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ON COMPLETE MOMENT CONVERGENCE FOR WEIGHTED SUMS UNDER NEGATIVELY ASSOCIATED SETUP

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Abstract. In this work, the complete moment convergence for weighted sums of negatively associated random variables is discussed without assumptions of identical distribution. Under the moment condition $E|X|^{\alpha}/(\log(1+|X|))^{\alpha/\gamma-1}<\infty$ for the case $0<\gamma<\alpha$ with $1<\alpha\leqslant 2$, the complete moment convergence theorem for weighted sums of negatively associated setup is presented. The main results obtained in this article extend and improve the corresponding ones of Chen and Sung (Stat. Probabil. Lett., 92: 45–52 (2014)), Sung (Stat. Pap., 52: 447–454 (2011)).

1. Introduction

Let $\{X_n;n\geqslant 1\}$ be a sequence of random variables and let $\{a_{ni};1\leqslant i\leqslant n,n\geqslant 1\}$ be an array of real constants. Since many useful linear statistics, such as least squares estimators, nonparametric regression function estimators and jackknife estimates are based on the weighted sums of $\sum_{i=1}^{n} a_{ni}X_i$, it is important and meaningful to deeply investigate the probability limiting behaviors for them. Many scholars devoted to study the limiting behaviors of the form of the weighted sums. We refer to the readers to Cuzick [8], Wu [19], Bai and Cheng [2], Chen and Gan [4], Sung [16] and among others.

In many probabilistic applications and stochastic models, the assumption of independent variables is not plausible. Hence, it is necessary to extend the results of independent random variables to dependent cases. One of dependence structures is negative association, which has attracted the interest by probabilists and statisticians. The concept of negatively associated (NA, for short) random variables, which was introduced by Alam and Saxena [1] and carefully studied by Joag-Dev and Proschan [10] is as follows.

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DEFINITION 1.1. Random variables $X_1, X_2, \cdots, X_n$ are said to be NA if for every pair of disjoint subsets A, B of $\{1, 2, \cdots, n\}$,

$$\operatorname{Cov}\left(f_1\left(X_i, i \in A\right), f_2\left(X_j, j \in B\right)\right) \leqslant 0, \tag{1.1}$$

whenever f_1 and f_2 are any real coordinatewise non-decreasing (or non-increasing) functions such that this covariance exists. A sequence of random variables $\{X_n; n \geqslant 1\}$ is NA if every finite subfamily is NA.

Joag-Dev and Proschan [10] pointed out and proved that many known multivariate distributions possess the NA property. Since the concept of NA random variables was introduced by Alam and Saxena [1], many applications have been found. For example, Shao [15] for the moment inequalities, Matula [14] for the almost sure convergence, Chen et al. [5] and Kuczmaszewska [11] for the complete convergence, Cai [3], Sung [18] and Liang et al. [13] for the strong convergence of weighed sums, and so forth.

A sequence of random variables $\{X_n; n \geqslant 1\}$ is said to converge completely to a constant λ if $\sum\limits_{n=1}^{\infty} P\left(|X_n - \lambda| > \varepsilon\right) < \infty$ for all $\varepsilon > 0$. This notion was firstly given by Hsu and Robbins [9]. In view of the Borel-Cantelli lemma, this implies that $X_n \to \lambda$ almost surely (a.s.). Hence, the complete convergence has been an important basic tool to investigate the convergence properties for summation of random variables as well as weighted sums.

Chow [7] introduced the complete moment convergence as follows: Let $\{Z_n; n \geqslant 1\}$ be a sequence of random variables, and $a_n > 0$, $b_n > 0$, $q > 0$. If $\sum\limits_{n=1}^{\infty} a_n E\left(b_n^{-1}|Z_n| - \varepsilon\right)_+^q < \infty$ for all $\varepsilon \geqslant 0$, then $\{Z_n; n \geqslant 1\}$ is called the complete moment convergence. It is clearly seen that the complete moment convergence implies the complete convergence. Thus, the complete moment convergence is the more general version of the complete convergence.

For $0 < \gamma < \alpha$ and $1 < \alpha \leqslant 2$, Chen and Sung [6] discussed the complete convergence for weighted sums of identically distributed NA random variables. They obtained the following result under the moment condition $E|X|^{\alpha}/(\log(1+|X|))^{\alpha/\gamma-1} < \infty$, which is weaker than that of Sung [18].

THEOREM A. *Let $\{X, X_n; n \geqslant 1\}$ be a sequence of identically distributed NA random variables with $EX_n = 0$, $\{a_{ni}; 1 \leqslant i \leqslant n, n \geqslant 1\}$ be an array of real constants such that $\sum\limits_{i=1}^{n} |a_{ni}|^{\alpha} = O(n)$ for $1 < \alpha \leqslant 2$. Set $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ for $0 < \gamma < \alpha$. If $E|X|^{\alpha}/(\log(1+|X|))^{\alpha/\gamma-1} < \infty$, then*

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\max_{1 \leqslant j \leqslant n}\left|\sum_{i=1}^{j} a_{ni} X_i\right| > \varepsilon b_n\right) < \infty \quad \text{for} \quad \forall \varepsilon > 0. \tag{1.2}$$

It is worthy noting that the main tool of Sung [18], Chen and Sung [6] is Theorem 1 of Chen et al. [5], which follows from an exponent inequality for NA random variables established by Shao [15]. In addition, Li et al. [12] extended the result of Chen and Sung [6] for NA random variables to ρ^*-mixing cases by using the different method from those of Chen and Sung [6].

