The following table provides the values of $\rho(p)$ and $\sigma(p)$ for few values of p.

p	$\rho(p)$	$\sigma(p)$
2	$\dfrac{4(21-2\pi^2)}{12-\pi^2}$	$\dfrac{2(\pi^2-6)}{12-\pi^2}$
4	$\dfrac{16(1395-14\pi^4)}{720-7\pi^4}$	$\dfrac{2(7\pi^4-360)}{720-7\pi^4}$
6	$\dfrac{32(120015-124\pi^6)}{30240-31\pi^6}$	$\dfrac{61\pi^6-30240}{30240-31\pi^6}$

Finally, we expect that the estimates of the remainder in Theorem 3.1 are valid for real numbers $p > 1$.

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