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QUANTUM SYMMETRIC ANALOGUE OF VARIOUS INTEGRAL INEQUALITIES OVER FINITE INTERVALS

MOHD BILAL, AKHLAD IQBAL* AND SACHIN RASTOGI

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Abstract. In this paper, we generalize the concept of Quantum calculus on finite intervals called symmetric quantum calculus over finite intervals. We define q_s-symmetric derivative and q_s-symmetric integral for a real valued function. We also prove quantum symmetric analogue of some integral inequalities over finite interval.

1. Introduction

In modern times quantum calculus is known as calculus without limits. In the beginning of twentieth century, Jackson [4, 3] started the serious work on q-calculus although previously Euler and Jacobi also worked on it. Nowadays, due to its vast applications on quantum computing, quantum mechanics and physics, it attracted many researchers, see [6, 11]. Quantum symmetric calculus was first discussed in the book of Kac and Cheung [5]. After that Brito da Cruz et al [2] did some work on q-symmetric variational calculus. Sun et. al. [8] introduced fractional q-symmetric calculus and studied some of its basic properties. Later, Tariboon et. al. [9] initiated the study of quantum calculus on finite interval and proved its basic properties.

In this paper, we extend the work of Toriboon et. al. [9, 10] to the q-symmetric calculus on finite interval and investigate some of its basic properties like symmetric derivarive of sum, product and quotient of two functions. We also define the q_s-integral over finite interval. Furthermore, we prove the q_s-symmetric analogue of some classical integral inequalities over finite interval.

2. Preliminaries

For the basic definitions and properties of q-symmetric calculus, see [2, 5].

Let $q \in (0,1)$ and let I be any interval of $\mathbb{R}$ containing 0, and denote by I_q the set

$$I_q = qI = \{qX : X \in I\}; \;\; I_q \subseteq I$$

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* Corresponding author.

DEFINITION 1. [5] Let $\phi : I \to \mathbb{R}$. The q-symmetric difference operator of ϕ is defined by

$$(\tilde{D}_q\phi)(t) = \frac{\phi(qt) - \phi(q^{-1}t)}{(q - q^{-1})t}; \quad t \in I_q - \{0\}$$

and

$$(\tilde{D}_q\phi)(t) = \phi'(0),\ t = 0,$$

provided ϕ is differentiable at 0. $\tilde{D}_q\phi$ is called the q -symmetric derivative of ϕ.

If ϕ is differentiable at $t \in I_q$, then

$$\lim_{q \to 1}(\tilde{D}_q\phi)(t) = \phi'(t).$$

THEOREM 1. [5] *Suppose that ϕ and g be q-symmetric differentiable on I, $\alpha, \beta \in \mathbb{R}$ and $t \in I_q$. Then*

1. $(\tilde{D}_q\phi)(t) = 0$ *iff* ϕ *is constant.*
2. $[\tilde{D}_q(\alpha\phi + \beta g)](t) = \alpha(\tilde{D}_q\phi)(t) + \beta(\tilde{D}_q g)(t)$
3. $(\tilde{D}_q\phi g)(t) = (\tilde{D}_q\phi)(t)g(qt) + \phi(q^{-1}t)(\tilde{D}_q g)(t)$
4. $\left(\tilde{D}_q(\frac{\phi}{g})\right)(t) = \frac{(\tilde{D}_q\phi)(t)g(q^{-1}t) - \phi(q^{-1}t)(\tilde{D}_q g)(t)}{g(qt)g(q^{-1}t)}$, *if* $g(qt)g(q^{-1}t) \neq 0$.

DEFINITION 2. [5] Suppose that $a,\ b \in I$ and $a < b$. For $\phi : I \to \mathbb{R}$ and for $q \in (0,1)$, the q-symmetric integral of ϕ is given by

$$\int_a^b \phi(t)\tilde{d}_q t = \int_0^b \phi(t)\tilde{d}_q t - \int_0^a \phi(t)\tilde{d}_q t$$

where

$$\int_0^x \phi(t)\tilde{d}_q t = x(1 - q^2)\sum_{n=0}^{\infty} q^{2n}\phi(q^{2n+1}x); \quad x \in I$$

provided that the series converges at $x = a$ and $x = b$.