Inspired by Chen and Sung [6], Li et al. [12], we further study the convergence behaviors for weighted sums of NA setup without assumptions of identical distribution. Under the moment condition $E|X|^{\alpha}/(\log(1+|X|))^{\alpha/\gamma-1} < \infty$ for $0<\gamma<\alpha$ with $1<\alpha\leqslant 2$, we establish a complete moment convergence theorem for weighted sums of NA random variables. As applications, the complete convergence theorem and the Marcinkiewicz-Zygmund type strong law of large numbers for weighted sums of NA cases are obtained. Our results extend and improve the corresponding ones of Chen and Sung [6], Sung [18].

DEFINITION 1.2. A sequence of random variables $\{X_n; n\geqslant 1\}$ is said to be stochastically dominated by a random variable X if there exists a positive constant C such that

$$P(|X_n|\geqslant x)\leqslant CP(|X|\geqslant x),$$

for all $x\geqslant 0$ and $n\geqslant 1$.

Throughout this paper, let $I(A)$ be the indicator function of the set A and $I(A,B)=I(A\bigcap B)$. The symbol $C, C_1, C_2, \ldots$ always present different positive constants in various places, and $a_n = O(b_n)$ stands for $a_n\leqslant Cb_n$.

2. Main results and proofs

In the following, the main results and proofs are given in this section.

THEOREM 2.1. *Let $\{X_n; n\geqslant 1\}$ be a sequence of mean zero NA random variables which is stochastically dominated by a random variable X, let $\{a_{ni}; 1\leqslant i\leqslant n, n\geqslant 1\}$ be an array of real constants such that $\sum_{i=1}^{n}|a_{ni}|^{\alpha}=O(n)$ for $0<\alpha\leqslant 2$. Set $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ for $0<\gamma<\alpha$ with $1<\alpha\leqslant 2$. If $E|X|^{\alpha}/(\log(1+|X|))^{\alpha/\gamma-1} < \infty$, then*

$$\sum_{n=1}^{\infty}\frac{1}{n}E\left(\frac{1}{b_n}\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}a_{ni}X_i\right|-\varepsilon\right)_+^{\alpha}<\infty \quad \textit{for} \quad \forall\varepsilon>0. \tag{2.1}$$

To prove this theorem, we will present the following important lemmas.

LEMMA 2.1. (Shao [15]) *Let $\{X_n, n\geqslant 1\}$ be a sequence of NA random variables. If $\{f_n, n\geqslant 1\}$ is a sequence of Borel functions all of which are monotone non-decreasing (or all monotone non-increasing), then $\{f_n(X_n), n\geqslant 1\}$ is still a sequence of NA random variables.*

LEMMA 2.2. (Shao [15]) *Let $\{X_n, n\geqslant 1\}$ be a sequence of NA random variables with $EX_n=0$ and $E|X_n|^M<\infty$ for $1\leqslant M\leqslant 2$ and all $n\geqslant 1$. Then there exists a positive constant depending only on M such that*

$$E\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}X_i\right|^{M}\right)\leqslant C\sum_{i=1}^{n}E|X_i|^{M}. \tag{2.2}$$

LEMMA 2.3. *Let $\{X_n, n \geqslant 1\}$ be a sequence of random variables which is stochastically dominated by a random variable X. For all $\alpha > 0$ and $b > 0$, the following two statements hold:*

$$E|X_n|^\alpha I(|X_n| \leqslant b) \leqslant C_1 \left(E|X|^\alpha I(|X| \leqslant b) + b^\alpha P(|X| > b)\right), \tag{2.3}$$

$$E|X_n|^\alpha I(|X_n| > b) \leqslant C_2 E|X|^\alpha I(|X| > b), \tag{2.4}$$

where C_1 and C_2 are different positive constants. Consequently, $E|X_n|^\alpha \leqslant CE|X|^\alpha$.

