Journal of Mathematical Inequalities
Volume 17, Number 2 (2023), 591–600 doi:10.7153/jmi-2023-17-38

A THEOREM AND AN ALGORITHM INVOLVING MUIRHEAD'S INEQUALITY

JIA XU, YONG YAO AND XIAO LING QIN

(*Communicated by M. Krnić*)

Abstract. Let $\mathbf{a},\mathbf{b}\in\mathbb{R}^n$ be column vectors, and $(\mathbf{u},\ \mathbf{v})$ be the inner product of vectors $\mathbf{u}$ and $\mathbf{v}$ on $\mathbb{R}^n$. Let $G\subset \mathrm{GL}(n,\mathbb{R})$ be a compact matrix group. For $A\in G$ and a continue function f on G, the integral $\int_G f(A)dA$ is the invariant integral of the compact group G. In this paper, we study the inequality

$$\forall \mathbf{x}\in\mathbb{R}^n \quad \int_G e^{(A\mathbf{a},\ \mathbf{x})}dA \geqslant \int_G e^{(A\mathbf{b},\ \mathbf{x})}dA.$$

We prove that the above inequality holds if and only if $\mathbf{b}\in\mathrm{Conv}(G\mathbf{a})$. This work follows a series of results, that is, Muirhead (1903), Hardy, Littlewood and Pòlya (1932), Rado (1952), Daykin (1971), Kimelfeld (1995) and Schulman (2009). Furthermore, We construct an determining algorithm when G is finite. Compared with other effective algorithms, this one is symbolic and easy to implement on computer.

1. Introduction

We continue to study the generalized Muirhead's inequality. First we give the definition of majorization in order to introduce Muirhead's theorem. Let $\mathbf{a},\mathbf{b}\in\mathbb{R}^n$, we say that $\mathbf{a}\succeq\mathbf{b}$ ($\mathbf{a}$ majorizes $\mathbf{b}$), if for $a_1\geqslant\ldots\geqslant a_n$ and $b_1\geqslant\ldots\geqslant b_n$, we have $\sum_{i=1}^n a_i=\sum_{i=1}^n b_i$, and $\sum_{i=1}^k a_i\geqslant\sum_{i=1}^k b_i$ $(k=1,\ldots,n-1)$.

In 1903, R. F. Muirhead proved the following theorem [1, 2].

THEOREM 1. (Muirhead's theorem) *If $\mathbf{x}\in\mathbb{R}^n_{>0}$ and $\mathbf{a}\succeq\mathbf{b}$ then we have the following inequality*

$$\frac{1}{n!}\sum_{\sigma\in S_n}(\sigma\mathbf{x})^{\mathbf{a}}\geqslant\frac{1}{n!}\sum_{\sigma\in S_n}(\sigma\mathbf{x})^{\mathbf{b}}, \tag{1}$$

where S_n is the symmetric group of degree n.

In 1952, Rado generalized the Muirhead's inequality [3] to an arbitrary permutation group. In 1971, D. E. Daykin gave an extension of the Muirhead-Rado inequality without using group action [4]. In 1995, B. Kimelfeld generalized the inequality (1) to finite groups [5].

Mathematics subject classification (2020): 26D15, 52A40.
Keywords and phrases: Compact matrix groups, inequality, convex hull.

THEOREM 2. (Kimelfeld's theorem) *Let* $\mathbf{a} = (a_1, \dots, a_n)^T \in \mathbb{R}^n$. *The inequality*

$$\frac{1}{|G|}\sum_{A\in G}\exp(\xi(\mathbf{A}\mathbf{a})) \geqslant \frac{1}{|G|}\sum_{A\in G}\exp(\xi(\mathbf{A}\mathbf{b})) \tag{2}$$

holds for every linear function ξ *on* $\mathbb{R}^n$ *if and only if* $\mathbf{b} \in \mathrm{Conv}(G\mathbf{a})$. *In this case equality occurs if and only if either* $\mathbf{b}$ *belongs to* G*-orbit of* $\mathbf{a}$ *or* ξ *is constant throughout the orbit.*

In 2009, based on the result of Rado, L. J., Schulman further generalized the inequality (2) to compact groups [6]. Before present Schulman's work, we need to recall the following notions and notations.