3. Symmetric quantum calculus on finite interval

Now, we extend the concepts of q-symmetric derivative and q-symmetric integral on finite intervals. For a fixed $s \in \mathbb{N} \cup \{0\}$, suppose $J_s = [t_s, t_{s+1}] \subset \mathbb{R}$ be an interval containing 0 and $0 < q_s < 1$ be a constant. The q_s-symmetric derivative of a function $\phi : J_s \to \mathbb{R}$, at a point $t \in J_s$ is defined as follows:

DEFINITION 3. Let $\phi : J_s \to \mathbb{R}$ be continuous and let $t \in J_s$, then the expression

$$(D_{q_s}\phi)(t) = \frac{\phi(q_s^{-1}t + (1 - q_s^{-1})t_s) - \phi(q_s t + (1 - q_s)t_s)}{(q_s^{-1} - q_s)(t - t_s)}; \quad t \neq t_s$$

$$(D_{q_s}\phi)(t_s) = \lim_{t \to t_s}(D_{q_s}\phi)(t)$$

is called the q_s-symmetric derivative of ϕ at t.

If $t_s = 0$ and $q_s = q$, then

$$D_{q_s}\phi = \tilde{D}_q\phi.$$

Now, we present some basic properties of q_s-symmetric derivatives as follows:

THEOREM 2. *Suppose that* $\phi,\ g : J_s \to R$ *be* q_s*-symmetric differentiable on* J_s*. Then*

(i) The sum $\phi + g : J_s \to R$ *is* q_s*-symmetric differentiable on* J_s *with*

$$D_{q_s}[\phi(t) + g(t)] = D_{q_s}\phi(t) + D_{q_s}g(t)$$

(ii) For any constant β*,* $\phi : J_s \to R$ *is* q_s*-symmetric differentiable on* J_s *with*

$$D_{q_s}(\beta\phi)(t) = \beta D_{q_s}\phi(t)$$

(iii) The product $\phi g : J_s \to R$ *is* q_s*-symmetric differentiable on* J_s *with*

$$D_{q_s}[\phi g](t) = \phi[q_s^{-1}t + (1-q_s^{-1})t_s]D_{q_s}g(t) + g[q_st + (1-q_s)t_s]D_{q_s}\phi(t)$$

or,

$$D_{q_s}[\phi g](t) = g[q_s^{-1}t + (1-q_s^{-1})t_s]D_{q_s}\phi(t) + \phi[q_st + (1-q_s)t_s]D_{q_s}g(t)$$

(iv) The quotient $\frac{\phi}{g} : J_s \to R$ *is* q_s*-symmetric differentiable on* J_s *with*

$$D_{q_s}\left[\frac{\phi}{g}\right](t) = \frac{g[q_st + (1-q_s)t_s]D_{q_s}\phi(t) - \phi[q_st + (1-q_s)t_s]D_{q_s}g(t)}{g[q_s^{-1}t + (1-q_s^{-1})t_s]g[q_st + (1-q_s)t_s]}$$

provided

$$g[q_s^{-1}t + (1-q_s^{-1})t_s]g[q_st + (1-q_s)t_s] \neq 0$$

Proof. The proofs of (i) and (ii) are straightforward.