LEMMA 2.4. *Let $\{X_n; n \geqslant 1\}$ be a sequence of NA random variables which is stochastically dominated by a random variable X, let $\{a_{ni}; 1 \leqslant i \leqslant n, n \geqslant 1\}$ be an array of real constants such that $\sum_{i=1}^{n} |a_{ni}|^\alpha = O(n)$ for $0 < \alpha \leqslant 2$. Set $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ for $0 < \gamma < \alpha$. Assume that $EX_n = 0$ for $1 < \alpha \leqslant 2$. If $E|X|^\alpha/(\log(1+|X|))^{\alpha/\gamma-1} < \infty$, then (1.2) holds.*

Proof. Analogous to the proof of Theorem 1.1 in Li et al. [12], we need to substitute the identical distribution condition of random variables with stochastic domination. The rest is similar to that of Theorem 1.1 in Li et al. [12]. So we omit the detail. □

LEMMA 2.5. *Under the conditions of Theorem 2.1, if $E|X|^\alpha/(\log(1+|X|))^{\alpha/\gamma-1} < \infty$ for $0 < \gamma < \alpha$ and $0 < \alpha \leqslant 2$, then*

$$\sum_{n=1}^{\infty} \frac{1}{n} \int_1^\infty \sum_{i=1}^{n} P\left(|a_{ni}X_i| > b_n t^{1/\alpha}\right) dt < \infty. \tag{2.5}$$

Proof. By the definition of the stochastic domination, it easily follows that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n} \int_1^\infty \sum_{i=1}^{n} P\left(|a_{ni}X_i| > b_n t^{1/\alpha}\right) dt &\leqslant C \sum_{n=1}^{\infty} \frac{1}{n} \int_1^\infty \sum_{i=1}^{n} P\left(|a_{ni}X| > b_n t^{1/\alpha}\right) dt \\ &\leqslant C \sum_{n=1}^{\infty} \frac{1}{n} \int_0^\infty \sum_{i=1}^{n} P\left(\frac{|a_{ni}X|^\alpha}{b_n^\alpha} > t\right) dt \\ &\leqslant C \sum_{n=1}^{\infty} n^{-1} b_n^{-\alpha} \sum_{i=1}^{n} E|a_{ni}X|^\alpha I(|a_{ni}X| > b_n). \end{aligned}$$

Obviously,

$$\begin{aligned} E|a_{ni}X|^\alpha I(|a_{ni}X| > b_n) = E|a_{ni}X|^\alpha I(|a_{ni}X| > b_n, |X| \leqslant b_n) \\ +E|a_{ni}X|^\alpha I(|a_{ni}X| > b_n, |X| > b_n). \end{aligned} \tag{2.6}$$

It is clearly shown that

$$E|a_{ni}X|^\alpha I(|a_{ni}X| > b_n, |X| \leqslant b_n) \leqslant C_3 E|a_{ni}X|^\alpha I(|X| \leqslant b_n), \tag{2.7}$$

$$E|a_{ni}X|^\alpha I(|a_{ni}X| > b_n, |X| > b_n) \leqslant C_4 E|a_{ni}X|^\alpha I(|X| > b_n). \tag{2.8}$$

Note that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{-1} b_n^{-\alpha} \sum_{i=1}^{n} |a_{ni}|^{\alpha} E|X|^{\alpha} I(|X| \leqslant b_n) \\
&\leqslant C \sum_{n=1}^{\infty} b_n^{-\alpha} E|X|^{\alpha} I(|X| \leqslant b_n) \\
&\leqslant C \sum_{n=1}^{\infty} b_n^{-\alpha} \sum_{j=1}^{n} E|X|^{\alpha} I\left(b_j < |X| \leqslant b_{j+1}\right) \\
&\leqslant C \sum_{j=1}^{\infty} E|X|^{\alpha} I\left(b_j < |X| \leqslant b_{j+1}\right) (\log j)^{1-(\alpha/\gamma)} \\
&\leqslant CE|X|^{\alpha} / (\log(1+|X|))^{(\alpha/\gamma)-1} < \infty,
\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{-1} b_n^{-\alpha} \sum_{i=1}^{n} |a_{ni}|^{\alpha} E|X|^{\alpha} I(|X| > b_n) \\
&\leqslant C \sum_{n=1}^{\infty} b_n^{-\alpha} E|X|^{\alpha} I(|X| > b_n) \\
&= C \sum_{n=1}^{\infty} b_n^{-\alpha} \sum_{j=n}^{\infty} E|X|^{\alpha} I\left(b_j < |X| \leqslant b_{j+1}\right) \\
&= C \sum_{j=1}^{\infty} E|X|^{\alpha} I\left(b_j < |X| \leqslant b_{j+1}\right) \sum_{n=1}^{j} n^{-1} (\log n)^{-\alpha/\gamma} \\
&\leqslant C \sum_{j=1}^{\infty} (\log j)^{1-(\alpha/\gamma)} E|X|^{\alpha} I\left(b_j < |X| \leqslant b_{j+1}\right) \\
&\leqslant CE|X|^{\alpha} / (\log(1+|X|))^{(\alpha/\gamma)-1} < \infty.
\end{aligned} \tag{2.10}$$

Then, (2.5) follows from (2.9) and (2.10). □

LEMMA 2.6. *Under the conditions of Theorem 2.1, if* $E|X|^{\alpha}/(\log(1+|X|))^{\alpha/\gamma-1} < \infty$ *for* $0 < \gamma < \alpha$ *with* $1 < \alpha \leqslant 2$*, then*

$$\sup_{t \geqslant 1} \frac{1}{b_n t^{1/\alpha}} \max_{1 \leqslant j \leqslant n} \left| \sum_{i=1}^{j} E a_{ni} X_i I\left(|a_{ni} X_i| \leqslant b_n t^{1/\alpha}\right) \right| \to 0. \tag{2.11}$$