Let $\mathbf{r}, \mathbf{s} \in \mathbb{R}^n$, and G be a compact subgroup of the group of invertible matrices $GL(n,\mathbb{R})$. We say that $\mathbf{r}$ G-majorizes $\mathbf{s}$, written as $\mathbf{r} \succeq_G \mathbf{s}$, if there is a probability measure t on G such that

$$\mathbf{s} = \int_G t(g)g\mathbf{r}d\eta(g).$$

For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, define

$$\langle \mathbf{u}, \mathbf{v}\rangle_G = \int_G e^{\mathbf{u}^\dagger g\mathbf{v}} d\eta(g),$$

where $\mathbf{u}^\dagger\mathbf{v}$ is the nondegenerate inner product on $\mathbb{R}^n$, and η is the left invariant Haar measure on G.

THEOREM 3. (Schulman's Theorem) *Let* $G \subset GL(n,\mathbb{R})$ *be a compact group acting unitarily on* $\mathbb{R}^n$ *and let* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. *Then we have*

$$\mathbf{u} \succeq_G \mathbf{v} \Longleftrightarrow \langle \mathbf{u}, \mathbf{w}\rangle_G \geqslant \langle \mathbf{v}, \mathbf{w}\rangle_G,\ \forall \mathbf{w} \in \mathbb{R}^n.$$

In this paper, we will generalize the Muirhead's inequality in two aspects.

1. We obtain an extension form of Muirhead inequality for compact groups which does not need the condition "acting unitarily". See the main Theorem 4 in Section 2.

2. We establish an algorithm for the finite group, which aims to eliminate the quantifier implied in the results of Rado, Kimelfeld and Schulman. See Algorithm 1 in Section 3.

Moreover, some corollaries and applications will be listed in Section 4.

2. Main Theorem

It is known that one can define an invariant integration on every compact topological group G (see [7, 8]). Let $\int_G f(A)\, dA$ denote the invariant integral of a continue function f on G, where $G \subset GL(n,\mathbb{R})$ is a compact matrix group. Consider the inequality

$$\forall \mathbf{x} \in \mathbb{R}^n \quad \int_G e^{(A\mathbf{a},\, \mathbf{x})}\, dA \geqslant \int_G e^{(A\mathbf{b},\, \mathbf{x})}\, dA, \tag{3}$$

where $(\mathbf{a},\mathbf{b})$ is the inner product. Let $w_i = \log x_i\ (x_i > 0)$, then $e^{(\mathbf{a},\,\mathbf{w})} = x_1^{a_1}\cdots x_n^{a_n} = \mathbf{x}^{\mathbf{a}}$. It is easy to see that the inequality (3) is equivalent to the following inequality

$$\forall \mathbf{x} \in \mathbb{R}^n_{>0} \quad \int_G \mathbf{x}^{A\mathbf{a}}\, dA \geqslant \int_G \mathbf{x}^{A\mathbf{b}}\, dA,\ (\mathbf{a},\mathbf{b} \in \mathbb{R}^n). \tag{4}$$

THEOREM 4. *Let $G \subset \mathrm{GL}(n,\mathbb{R})$ be a compact matrix group. The inequality (3) or (4) holds if and only if $\mathbf{b} \in \mathrm{Conv}(G\mathbf{a})$.*

We need several lemmas for the proof of Theorem 4.

LEMMA 1. ([5]) *For any $\mathbf{a} \in \mathbb{R}^n$ the set of extrema points of $\mathrm{Conv}(\mathrm{G}\mathbf{a})$ is $G\mathbf{a}$.*

REMARK 1. The proof of Lemma 1 in [5] is limited to finite groups, it is also valid for compact groups.

LEMMA 2. (Carathéodory's Theorem [9]) *Let $S \subset \mathbb{R}^n$ and a point $p \in Conv(S)$. Then there is a set $Y \subseteq S$ consisting of $n+1$ or fewer points such that $p \in Conv(Y)$.*

LEMMA 3. (Hyperplane separation theorem [10]) *Let S_1 and S_2 be two disjoint closed convex sets of $\mathbb{R}^n$, one of which is compact. Then there is $\mathbf{v} \neq \mathbf{0}$ and $c \in \mathbb{R}$ such that $(\mathbf{v},\mathbf{u}_1) > c$ and $(\mathbf{v},\mathbf{u}_2) < c$ for $\mathbf{u}_1 \in S_1$ and $\mathbf{u}_2 \in S_2$.*

Now it is time to prove Theorem 4.