(iii) We have

$$\begin{aligned}
&D_{q_s}[\phi g](t)\\
&= \frac{\phi[q_s^{-1}t+(1-q_s^{-1})t_s]g[q_s^{-1}t+(1-q_s^{-1})t_s]-\phi[q_st+(1-q_s)t_s]g[q_st+(1-q_s)t_s]}{(q_s^{-1}-q_s)(t-t_s)}\\
&= \frac{\phi[q_s^{-1}t+(1-q_s^{-1})t_s]g[q_s^{-1}t+(1-q_s^{-1})t_s]-\phi[q_st+(1-q_s)t_s]g[q_st+(1-q_s)t_s]}{(q_s^{-1}-q_s)(t-t_s)}\\
&\quad+\frac{\phi[q_s^{-1}t+(1-q_s^{-1})t_s]g[q_st+(1-q_s)t_s]-\phi[q_s^{-1}t+(1-q_s^{-1})t_s]g[q_st+(1-q_s)t_s]}{(q_s^{-1}-q_s)(t-t_s)}\\
&= \phi[q_s^{-1}t+(1-q_s^{-1})t_s]\left[\frac{g[q_s^{-1}t+(1-q_s^{-1})t_s]-g[q_st+(1-q_s)t_s]}{(q_s^{-1}-q_s)(t-t_s)}\right]\\
&\quad+g[q_st+(1-q_s)t_s]\left[\frac{\phi[q_s^{-1}t+(1-q_s^{-1})t_s]-\phi[q_st+(1-q_s)t_s]}{(q_s^{-1}-q_s)(t-t_s)}\right].
\end{aligned}$$

After arranging the terms, we get

$$D_{q_s}[\phi g](t) = \phi[q_s^{-1}t + (1-q_s^{-1})t_s]D_{q_s}g(t) + g[q_st + (1-q_s)t_s]D_{q_s}\phi(t).$$

By inter-changing the functions ϕ and g, we can obtain the other equation of part (iii).

(iv) We have

$$\begin{aligned}
&D_{q_s}\left[\frac{\phi}{g}\right](t) \\
&= \frac{\dfrac{\phi[q_s^{-1}t + (1-q_s^{-1})t_s]}{g[q_s^{-1}t + (1-q_s^{-1})t_s]} - \dfrac{\phi[q_st + (1-q_s)t_s]}{g[q_st + (1-q_s)t_s}}{(q_s^{-1} - q_s)(t-t_s)} \\
&= \frac{\phi[q_s^{-1}t+(1-q_s^{-1})t_s]g[q_st+(1-q_s)t_s] - \phi[q_st+(1-q_s)t_s]g[q_s^{-1}t+(1-q_s^{-1})t_s]}{g[q_s^{-1}t+(1-q_s^{-1})t_s]g[q_st+(1-q_s)t_s](q_s^{-1}-q_s)(t-t_s)}.
\end{aligned}$$

Adding and subtracting the term $\phi[q_st + (1-q_s)t_s]g[q_st + (1-q_s)t_s]$ in the numerator and after rearranging the terms, we obtained

$$D_{q_s}\left[\frac{\phi}{g}\right](t) = \frac{g[q_st + (1-q_s)t_s]D_{q_s}\phi(t) - \phi[q_st + (1-q_s)t_s]D_{q_s}g(t)}{g[q_s^{-1}t + (1-q_s^{-1})t_s]g[q_st + (1-q_s)t_s]}.$$

To construct the q_s-symmetric integral $\phi(t)$, a shifting operator is defined as

$$E_{q_s}\phi(t) = \phi(q_st + (1-q_s)t_s),$$

and

$$E_{q_s^{-1}}\phi(t) = \phi(q_s^{-1}t + (1-q_s^{-1})t_s).$$

Now,

$$\begin{aligned}
E_{q_s}^2\phi(t) &= E_{q_s}(E_{q_s}\phi(t)) \\
&= E_{q_s}\phi(q_st + (1-q_s)t_s) \\
&= \phi(q_s(q_st + (1-q_s)t_s) + (1-q_s)t_s) \\
&= \phi(q_s^2t + q_st_s - q_s^2t_s + t_s - t_sq_s) \\
&= \phi(q_s^2t + (1-q_s^2)t_s).
\end{aligned}$$

Using mathematical induction, we can prove that

$$E_{q_s}^n\phi(t) = \phi(q_s^nt + (1-q_s^n)t_s).$$

Let us compute

$$\begin{aligned}
E_{q_s}E_{q_s^{-1}}\phi(t) &= E_{q_s}(\phi(q_s^{-1}t + (1-q_s^{-1})t_s)) \\
&= \phi(q_s(q_s^{-1}t + (1-q_s^{-1})t_s) + (1-q_s)t_s) \\
&= \phi(t + q_st_s - t_s + t_s - q_st_s) \\
&= \phi(t).
\end{aligned}$$