Proof. By $EX_n = 0$ and (2.4) of Lemma 2.3, we have

$$\begin{aligned}
&\sup_{t\geqslant 1}\frac{1}{b_n t^{1/\alpha}}\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} Ea_{ni}X_iI\left(|a_{ni}X_i|\leqslant b_n t^{1/\alpha}\right)\right.\\
&=\sup_{t\geqslant 1}\frac{1}{b_n t^{1/\alpha}}\max_{1\leqslant j\leqslant n}\left.\sum_{i=1}^{j} Ea_{ni}X_iI\left(|a_{ni}X_i|> b_n t^{1/\alpha}\right)\right|\\
&\leqslant C\sup_{t\geqslant 1}\frac{1}{b_n t^{1/\alpha}}\sum_{i=1}^{n}E\left|a_{ni}X_i\right|I\left(|a_{ni}X_i|> b_n t^{1/\alpha}\right)\\
&\leqslant C\sup_{t\geqslant 1}\frac{1}{b_n t^{1/\alpha}}\sum_{i=1}^{n}E\left|a_{ni}X\right|I\left(|a_{ni}X|> b_n t^{1/\alpha}\right).
\end{aligned}$$

Obviously,

$$\begin{aligned}
&E\left|a_{ni}X\right|I\left(|a_{ni}X|> b_n t^{1/\alpha}\right)\\
&=E\left|a_{ni}X\right|I\left(|a_{ni}X|> b_n t^{1/\alpha}, |X|\leqslant b_n\right)\\
&\quad+E\left|a_{ni}X\right|I\left(|a_{ni}X|> b_n t^{1/\alpha}, |X|> b_n\right).
\end{aligned}\tag{2.12}$$

For $0<\gamma<\alpha$ and $1<\alpha\leqslant 2$, it is clearly shown that

$$\begin{aligned}
&E\left|a_{ni}X\right|I\left(|a_{ni}X|> b_n t^{1/\alpha}, |X|\leqslant b_n\right)\\
&\leqslant C_5 b_n^{1-\alpha}t^{(1/\alpha)-1}|a_{ni}|^{\alpha}E|X|^{\alpha}I\left(|X|\leqslant b_n\right)\\
&\leqslant C_5 b_n^{1-\alpha}t^{(1/\alpha)-1}|a_{ni}|^{\alpha}E\left(\frac{|X|^{\alpha}}{(\log(1+|X|))^{\alpha/\gamma-1}}(\log(1+|X|))^{\alpha/\gamma-1}\right)I\left(|X|\leqslant b_n\right)\\
&\leqslant C_5 t^{(1/\alpha)-1}n^{-1+(1/\alpha)}|a_{ni}|^{\alpha}(\log n)^{(1/\gamma)-1},
\end{aligned}\tag{2.13}$$

and

$$\begin{aligned}
&E\left|a_{ni}X\right|I\left(|a_{ni}X|> b_n t^{1/\alpha}, |X|> b_n\right)\\
&\leqslant C_6\left|a_{ni}\right|E\left|X\right|I\left(|X|> b_n\right)\\
&\leqslant C_6 b_n^{1-\alpha}(\log(1+b_n))^{(\alpha/\gamma)-1}\left|a_{ni}\right|\\
&\leqslant C_6 n^{-1+(1/\alpha)}(\log n)^{-1+(1/\gamma)}\left|a_{ni}\right|.
\end{aligned}\tag{2.14}$$

Hence,

$$\begin{aligned}
&\sup_{t\geqslant 1}\frac{1}{b_n t^{1/\alpha}}\sum_{i=1}^{n}E\left|a_{ni}X\right|I\left(|a_{ni}X|> b_n t^{1/\alpha}, |X|\leqslant b_n\right)\\
&\leqslant C b_n^{-1}n^{-1+(1/\alpha)}(\log n)^{(1/\gamma)-1}\sum_{i=1}^{n}|a_{ni}|^{\alpha}\\
&\leqslant C(\log n)^{-1}\to 0,
\end{aligned}\tag{2.15}$$

and

$$\begin{aligned}
&\sup_{t\geqslant 1}\frac{1}{b_n t^{1/\alpha}}\sum_{i=1}^{n}E\left|a_{ni}X\right|I\left(\left|a_{ni}X\right|>b_n t^{1/\alpha},|X|>b_n\right)\\
&\leqslant Cb_n^{-1}n^{-1+(1/\alpha)}(\log n)^{-1+(1/\gamma)}\sum_{i=1}^{n}|a_{ni}|\\
&\leqslant C(\log n)^{-1}\to 0. \qquad (2.16)
\end{aligned}$$

Thus, (2.11) follows immediately from the above statements. □

LEMMA 2.7. *Under the conditions of Theorem 2.1, if* $E|X|^{\alpha}/(\log(1+|X|))^{\alpha/\gamma-1}<\infty$ *for* $0<\gamma<\alpha$ *and* $0<\alpha\leqslant 2$, *then*

$$\sum_{n=1}^{\infty}\frac{1}{n}\frac{1}{b_n^{\alpha}}\sum_{i=1}^{n}E|a_{ni}X_i|^{\alpha}I\left(|a_{ni}X_i|\leqslant b_n\right)<\infty. \qquad (2.17)$$