Proof. Firstly, we prove that if $\mathbf{b} \in \mathrm{Conv}(G\mathbf{a})$ then the inequality (4) holds.
By Lemma 1 and Lemma 2, we have $\mathbf{b} \in \mathrm{Conv}(G\mathbf{a}) \Longleftrightarrow \underset{\lambda \in \mathbb{R}^m_{\geqslant 0}}{\exists}\ \mathbf{b} = \sum_{i=1}^m \lambda_i (A_i\mathbf{a})$,

where $\sum_{i=1}^m \lambda_i = 1$ and $A_i \in G$. Thus

$$\begin{aligned}
\int_G \mathbf{x}^{(A\mathbf{b})}\, dA &= \int_G \mathbf{x}^{(A\sum_{i=1}^m \lambda_i (A_i\mathbf{a})}\, dA && \text{(from } \mathbf{b} = \textstyle\sum \lambda_i (A_i\mathbf{a})\text{)}\\
&= \int_G \mathbf{x}^{(\sum_{i=1}^m \lambda_i (AA_i\mathbf{a})}\, dA && \text{(by distributing the } A\text{)}\\
&\leqslant \int_G \sum_{i=1}^m \lambda_i \mathbf{x}^{(AA_i\mathbf{a})}\, dA && \text{(from generalized mean inequality)}\\
&= \sum_{i=1}^m \left(\int_G \lambda_i \mathbf{x}^{(AA_i\mathbf{a})}\, dA \right) && \text{(by exchanging integral and sum)}\\
&= \sum_{i=1}^m \left(\lambda_i \int_G \mathbf{x}^{(A\mathbf{a})}\, dA \right) && \text{(from the difinition of invariant integral)}\\
&= \left(\sum_{i=1}^m \lambda_i \right) \left(\int_G \mathbf{x}^{(A\mathbf{a})}\, dA \right) && \text{(by separating integral and sum)}\\
&= \int_G \mathbf{x}^{(A\mathbf{a})}\, dA.
\end{aligned}$$

The next step is to prove that if the inequality (4) holds then $\mathbf{b} \in \mathrm{Conv}(G\mathbf{a})$. Suppose $\mathbf{b} \notin \mathrm{Conv}(G\mathbf{a})$. We will show the contradiction. Let $S_1 = \mathbf{b}$, $S_2 = \mathrm{Conv}(G\mathbf{a})$. Then by Lemma 3, there is a vector $\mathbf{v} = (v_1, \dots, v_n)^T \in \mathbb{R}^n$ such that $(\mathbf{v}, \mathbf{b}) > \max\limits_{A \in G}(\mathbf{v}, A\mathbf{a})$ holds. Because of the continuity of the inner product on the compact group G, there exists a closed subset $S_{\mathbf{b}}$ of G such that $\forall\limits_{A' \in S_{\mathbf{b}}} \max_{A \in G}(\mathbf{v}, A\mathbf{a}) < (\mathbf{v}, A'\mathbf{b})$.

Let $w = \min\limits_{A' \in S_{\mathbf{b}}} (\mathbf{v}, A'\mathbf{b})$ and $k = \int_{A' \in S_{\mathbf{b}}} dA'$. Thus we have

$$\max_{A \in G}(\mathbf{v}, A\mathbf{a}) < w \quad \text{and} \quad 0 < k < 1. \tag{5}$$

Assume that $f(\mathbf{x}) = \int_{A \in G} \mathbf{x}^{A\mathbf{a}}\, dA - \int_{A' \in S_{\mathbf{b}}} \mathbf{x}^{A'\mathbf{b}}\, dA'$, then

$$\begin{aligned}
&f(e^{v_1 t}, \dots, e^{v_n t}) \\
&= \int_{A \in G} e^{(\mathbf{v},\, A\mathbf{a})t}\, dA - \int_{A' \in S_{\mathbf{b}}} e^{(\mathbf{v},\, A'\mathbf{b})t} dA' \\
&= \left(\int_{A \in G} e^{(\mathbf{v},\, A\mathbf{a})t}\, dA - \int_{A' \in S_{\mathbf{b}}} e^{wt} dA' \right) + \left(\int_{A' \in S_{\mathbf{b}}} e^{wt}\, dA' - \int_{A' \in S_{\mathbf{b}}} e^{(\mathbf{v},\, A'\mathbf{b})t} dA' \right) \\
&< \int_{A \in G} e^{(\mathbf{v},\, A\mathbf{a})t}\, dA - e^{wt} \int_{A' \in S_{\mathbf{b}}} dA' \\
&= e^{wt} \left(\int_{A \in G} e^{((\mathbf{v},\, A\mathbf{a}) - w)t}\, dA - k \right).
\end{aligned}$$

Thus

$$\frac{f(e^{v_1 t}, \dots, e^{v_n t})}{e^{wt}} < \int_{A \in G} e^{((\mathbf{v},\, A\mathbf{a}) - w)t}\, dA - k. \tag{6}$$