Hence,

$$E_{q_s^{-1}} = \frac{1}{E_{q_s}}.$$

EXAMPLE 1. Let $\phi(t) = (t - t_s)^n$; $t \in J_s$. Then

$$\begin{aligned}
(D_{q_s}\phi)(t) &= \frac{(q_s^{-1}t + (1-q_s^{-1})t_s - t_s)^n - (q_s t + (1-q_s)t_s - t_s)^n}{(q_s^{-1} - q_s)(t - t_s)} \\
&= \frac{(q_s^{-1}t + t_s - t_s q_s^{-1} - t_s)^n - (q_s t + t_s - t_s q_s - t_s)^n}{(q_s^{-1} - q_s)(t - t_s)} \\
&= \frac{(q_s^{-1})^n (t - t_s)^n - (q_s)^n (t - t_s)^n}{(q_s^{-1} - q_s)(t - t_s)} \\
&= \frac{[(q_s^{-1})^n - (q_s)^n](t - t_s)^{n-1}}{(q_s^{-1} - q_s)} \\
&= [n]_{q_s}(t - t_s)^{n-1},
\end{aligned}$$

where $[n]_{q_s} = \dfrac{(q_s^{-1})^n - (q_s)^n}{(q_s^{-1} - q_s)}$.

In q-symmetric calculus,

$$\tilde{D}_q t^n = [n]_q (t)^{n-1},$$

where $[n]_q = \dfrac{(q^{-1})^n - (q)^n}{(q^{-1} - q)}$, [see Equation 26.4 and 26.7 from [5]].

Now, from the definition of q_s-symmetric derivative, we have

$$\frac{(E_{q_s^{-1}} - E_{q_s})\phi(t)}{(q_s^{-1} - q_s)(t - t_s)} = \psi(t)$$

Therefore, the q_s-symmetric integral can be expressed as

$$\phi(t) = \frac{(q_s^{-1} - q_s)(t - t_s)\psi(t)}{(E_{q_s^{-1}} - E_{q_s})}$$

$$\begin{aligned}
\phi(t) &= \frac{(q_s^{-1} - q_s)E_{q_s}(t - t_s)\psi(t)}{(1 - E_{q_s}^2)} \\
&= (q_s^{-1} - q_s)E_{q_s}(1 - E_{q_s}^2)^{-1}(t - t_s)\psi(t) \\
&= (q_s^{-1} - q_s)(E_{q_s}^1 + E_{q_s}^3 + E_{q_s}^5 + \ldots)(t - t_s)\psi(t) \\
&= (q_s^{-1} - q_s)\sum_{n=0}^{\infty} E_{q_s}^{2n+1}(t - t_s)\psi(t) \\
&= (q_s^{-1} - q_s)\sum_{n=0}^{\infty} (q_s^{2n+1}t + (1 - q_s^{2n+1})t_s - t_s)\psi(q_s^{2n+1}t + (1 - q_s^{2n+1})t_s)
\end{aligned}$$

$$= (t-t_s)(q_s^{-1}-q_s)\sum_{n=0}^{\infty} q_s^{2n+1}\psi(q_s^{2n+1}t+(1-q_s^{2n+1})t_s)$$
$$= (t-t_s)(1-q_s^2)\sum_{n=0}^{\infty} q_s^{2n}\psi(q_s^{2n+1}t+(1-q_s^{2n+1})t_s).$$

DEFINITION 4. Let $\phi : J_s \to R$ be a continuous function, then q_s-symmetric integral is defined by

$$\int_{t_s}^{t}\phi(s)d_{q_s}s = (t-t_s)(1-q_s^2)\sum_{n=0}^{\infty} q_s^{2n}\phi(q_s^{2n+1}t+(1-q_s^{2n+1})t_s).$$

It is to be noted that for $t_s = 0$ and $q_s = q$, the definition 4 is equivalent to Definition 2.