Proof. Analogous to the proof of Lemma 2.2 in Li et al. [12], it follows that

$$\begin{aligned}
E|a_{ni}X_i|^{\alpha}I\left(|a_{ni}X_i|\leqslant b_n\right)=&\,E|a_{ni}X_i|^{\alpha}I\left(|a_{ni}X_i|\leqslant b_n,|X_i|\leqslant b_n\right)\\
&+E|a_{ni}X_i|^{\alpha}I\left(|a_{ni}X_i|\leqslant b_n,|X_i|>b_n\right). \qquad (2.18)
\end{aligned}$$

By Lemma 2.3, it clearly follows that

$$\begin{aligned}
E|a_{ni}X_i|^{\alpha}I\left(|a_{ni}X_i|\leqslant b_n,|X_i|\leqslant b_n\right)&\leqslant C_7|a_{ni}|^{\alpha}E|X_i|^{\alpha}I\left(|X_i|\leqslant b_n\right)\\
&\leqslant C_7|a_{ni}|^{\alpha}E|X|^{\alpha}I\left(|X|\leqslant b_n\right)\\
&\quad+C_7|a_{ni}|^{\alpha}b_n^{\alpha}P\left(|X|>b_n\right). \qquad (2.19)
\end{aligned}$$

For $\forall 0<\theta<\alpha$,

$$\begin{aligned}
E|a_{ni}X_i|^{\alpha}I\left(|a_{ni}X_i|\leqslant b_n,|X_i|>b_n\right)&\leqslant C_8 b_n^{\alpha-\theta}|a_{ni}|^{\theta}E|X_i|^{\theta}I\left(|X_i|>b_n\right)\\
&\leqslant C_8 b_n^{\alpha-\theta}|a_{ni}|^{\theta}E|X|^{\theta}I\left(|X|>b_n\right). \qquad (2.20)
\end{aligned}$$

Hence,

$$\begin{aligned}
&\sum_{n=1}^{\infty}n^{-1}b_n^{-\alpha}\sum_{i=1}^{n}|a_{ni}|^{\alpha}E|X|^{\alpha}I\left(|X|\leqslant b_n\right)\\
&\leqslant C\sum_{n=1}^{\infty}b_n^{-\alpha}E|X|^{\alpha}I\left(|X|\leqslant b_n\right)\\
&\leqslant C\sum_{n=1}^{\infty}b_n^{-\alpha}\sum_{k=1}^{n}E|X|^{\alpha}I\left(b_k<|X|\leqslant b_{k+1}\right)\\
&\leqslant C\sum_{k=1}^{\infty}E|X|^{\alpha}I\left(b_k<|X|\leqslant b_{k+1}\right)(\log k)^{-(\alpha/\gamma)+1}\\
&\leqslant CE|X|^{\alpha}/(\log(1+|X|))^{(\alpha/\gamma)-1}<\infty. \qquad (2.21)
\end{aligned}$$

Similarly, we have

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{-1} \sum_{i=1}^{n} |a_{ni}|^{\alpha} P(|X| > b_n) \\
&= \sum_{n=1}^{\infty} P(|X| > b_n) \\
&= \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} P\left(k^{1/\alpha}(\log k)^{1/\gamma} < |X| \leqslant (k+1)^{1/\alpha}(\log(k+1))^{1/\gamma}\right) \\
&= \sum_{k=1}^{\infty} P\left(k^{1/\alpha}(\log k)^{1/\gamma} < |X| \leqslant (k+1)^{1/\alpha}(\log(k+1))^{1/\gamma}\right) k \\
&\leqslant CE|X|^{\alpha}/(\log(1+|X|))^{\alpha/\gamma} < \infty. \qquad (2.22)
\end{aligned}
$$

By $\sum_{i=1}^{n} |a_{ni}|^{\alpha} = O(n)$, we have $\sum_{i=1}^{n} |a_{ni}|^{\theta} = O(n)$ for $\forall 0 < \theta < \alpha$. Therefore,

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{-1} \sum_{i=1}^{n} b_n^{-\theta} |a_{ni}|^{\theta} E|X|^{\theta} I(|X| > b_n) \\
&\leqslant C \sum_{n=1}^{\infty} b_n^{-\theta} E|X|^{\theta} I(|X| > b_n) \\
&\leqslant C \sum_{k=1}^{\infty} E|X|^{\theta} I\left(k^{1/\alpha}(\log k)^{1/\gamma} < |X| \leqslant (k+1)^{1/\alpha}(\log(k+1))^{1/\gamma}\right) \\
&\quad \times \sum_{n=1}^{k} n^{-\theta/\alpha}(\log n)^{-\theta/\gamma} \\
&\leqslant C \sum_{k=1}^{\infty} k^{1-(\theta/\alpha)}(\log k)^{-\theta/\gamma} E|X|^{\theta} \\
&\quad \times I\left(k^{1/\alpha}(\log k)^{1/\gamma} < |X| \leqslant (k+1)^{1/\alpha}(\log(k+1))^{1/\gamma}\right) \\
&\leqslant C \sum_{k=1}^{\infty} (\log k)^{-\alpha/\gamma} E|X|^{\alpha} I\left(k^{1/\alpha}(\log k)^{1/\gamma} < |X| \leqslant (k+1)^{1/\alpha}(\log(k+1))^{1/\gamma}\right) \\
&\leqslant CE|X|^{\alpha}/(\log(1+|X|))^{\alpha/\gamma} < \infty. \qquad (2.23)
\end{aligned}
$$