Note that the set $G\mathbf{a}$ is a compact set over $\mathbb{R}^n$. According to the Weierstrass extreme value theorem, there is a maximum R and a minimum r, such that

$$\forall_{\mathbf{x} \in G\mathbf{a}}\; r \leqslant (\mathbf{v}, \mathbf{x}) - w \leqslant R.$$

From inequality (5), we have $r < R < 0$ and

$$e^{rt} = \int_{A \in G} e^{rt} dA \leqslant \int_{A \in G} e^{((\mathbf{v},\, A\mathbf{a}) - w)t}\, dA \leqslant \int_{A \in G} e^{Rt} dA = e^{Rt}. \tag{7}$$

Taking limit for $t \to +\infty$ on the formula (7), we have

$$0 \leqslant \lim_{t \to +\infty} \int_{A \in G} e^{((\mathbf{v},\, A\mathbf{a}) - w)t} dA \leqslant 0.$$

From the inequality (6), we have

$$\lim_{t \to +\infty} \frac{f(e^{v_1 t}, \dots, e^{v_n t})}{e^{wt}} \leqslant \lim_{t \to +\infty} \int_{A \in G} e^{((\mathbf{v},\, A\mathbf{a} - w)t} dA - k = -k < 0 \tag{8}$$

But, we know for all $\mathbf{x} \in \mathbb{R}^n_{>0}$

$$f(\mathbf{x}) = \int_{A \in G} \mathbf{x}^{A\mathbf{a}}\, dA - \int_{A' \in S_{\mathbf{b}}} \mathbf{x}^{A'\mathbf{b}}\, dA' \geqslant \int_{A \in G} \mathbf{x}^{A\mathbf{a}}\, dA - \int_{A' \in G} \mathbf{x}^{A'\mathbf{b}}\, dA' = 0. \tag{9}$$

Inequalities (8) and (9) are contradictory. The proof is completed. $\square$

3. Algorithm

It is most basic for convex problems to determine wether $\mathbf{b} \in \mathrm{Conv}(G\mathbf{a})$. As we known, there are various algorithms for convex problems [9, 11]. Here we construct an algorithm for the orbit $G\mathbf{a}$. Firstly, we need a lemma. Let us start from a definition.

DEFINITION 1. A set of finite points $S = \{\mathbf{a}_1, \cdots, \mathbf{a}_d\} \subset \mathbb{R}^n$ $(1 \leqslant d \leqslant n+1)$ is called affinely independent, if the vectors $\mathbf{a}_2 - \mathbf{a}_1, \cdots, \mathbf{a}_d - \mathbf{a}_1$ are independent. Furthermore, the matrix

$$B_S = \begin{pmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_d \\ 1 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} S \\ \mathbf{1}_{1\times d} \end{pmatrix}$$

is named as a bordered matrix of S.

LEMMA 4. *Let $S = \{\mathbf{a}_1, \ldots, \mathbf{a}_d\} \subset \mathbb{R}^n$ be affinely independent. Let $\mathbf{b} \in \mathbb{R}^n$. Then $\mathbf{b} \in \mathrm{Conv}(S)$ if and only if*

$$\mathrm{Rank}(B_S) = \mathrm{Rank}(B_{S\cup\mathbf{b}}) \ \textit{and} \ (B_S^T B_S)^{-1} B_S^T \begin{pmatrix} \mathbf{b} \\ 1 \end{pmatrix} \geqslant 0.$$

Proof. "$\Longrightarrow$": Firstly, we have

$$\begin{aligned} & \mathbf{b} \in \mathrm{Conv}(\mathbf{a}_1, \ldots, \mathbf{a}_d) \\ \Longleftrightarrow \ & \underset{\lambda \in \mathbb{R}^d_{\geqslant 0}}{\exists} \begin{pmatrix} \mathbf{b} \\ 1 \end{pmatrix} = B_S \lambda, \ \lambda_1 + \cdots + \lambda_d = 1 \\ \Longrightarrow \ & \mathrm{Rank}(B_S) = \mathrm{Rank}(B_{S\cup\mathbf{b}}). \end{aligned}$$