4. q_s-symmetric analogue of some integral inequalities

Now, we prove q_s-symmetric analogue of some integral inequalities.

THEOREM 3. [q_s-symmetric analogue of Hermite-Hadamard inequality] *Suppose that* $\phi : [a,b] \to \mathbb{R}$ *be a convex differentiable function on* (a,b) *and* $0 < q_s < 1$. *Then*

$$\phi\Big(\frac{q_s^2a+b}{1+q_s^2}\Big)+\frac{(q_s-1)(b-a)}{(1+q_s^2)}\phi'\Big(\frac{q_s^2a+b}{1+q_s^2}\Big)$$
$$\leqslant \frac{1}{(b-a)}\int_a^b \phi(x)d_{q_s}x \leqslant \frac{(1+q_s^2-q_s)\phi(a)+q_s\phi(b)}{(1+q_s^2)}.$$

Proof. Given that ϕ is a differentiable function on $[a,b]$, hence there exists a tangent line for the function ϕ at the point $\frac{q_s^2a+b}{1+q_s^2} \in (a,b)$. It can be expressed as

$$h(x) = \phi\Big(\frac{q_s^2a+b}{1+q_s^2}\Big)+\phi'\Big(\frac{q_s^2a+b}{1+q_s^2}\Big)\Big(x-\frac{q_s^2a+b}{1+q_s^2}\Big)$$

Since, ϕ is convex on [a,b], we have

$$h(x) \leqslant \phi(x), \quad \forall x \in [a,b].$$

Now, q_s-symmetric integrate the above inequality on $[a,b]$, we get

$$\int_a^b h(x)d_{q_s}x$$
$$= \int_a^b \Big\{\phi\Big(\frac{q_s^2a+b}{1+q_s^2}\Big)+\phi'\Big(\frac{q_s^2a+b}{1+q_s^2}\Big)\Big(x-\frac{q_s^2a+b}{1+q_s^2}\Big)\Big\}d_{q_s}x$$
$$= (b-a)\phi\Big(\frac{q_s^2a+b}{1+q_s^2}\Big)+\phi'\Big(\frac{q_s^2a+b}{1+q_s^2}\Big)\int_a^b\Big(x-\frac{q_s^2a+b}{1+q_s^2}\Big)d_{q_s}x$$
$$= (b-a)\phi\Big(\frac{q_s^2a+b}{1+q_s^2}\Big)+\phi'\Big(\frac{q_s^2a+b}{1+q_s^2}\Big)\Big[\int_a^b x d_{q_s}x-(b-a)\frac{q_s^2a+b}{1+q_s^2}\Big]$$

$$\begin{aligned}\int_a^b x d_{q_s} x &= (1-q_s^2)(b-a)\sum_{n=0}^{\infty} q_s^{2n}[(q_s^{2n+1})b+(1-q_s^{2n+1})a]\\ &= (1-q_s^2)(b-a)\Big[b\sum_{n=0}^{\infty}(q_s^{4n+1})+a\Big[\sum_{n=0}^{\infty}q_s^{2n}-\sum_{n=0}^{\infty}q_s^{4n+1}\Big]\Big]\\ &= (1-q_s^2)(b-a)\Big[\frac{bq_s}{1-q_s^4}+a\Big[\frac{1}{1-q_s^2}-\frac{q_s}{1-q_s^4}\Big]\Big]\\ &= (1-q_s^2)(b-a)\Big[\frac{bq_s+a(1+q_s^2)-aq_s)}{(1-q_s^4)}\Big]\\ &= \frac{(b-a)}{(1+q_s^2)}[(b-a)q_s+a(1+q_s^2)].\end{aligned}$$