Based on the above statements, the desired result (2.17) follows immediately. □

Proof of Theorem 2.1. For $\forall \varepsilon > 0$, it follows that

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{n} E\left(\frac{1}{b_n} \max_{1\leqslant j\leqslant n} \left|\sum_{i=1}^{j} a_{ni}X_i\right| - \varepsilon\right)_+^{\alpha} \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\infty} P\left(\frac{1}{b_n} \max_{1\leqslant j\leqslant n} \left|\sum_{i=1}^{j} a_{ni}X_i\right| - \varepsilon > t^{1/\alpha}\right) dt
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 P\left(\frac{1}{b_n} \max_{1\leqslant j\leqslant n} \left|\sum_{i=1}^{j} a_{ni}X_i\right| > \varepsilon + t^{1/\alpha}\right) dt \\
&\quad + \sum_{n=1}^{\infty} \frac{1}{n} \int_1^{\infty} P\left(\frac{1}{b_n} \max_{1\leqslant j\leqslant n} \left|\sum_{i=1}^{j} a_{ni}X_i\right| > \varepsilon + t^{1/\alpha}\right) dt \\
&\leqslant \sum_{n=1}^{\infty} \frac{1}{n} P\left(\max_{1\leqslant j\leqslant n} \left|\sum_{i=1}^{j} a_{ni}X_i\right| > \varepsilon b_n\right) \\
&\quad + \sum_{n=1}^{\infty} \frac{1}{n} \int_1^{\infty} P\left(\max_{1\leqslant j\leqslant n} \left|\sum_{i=1}^{j} a_{ni}X_i\right| > b_n t^{1/\alpha}\right) dt \\
&\triangleq I + J.
\end{aligned} \tag{2.24}
$$

To prove (2.1), it suffices to show that $I<\infty$ and $J<\infty$. By Lemma 2.4 (or Theorem A for identical distribution condition of random variables substituted by stochastic domination), $I<\infty$ clearly follows.

Without loss of generality, we can assume that $a_{ni}\geqslant 0$. For any $t\geqslant 1$ and all $1\leqslant i\leqslant n$, $n\in\mathbb{N}$, define

$$
Y_i = -b_n t^{1/\alpha} I\left(a_{ni}X_i < -b_n t^{1/\alpha}\right) + a_{ni}X_i I\left(|a_{ni}X_i| \leqslant b_n t^{1/\alpha}\right) + b_n t^{1/\alpha} I\left(a_{ni}X_i > b_n t^{1/\alpha}\right).
$$

It is easily seen that

$$
\left(\max_{1\leqslant j\leqslant n} \left|\sum_{i=1}^{j} a_{ni}X_i\right| > b_n t^{1/\alpha}\right) \subset \left(\max_{1\leqslant j\leqslant n} \left|\sum_{i=1}^{j} Y_i\right| > b_n t^{1/\alpha}\right) \bigcup \left(\bigcup_{i=1}^{n} \left(|a_{ni}X_i| > b_n t^{1/\alpha}\right)\right),
$$

which implies

$$
\begin{aligned}
P\left(\max_{1\leqslant j\leqslant n} \left|\sum_{i=1}^{j} a_{ni}X_i\right| > b_n t^{1/\alpha}\right) &\leqslant P\left(\max_{1\leqslant j\leqslant n} \left|\sum_{i=1}^{j} Y_i\right| > b_n t^{1/\alpha}\right) \\
&\quad + P\left(\bigcup_{i=1}^{n} \left(|a_{ni}X_i| > b_n t^{1/\alpha}\right)\right).
\end{aligned} \tag{2.25}
$$

Hence, it suffices to show that

$$
J_1 = \sum_{n=1}^{\infty} \frac{1}{n} \int_1^{\infty} P\left(\max_{1\leqslant j\leqslant n} \left|\sum_{i=1}^{j} Y_i\right| > b_n t^{1/\alpha}\right) dt < \infty,
$$

$$
J_2 = \sum_{n=1}^{\infty} \frac{1}{n} \int_1^{\infty} P\left(\bigcup_{i=1}^{n} \left(|a_{ni}X_i| > b_n t^{1/\alpha}\right)\right) dt < \infty.
$$

By Lemma 2.4, we have

$$
J_2 \leqslant C \sum_{n=1}^{\infty} \frac{1}{n} \int_1^{\infty} \sum_{i=1}^{n} P\left(|a_{ni}X_i| > b_n t^{1/\alpha}\right) dt < \infty.
$$