Furthermore,

$$\begin{aligned} & \mathbf{b} \in \mathrm{Conv}(\mathbf{a}_1, \cdots, \mathbf{a}_d) \\ \Longleftrightarrow \ & \underset{\lambda \in \mathbb{R}^d_{\geqslant 0}}{\exists} \begin{pmatrix} \mathbf{b} \\ 1 \end{pmatrix} = B_S \lambda, \ \lambda_1 + \cdots + \lambda_d = 1 \\ \Longrightarrow \ & \underset{\lambda \in \mathbb{R}^d_{\geqslant 0}}{\exists} B_S^T \begin{pmatrix} \mathbf{b} \\ 1 \end{pmatrix} = B_S^T B_S \lambda \\ \Longrightarrow \ & \underset{\lambda \in \mathbb{R}^d_{\geqslant 0}}{\exists} \left(B_S^T B_S\right)^{-1} B_S^T \begin{pmatrix} \mathbf{b} \\ 1 \end{pmatrix} = \lambda \ \ (\text{since } S \text{ is affinely independent}) \\ \Longrightarrow \ & \left(B_S^T B_S\right)^{-1} B_S^T \begin{pmatrix} \mathbf{b} \\ 1 \end{pmatrix} \geqslant 0. \end{aligned}$$

"$\Longleftarrow$": Obversely, $\mathrm{Rank}(B_S) = \mathrm{Rank}(B_{S\cup\mathbf{b}}) \Longrightarrow \underset{\lambda \in \mathbb{R}^d}{\exists} \begin{pmatrix} \mathbf{b} \\ 1 \end{pmatrix} = B_S \lambda$. Denote this λ as λ'. Next it suffice to show that $\lambda' \geqslant \mathbf{0}$. Note that $B_S^T B_S$ is invertible due to S

being affinely independent. Hence,

$$\begin{aligned}\lambda' &= \left(B_S^T B_S\right)^{-1}\left(B_S^T B_S\right)\lambda' \\ &= \left(B_S^T B_S\right)^{-1} B_S^T\left(B_S\lambda'\right) \\ &= \left(B_S^T B_S\right)^{-1} B_S^T\begin{pmatrix}\mathbf{b}\\1\end{pmatrix} \geqslant \mathbf{0}. \quad \square\end{aligned}$$

It is well known that $\mathrm{Conv}(S)$ is a simplex when S is affinely independent. Furthermore, a convex polyhedron can always be decomposed as the union of simplexes [12, 13, 14]. Thus we can design the following algorithm based on Lemma 1 and Lemma 4. The algorithm can determine whether $\mathbf{b} \in \mathrm{Conv}(G\mathbf{a})$.

ALGORITHM 1. (FMGI)

In: *G a finite subgroup of $GL(n,\mathbb{R})$, $\mathbf{a},\mathbf{b}$ elements of $\mathbb{R}^n$*

Out: $\mathbf{b} \notin \mathrm{Conv}(G\mathbf{a})$ *or a formula*

$$\mathbf{b} = \sum_{A\in G}\lambda_i(A\mathbf{a}), \ \sum\lambda_i = 1.$$

1. $G\mathbf{a} \leftarrow \{A\mathbf{a} :\ A \in G\}$
2. $r_1 \leftarrow Rank\begin{pmatrix}G\mathbf{a}\\ \mathbf{1}\end{pmatrix}$, $r_2 \leftarrow Rank\begin{pmatrix}G\mathbf{a} & \mathbf{b}\\ \mathbf{1} & 1\end{pmatrix}$
3. *If* $r_1 \neq r_2$, *then return* $\mathbf{b} \notin \mathrm{Conv}(G\mathbf{a})$.
4. *If there exists* $S \subseteq G\mathbf{a}$ *such that* $|S| = r_1$, $\mathbf{a} \in S$, S *is affinely independent, and*

$$(B_S^T B_S)^{-1} B_S^T\begin{pmatrix}\mathbf{b}\\1\end{pmatrix} \geqslant 0,$$

then return the formula

$$\mathbf{b} = S(B_S^T B_S)^{-1} B_S^T\begin{pmatrix}\mathbf{b}\\1\end{pmatrix}.$$

5. *Return* $\mathbf{b} \notin \mathrm{Conv}(G\mathbf{a})$.

The above algorithm is implemented by the symbolic mathematics software Maple.

4. Applications

Some special results based on Theorem 4 are presented at this section.