Now,

$$\begin{aligned}&\int_a^b h(x) d_{q_s} x\\ &= (b-a)\phi\Big(\frac{q_s^2a+b}{1+q_s^2}\Big)+\phi'\Big(\frac{q_s^2a+b}{1+q_s^2}\Big)(b-a)\Big[\frac{(b-a)q_s+a(1+q_s^2)}{1+q_s^2}-\frac{q_s^2a+b}{1+q_s^2}\Big]\\ &= (b-a)\Big[\phi\Big(\frac{q_s^2a+b}{1+q_s^2}\Big)+\phi'\Big(\frac{q_s^2a+b}{1+q_s^2}\Big)\Big[\frac{bq_s-aq_s+a+q_s^2a-q_s^2a-b}{1+q_s^2}\Big]\Big]\\ &= (b-a)\Big[\phi\Big(\frac{q_s^2a+b}{1+q_s^2}\Big)+\phi'\Big(\frac{q_s^2a+b}{1+q_s^2}\Big)\frac{(q_s-1)(b-a)}{1+q_s^2}\Big]\end{aligned}$$

which implies

$$\phi\Big(\frac{q_s^2a+b}{1+q_s^2}\Big)+\frac{(q_s-1)(b-a)}{(1+q_s^2)}\phi'\Big(\frac{q_s^2a+b}{1+q_s^2}\Big)\leqslant\frac{1}{(b-a)}\int_a^b\phi(x)d_{q_s}x.$$

Although the line containing points $(a,\phi(a))$ and $(b,\phi(b))$ can be expressed as

$$\phi(x)\leqslant k(x)=\phi(a)+\frac{\phi(b)-\phi(a)}{b-a}(x-a),\forall x\in[a,b]$$

Now, q_s-symmetric integrate the above inequality, we get

$$\begin{aligned}\int_a^b\phi(x)d_{q_s}x &\leqslant \int_a^b\phi(a)d_{q_s}x+\frac{\phi(b)-\phi(a)}{b-a}\int_a^b(x-a)d_{q_s}x\\ &= (b-a)\phi(a)+\frac{\phi(b)-\phi(a)}{b-a}\Big[\int_a^b x d_{q_s}x-a(b-a)\Big]\\ &= (b-a)\phi(a)+\frac{\phi(b)-\phi(a)}{b-a}\Big[\frac{(b-a)[(b-a)q_s+a(1+q_s^2)]}{(1+q_s^2)}-a(b-a)\Big]\\ &= (b-a)\phi(a)+[\phi(b)-\phi(a)]\Big[\frac{(b-a)q_s+a(1+q_s^2)-a(1+q_s^2)}{(1+q_s^2)}\Big]\end{aligned}$$

$$= (b-a)\phi(a) + \frac{[\phi(b)-\phi(a)](b-a)q_s}{(1+q_s^2)}$$
$$= (b-a)\Big[\frac{\phi(a)+\phi(a)q_s^2+q_s\phi(b)-q_s\phi(a)}{(1+q_s^2)}\Big]$$
$$= (b-a)\Big[\frac{(1+q_s^2-q_s)\phi(a)+q_s\phi(b)}{(1+q_s^2)}\Big]$$

$$\frac{1}{b-a}\int_a^b \phi(x)d_{q_s}x \leqslant \frac{(1+q_s^2-q_s)\phi(a)+q_s\phi(b)}{(1+q_s^2)}.$$

Finally, we have

$$\phi\Big(\frac{q_s^2a+b}{1+q_s^2}\Big) + \frac{(q_s-1)(b-a)}{(1+q_s^2)}\phi'\Big(\frac{q_s^2a+b}{1+q_s^2}\Big) \leqslant \frac{1}{(b-a)}\int_a^b \phi(x)d_{q_s}x \leqslant \frac{(1+q_s^2-q_s)\phi(a)+q_s\phi(b)}{(1+q_s^2)}. \quad \square$$

REMARK 1. If we take $q_s = 1$, we obtain classical Hermite-Hadamard inequality

$$\phi\Big(\frac{a+b}{2}\Big) \leqslant \frac{1}{(b-a)}\int_a^b \phi(x)dx \leqslant \frac{\phi(a)+\phi(b)}{2}$$