By Lemma 2.5, for n large enough, $\max\limits_{1\leqslant j\leqslant n}\left|\sum\limits_{i=1}^{j} EY_i\right| \leqslant \frac{b_n t^{1/\alpha}}{2}$ holds uniformly for all $t\geqslant 1$. Hence, by the Markov's inequality, Lemma 2.2 and (2.3) of Lemma 2.3, we have

$$\begin{aligned}
J_1 &\leqslant C\sum_{n=1}^{\infty}\frac{1}{n}\int_1^{\infty} P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}(Y_i-EY_i)\right| > \frac{b_n t^{1/\alpha}}{2}\right)dt\\
&\leqslant C\sum_{n=1}^{\infty}\frac{1}{n}\int_1^{\infty}\frac{1}{b_n^2 t^{2/\alpha}}E\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}(Y_i-EY_i)\right|^2\right)dt\\
&\leqslant C\sum_{n=1}^{\infty}\frac{1}{n}\int_1^{\infty}\frac{1}{b_n^2 t^{2/\alpha}}\left(\sum_{i=1}^{n}E|Y_i-EY_i|^2\right)dt\\
&\leqslant C\sum_{n=1}^{\infty}\frac{1}{n}\int_1^{\infty}\frac{1}{b_n^2 t^{2/\alpha}}\left(\sum_{i=1}^{n}E|a_{ni}X_i|^2 I\left(|a_{ni}X_i|\leqslant b_n t^{1/\alpha}\right)\right)dt\\
&\quad +C\sum_{n=1}^{\infty}\frac{1}{n}\int_1^{\infty}\sum_{i=1}^{n}P\left(|a_{ni}X_i|>b_n t^{1/\alpha}\right)dt\\
&\leqslant C\sum_{n=1}^{\infty}\frac{1}{n}\int_1^{\infty}\frac{1}{b_n^2 t^{2/\alpha}}\left(\sum_{i=1}^{n}E|a_{ni}X|^2 I\left(|a_{ni}X|\leqslant b_n t^{1/\alpha}\right)\right)dt\\
&\quad +C\sum_{n=1}^{\infty}\frac{1}{n}\int_1^{\infty}\sum_{i=1}^{n}P\left(|a_{ni}X|>b_n t^{1/\alpha}\right)dt\\
&\leqslant C\sum_{n=1}^{\infty}\frac{1}{n}\int_1^{\infty}\frac{1}{b_n^2 t^{2/\alpha}}\left(\sum_{i=1}^{n}E|a_{ni}X|^2 I\left(|a_{ni}X|\leqslant b_n\right)\right)dt\\
&\quad +C\sum_{n=1}^{\infty}\frac{1}{n}\int_1^{\infty}\frac{1}{b_n^2 t^{2/\alpha}}\left(\sum_{i=1}^{n}E|a_{ni}X|^2 I\left(b_n<|a_{ni}X|\leqslant b_n t^{1/\alpha}\right)\right)dt\\
&\quad +C\sum_{n=1}^{\infty}\frac{1}{n}\int_1^{\infty}\sum_{i=1}^{n}P\left(|a_{ni}X|>b_n t^{1/\alpha}\right)dt\\
&= J_{11}+J_{12}+J_{13}.
\end{aligned}\tag{2.26}$$

For $1<\alpha\leqslant 2$ and Lemma 2.7, note that

$$\begin{aligned}
J_{11} &= \sum_{n=1}^{\infty}\frac{1}{n}\int_1^{\infty}\frac{1}{b_n^2 t^{2/\alpha}}\left(\sum_{i=1}^{n}E|a_{ni}X|^2 I\left(|a_{ni}X|\leqslant b_n\right)\right)dt\\
&\leqslant \sum_{n=1}^{\infty}\frac{1}{n}\frac{1}{b_n^{\alpha}}\left(\sum_{i=1}^{n}E|a_{ni}X|^{\alpha} I\left(|a_{ni}X|\leqslant b_n\right)\right)<\infty.
\end{aligned}\tag{2.27}$$

Letting $t=x^{\alpha}$. By (2.3) of Lemma 2.3, the Markov's inequality and Lemma 2.5, we have

$$J_{12} = \sum_{n=1}^{\infty}\frac{1}{n}\int_1^{\infty}\frac{1}{b_n^2 t^{2/\alpha}}\left(\sum_{i=1}^{n}E|a_{ni}X|^2 I\left(b_n<|a_{ni}X|\leqslant b_n t^{1/\alpha}\right)\right)dt$$