COROLLARY 1. *Consider the following $n\times n$ cyclic matrices*

$$C=\begin{pmatrix}0&0&\cdots&1\\1&0&\cdots&0\\0&1&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&\cdots&1&0\end{pmatrix}_{n\times n},\quad D=\begin{pmatrix}a_1&a_2&\dots&a_n\\a_2&a_3&\dots&a_1\\\vdots&\vdots&&\vdots\\a_n&a_1&\dots&a_{n-1}\end{pmatrix}.$$

C generates the cyclic group $G=\{C,C^2,\cdots,C^n\}$. The determinant $|D|\neq 0$. Thus

$$\forall_{\mathbf{x}\in\mathbb{R}^n_{\geqslant 0}}\ \sum_{i=1}^{n}\mathbf{x}^{(C^i\mathbf{a})}\geqslant\sum_{i=1}^{n}\mathbf{x}^{(C^i\mathbf{b})}\Longleftrightarrow\sum a_i=\sum b_i\ \wedge\ D^{-1}\mathbf{b}\geqslant 0.$$

Proof. Note that $|D|\neq 0$ means D is a invertible matrix. Then

$$\begin{aligned}&\mathbf{b}\in\mathrm{Conv}(G\mathbf{a})\\\Longleftrightarrow\ &\exists_{\lambda\in\mathbb{R}^d_{\geqslant 0}}\begin{pmatrix}a_1&a_2&\dots&a_n\\a_2&a_3&\dots&a_1\\\vdots&\vdots&&\vdots\\a_n&a_1&\dots&a_{n-1}\end{pmatrix}\begin{pmatrix}\lambda_1\\\lambda_2\\\vdots\\\lambda_n\end{pmatrix}=\begin{pmatrix}b_1\\b_2\\\vdots\\b_n\end{pmatrix},\quad\lambda_1+\cdots+\lambda_n=1\\\Longleftrightarrow\ &\sum a_i=\sum b_i\ \wedge\ D^{-1}\mathbf{b}\geqslant 0.\quad\square\end{aligned}$$

COROLLARY 2. *Let a,b,r,s,t,x,y,z be positive real numbers and satisfy $a+b=r+s+t$. Then*

$$\begin{aligned}&x^ay^b+y^az^b+z^ax^b\geqslant x^ry^sz^t+y^rz^sx^t+z^rx^sy^t\\&\Longleftrightarrow\min\{ar+bs,as+bt,at+br\}\geqslant ab.\end{aligned}\tag{10}$$

Proof. Let $D=\begin{pmatrix}a&b&0\\b&0&a\\0&a&b\end{pmatrix}$. We have $\begin{pmatrix}a&b&0\\b&0&a\\0&a&b\end{pmatrix}^{-1}=\frac{1}{a^3+b^3}\begin{pmatrix}a^2&b^2&-ab\\b^2&-ab&a^2\\-ab&a^2&b^2\end{pmatrix}$.

Since $a+b=r+s+t$, (10) is equivalent to the following inequality by Corollary 1.

$$\begin{pmatrix}a^2&b^2&-ab\\b^2&-ab&a^2\\-ab&a^2&b^2\end{pmatrix}\begin{pmatrix}r\\s\\t\end{pmatrix}\geqslant 0.$$

Again using $a+b=r+s+t$, the above inequality can be transformed as follows.

$$\begin{pmatrix}a^2r+b^2s-abt\\b^2r-abs+a^2t\\-abr+a^2s+b^2t\end{pmatrix}=(a+b)\begin{pmatrix}ar+bs-ab\\br-ab+at\\-ab+as+bt\end{pmatrix}\geqslant 0.$$

That means $\min\{ar+bs, as+bt, at+br\} \geqslant ab.$ $\square$

COROLLARY 3. *Let a,b,b_1,b_2,b_3,b_4 be positive real numbers. Let $\mathbf{a}=(a,a+b,b,0)^T$, $\mathbf{b}=(b_1,b_2,b_3,b_4)^T$. Then $\forall x_i>0,\ i=1,\dots,4$, the following cyclic inequality*

$$\begin{aligned} &x_1^a x_2^{a+b} x_3^b + x_2^a x_3^{a+b} x_4^b + x_3^a x_4^{a+b} x_1^b + x_4^a x_1^{a+b} x_2^b \\ &\geqslant x_1^{b_1} x_2^{b_2} x_3^{b_3} x_4^{b_4} + x_2^{b_1} x_3^{b_2} x_4^{b_3} x_1^{b_4} + x_3^{b_1} x_4^{b_2} x_1^{b_3} x_2^{b_4} + x_4^{b_1} x_1^{b_2} x_2^{b_3} x_3^{b_4} \end{aligned} \tag{11}$$

holds if and only if

$$\begin{aligned} &(b_1+b_3=b_2+b_4=a+b) \\ \wedge\,&(ab \leqslant ab_1+bb_2 \leqslant a^2+ab+b^2) \\ \wedge\,&(-b^2 \leqslant ab_2-bb_1 \leqslant a^2). \end{aligned} \tag{12}$$