$$
\begin{aligned}
&\leqslant C\sum_{n=1}^{\infty}\frac{1}{nb_n^2}\int_1^{\infty}x^{\alpha-3}\sum_{i=1}^{n}E|a_{ni}X|^2I(b_n<|a_{ni}X|\leqslant b_nx)dx\\
&\leqslant C\sum_{n=1}^{\infty}\frac{1}{nb_n^2}\sum_{m=1}^{\infty}\int_m^{m+1}x^{\alpha-3}\sum_{i=1}^{n}E|a_{ni}X|^2I(b_n<|a_{ni}X|\leqslant b_nx)dx\\
&\leqslant C\sum_{n=1}^{\infty}\frac{1}{nb_n^2}\sum_{m=1}^{\infty}m^{\alpha-3}\sum_{i=1}^{n}E|a_{ni}X|^2I(b_n<|a_{ni}X|\leqslant b_n(m+1))\\
&= C\sum_{n=1}^{\infty}\frac{1}{nb_n^2}\sum_{i=1}^{n}\sum_{m=1}^{\infty}\sum_{s=1}^{m}m^{\alpha-3}E|a_{ni}X|^2I(b_ns<|a_{ni}X|\leqslant b_n(s+1))\\
&= C\sum_{n=1}^{\infty}\frac{1}{nb_n^2}\sum_{i=1}^{n}\sum_{s=1}^{\infty}E|a_{ni}X|^2I(b_ns<|a_{ni}X|\leqslant b_n(s+1))\sum_{m=s}^{\infty}m^{\alpha-3}\\
&\leqslant C\sum_{n=1}^{\infty}\frac{1}{nb_n^2}\sum_{i=1}^{n}\sum_{s=1}^{\infty}E|a_{ni}X|^2I(b_ns<|a_{ni}X|\leqslant b_n(s+1))s^{\alpha-2}\\
&\leqslant C\sum_{n=1}^{\infty}\frac{1}{nb_n^{\alpha}}\sum_{i=1}^{n}E|a_{ni}X|^{\alpha}I(|a_{ni}X|>b_n)\\
&\leqslant CE|X|^{\alpha}/(\log(1+|X|))^{\alpha/\gamma-1}<\infty. \qquad (2.28)
\end{aligned}
$$

Analogous to the proof of Lemma 2.5, it follows that

$$
\begin{aligned}
J_{13}&=\sum_{n=1}^{\infty}\frac{1}{n}\int_1^{\infty}\sum_{i=1}^{n}P\left(|a_{ni}X|>b_nt^{1/\alpha}\right)dt\\
&\leqslant CE|X|^{\alpha}/(\log(1+|X|))^{\alpha/\gamma-1}<\infty. \qquad (2.29)
\end{aligned}
$$

Hence, the desired result $J_1<\infty$ follows immediately. The proof of Theorem 2.1 is completed. □

REMARK 2.1. Under the conditions of Theorem 2.1, it is easy to check that

$$
\begin{aligned}
\infty&>\sum_{n=1}^{\infty}\frac{1}{n}E\left(\frac{1}{b_n}\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}a_{ni}X_i\right|-\varepsilon\right)_+^{\alpha}\\
&=\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\infty}P\left(\frac{1}{b_n}\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}a_{ni}X_i\right|-\varepsilon>t^{1/\alpha}\right)dt\\
&\geqslant\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\varepsilon^{\alpha}}P\left(\frac{1}{b_n}\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}a_{ni}X_i\right|>\varepsilon+t^{1/\alpha}\right)dt\\
&=\varepsilon^{\alpha}\sum_{n=1}^{\infty}\frac{1}{n}P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}a_{ni}X_i\right|>2\varepsilon b_n\right). \qquad (2.30)
\end{aligned}
$$

Since $\varepsilon>0$ is arbitrary. Therefore, from (2.30), we obtain that the complete moment convergence implies the complete convergence. Compared with the corresponding ones of Sung [18], Chen and Sung [6], it is worth pointing out that the main result is an extension and improvement under the same moment condition.

Taking $a_{ni} = a_i$ in Theorem 2.1, we can get the following result.

COROLLARY 2.1. *Let $\{X_n; n \geqslant 1\}$ be a mean zero sequence of NA random variables which is stochastically dominated by a random variable X, let $\{a_n; n \geqslant 1\}$ be a sequence of real constants such that $\sum_{i=1}^{n} |a_i|^\alpha = O(n)$ for $0 < \alpha \leqslant 2$. Set $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ for $0 < \gamma < \alpha$ with $1 < \alpha \leqslant 2$. If $E|X|^\alpha/(\log(1+|X|))^{\alpha/\gamma-1} < \infty$, then*

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\max_{1 \leqslant j \leqslant n} \left| \sum_{i=1}^{j} a_i X_i \right| > \varepsilon b_n \right) < \infty \quad \textit{for} \quad \forall \varepsilon > 0, \tag{2.31}$$

and

$$\frac{1}{b_n} \sum_{i=1}^{n} a_i X_i \to 0 \quad a.s., \quad n \to \infty. \tag{2.32}$$

Proof of Corollary 2.1. By Remark 2.1, it is easily to show that

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\max_{1 \leqslant j \leqslant n} \left| \sum_{i=1}^{j} a_i X_i \right| > \varepsilon b_n \right) < \infty \quad \text{for} \quad \forall \varepsilon > 0,$$

which implies (2.32) by a standard computation method (see for example, Lemma 2.4 in Sung [17]). $\square$

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