Proof. For matrix $C_4=\begin{pmatrix}0&0&0&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix}$, consider its cyclic group $G=\{C_4,C_4^2,C_4^3,C_4^4\}$. The inequality (11) can be presented as follows.

$$\sum_{i=1}^{4}\mathbf{x}^{(C_4^i\mathbf{a})} \geqslant \sum_{i=1}^{4}\mathbf{x}^{(C_4^i\mathbf{b})}.$$

According to Theorem 4, it is left to prove (12) is equivalent to $\mathbf{b}\in\mathrm{Conv}(G\mathbf{a})$. Let

$$\begin{aligned} &A=C_4\mathbf{a}=(0,a,a+b,b)^T,\ \ B=C_4^2\mathbf{a}=(b,0,a,a+b)^T, \\ &C=C_4^3\mathbf{a}=(a+b,b,0,a)^T,\ \ D=\mathbf{a}=(a,a+b,b,0)^T. \end{aligned}$$

It is easy to verify that $\mathrm{Conv}(G\mathbf{a})$, namely $\mathrm{Conv}(A,B,C,D)$, is on a two-dimensional plane of $\mathbb{R}^4$. Denote the plane by $P_{\mathbf{a}}$. The equation of the plane $P_{\mathbf{a}}$ is $y_1+y_3=y_2+y_4=a+b$. Furthermore,

$$\begin{aligned} &|\overrightarrow{AB}|=|\overrightarrow{BC}|=|\overrightarrow{CD}|=|\overrightarrow{DA}|=\sqrt{2(a^2+b^2)}. \\ &\overrightarrow{AB}\cdot\overrightarrow{BC}=\overrightarrow{BC}\cdot\overrightarrow{CD}=\overrightarrow{CD}\cdot\overrightarrow{DA}=\overrightarrow{DA}\cdot\overrightarrow{AB}=0\ \ \text{(inner product)}. \end{aligned}$$

So the $\mathrm{Conv}(A,B,C,D)$ is a square. Consider a projection mapping ρ from $P_{\mathbf{a}}$ to the y_1y_2-coordinate plane $P_{y_1y_2}$.

$$\begin{aligned} \rho: P_{\mathbf{a}} &\longmapsto P_{y_1y_2} \\ (y_1,y_2,y_3,y_4)^T &\longrightarrow (y_1,y_2)^T. \end{aligned}$$

Obviously, ρ is an invertible mapping, and images of points $A,B,C,D,\mathbf{b}$ are points $A',B',C',D',\mathbf{b}'$,

$$A'=(0,a)^T,\ \ B'=(b,0)^T,\ \ C'=(a+b,b)^T,\ \ D'=(a,a+b)^T,\ \ \mathbf{b}'=(b_1,b_2)^T.$$

In view of geometry, $\mathbf{b} \in \mathrm{Conv}(A,B,C,D) \Longleftrightarrow \mathbf{b} \in P_{\mathbf{a}} \ \wedge \ \mathbf{b}' \in \mathrm{Conv}(A',B',C',D')$.

Firstly, we have

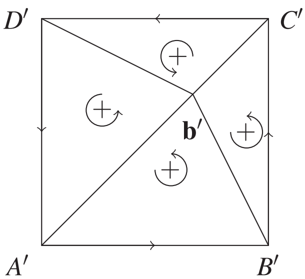
$$\mathbf{b} \in P_{\mathbf{a}} \Longleftrightarrow (b_1+b_3 = b_2+b_4 = a+b),$$

which is the first part of the formula (12).

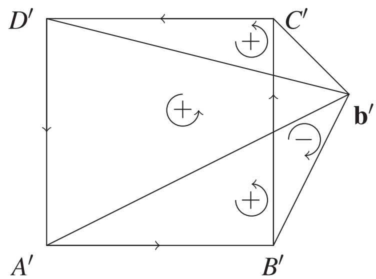
Next, by calculating the determinant of

$$\begin{vmatrix} 0 & b & a+b \\ a & 0 & b \\ 1 & 1 & 1 \end{vmatrix} = a^2+b^2 > 0, \tag{13}$$

we know A',B',C',D' are arranged in counter clockwise.



$\mathbf{b}'$ is in $\square A'B'C'D'$



$\mathbf{b}'$ is out of $\square A'B'C'D'$

Then $\mathbf{b}' \in \mathrm{Conv}(A',B',C',D')$ if and only if the four triples $(A',B',\mathbf{b}')$, $(B',C',\mathbf{b}')$, $(C',D',\mathbf{b}')$, $(D',A',\mathbf{b}')$ are all arranged in counter clockwise. Its formula is expressed as follows

$$\begin{vmatrix} 0 & b & b_1 \\ a & 0 & b_2 \\ 1 & 1 & 1 \end{vmatrix} \geqslant 0, \quad \begin{vmatrix} b & a+b & b_1 \\ 0 & b & b_2 \\ 1 & 1 & 1 \end{vmatrix} \geqslant 0, \quad \begin{vmatrix} a+b & a & b_1 \\ b & a+b & b_2 \\ 1 & 1 & 1 \end{vmatrix} \geqslant 0, \quad \begin{vmatrix} a & 0 & b_1 \\ a+b & a & b_2 \\ 1 & 1 & 1 \end{vmatrix} \geqslant 0.$$

After simplification, they are the rest of the formula (12). $\square$

COROLLARY 4. *Let a,b be real numbers. Then $\forall\ (x>0, y>0)$, the following inequality*

$$\int_0^{2\pi} x^{\cos(t)} y^{\sin(t)}\, dt \geqslant \int_0^{2\pi} x^{a\cos(t)-b\sin(t)} y^{a\sin(t)+b\cos(t)}\, dt$$

holds if and only if $a^2+b^2 \leqslant 1$.

Proof. Let $\mathbf{a} = (1,0)^T$, $\mathbf{b} = (a,b)^T$, and

$$G = \left\{ \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} : \quad t \in [0,2\pi) \right\}.$$

Then $\operatorname{Conv}(G\mathbf{a}) = \{(x,y)^T : x^2+y^2 \leqslant 1\}$. So $\mathbf{b} \in \operatorname{Conv}(G\mathbf{a})$ if and only if $a^2+b^2 \leqslant 1$. According to Theorem 4, we completed the proof. □

Acknowledgements. The authors are grateful to the comments given by anonymous referees and editors, which have improved the final version of the paper. This research is supported by the Fundamental Research Funds for the Central Universities, Southwest Minzu University (2020NYB40), Science and Technology Service Network Initiative (KFJ-STS-QYZD-2021-21-001).

REFERENCES

[1] R. F. MUIRHEAD, *Some methods applicable to identities and inequalities of symmetric algebraic functions of n letters*, Proc. Edinburgh Math. Soc., **21**, (1903), 144–157.
[2] G. H. HARDY, J. E. LITTLEWOOD AND G. POLYA, *Inequalities*, Cambridge University Press, Cambridge, 1934.
[3] R. RADO, *An inequality*, J. Lond. Math. Soc., **27**, (1952), 1–6.
[4] D.E. DAYKIN, *Generalisation of the Muirhead-Rado inequality*, Proc. Am. Math. Soc., **30**, (1971), 84–96.
[5] B. KIMELFELD, *A generalization of Muirhead's theorem*, Linear Algebra and its Applications, **216**, (1995), 205–209.
[6] L. J. SCHULMAN, *Muirhead-Rado inequality for compact groups*, Positivity, **13**, (2009), 559–574.
[7] E. B. VINBERG, *Linear Representations of Groups*, Springer-Verlag, New York, 1989.
[8] L. PONTRJAGIN, *Topological Groups*, Princeton University press, Princeton, 1946.
[9] J. E. GOODMAN, J. O'ROURKE AND C. D. TÓTH (Eds.), *Handbook of Discrete and Computational Geometry*, 3rd Edition, Chapman & Hall Boca Raton, 2004.
[10] S. BOYD AND L. VANDENBERGHE, *Convex Optimization*, Cambridge University Press, Cambridge, 2004.
[11] BAHMAN KALANTARI, *Randomized triangle algorithms for convex hull membership*, arXiv:1410.3564v1, 2014.
[12] B. STURMFELS, *Polynomial Equations and Convex Polytopes*, American Mathematical Monthly, **105**, 10 (1998), 907–922.
[13] G. M. ZIEGLER, *Lectures on Polytopes*, Springer-Verlag, New York, 1995.
[14] M. JOSWIG AND T. THEOBALD, *Polyhedral and algebraic methods in computation geometry*, Springer-Verlag, London, 2013.

(Received May 12, 2020)

Jia Xu
Department of Mathematics
Southwest Minzu University
Chengdu, China
e-mail: xufine@163.com

Yong Yao
Chengdu Institute of Computer Applications
Chinese Academy of Sciences
Chengdu, China
e-mail: yaoyong@casit.ac.cn

Xiao Ling Qin
Chengdu Institute of Computer Applications
Chinese Academy of Sciences
Chengdu, China
e-mail: qinxl@casit.ac.cn