THEOREM 4. [q_s-symmetric analogue of Holder's inequality] *Let $x \in J = [a,b]$, $0 < q_s < 1$, $p_1, p_2 > 1$ such that $\frac{1}{p_1}+\frac{1}{p_2} = 1$. Then, we have*

$$\int_a^x |\phi(t)|\ |g(t)|d_{q_s}t \ \leqslant \Big(\int_a^x |\phi(t)|^{p_1}d_{q_s}t\Big)^{\frac{1}{p_1}}\Big(\int_a^x |g(t)|^{p_2}d_{q_s}t\Big)^{\frac{1}{p_2}}$$

Proof. From the definition of q_s-symmetric integral, we have

$$\begin{aligned}
&\int_a^x |\phi(t)||g(t)|d_{q_s}t \\
&= (1-q_s^2)(x-a)\sum_{n=0}^{\infty} q_s^{2n}|\phi(q_s^{2n+1}x+(1-q_s^{2n+1})a)| \\
&\quad \times |g(q_s^{2n+1}x+(1-q_s^{2n+1})a)| \\
&= (1-q_s^2)^{\frac{1}{p_1}+\frac{1}{p_2}}(x-a)^{\frac{1}{p_1}+\frac{1}{p_2}}\sum_{n=0}^{\infty}(q_s^{2n})^{\frac{1}{p_1}+\frac{1}{p_2}}|\phi(q_s^{2n+1}x+(1-q_s^{2n+1})a)| \\
&\quad \times |g(q_s^{2n+1}x+(1-q_s^{2n+1})a)| \\
&\leqslant \Big[(1-q_s^2)(x-a)\sum_{n=0}^{\infty} q_s^{2n}|\phi(q_s^{2n+1}x+(1-q_s^{2n+1})a)|^{p_1}\Big]^{\frac{1}{p_1}} \\
&\quad \times \Big[(1-q_s^2)(x-a)\sum_{n=0}^{\infty} q_s^{2n}|g(q_s^{2n+1}x+(1-q_s^{2n+1})a)|^{p_2}\Big]^{\frac{1}{p_2}} \\
&= \Big(\int_a^x |\phi(t)|^{p_1}d_{q_s}t\Big)^{\frac{1}{p_1}}\Big(\int_a^x |g(t)|^{p_2}d_{q_s}t\Big)^{\frac{1}{p_2}}. \quad \square
\end{aligned}$$

THEOREM 5. [q_s-symmetric analogue of Ostrowski inequality] *Let $\phi : J = [a,b] \to \mathbb{R}$ be a q_s-symmetric differentiable function with $D_{q_s}\phi$ is continuous on $[a,b]$ and $0 < q_s < 1$, then*

$$|\phi(x) - \frac{1}{b-a}\int_a^b \phi(t)d_{q_s}t| \leqslant \frac{||D_{q_s}\phi||}{(b-a)}\Bigg[\Big(x - \frac{a(1-q_s+{q_s}^2)+bq_s}{1+{q_s}^2}\Big)^2 + \frac{a^2(1-q_s+{q_s}^2)+b^2q_s}{1+{q_s}^2} - \Big(\frac{a(1-q_s+{q_s}^2)+bq_s}{1+{q_s}^2}\Big)^2\Bigg].$$

Proof. We have

$$\begin{aligned}
&\left|\phi(x) - \frac{1}{b-a}\int_a^b \phi(t)d_{q_s}t\right| \\
&= \left|\frac{1}{b-a}\int_a^b \phi(x) - \phi(t)d_{q_s}t\right| \\
&\leqslant \frac{1}{b-a}\int_a^b |\phi(x) - \phi(t)|d_{q_s}t \\
&\leqslant \frac{||D_{q_s}\phi||}{b-a}\int_a^b |x-t|d_{q_s}t \quad \text{(By the Lagrange mean value theorem)} \\
&= \frac{||D_{q_s}\phi||}{b-a}\Big[\int_a^x (x-t)d_{q_s}t + \int_x^b (t-x)d_{q_s}t\Big] \\
&= \frac{||D_{q_s}\phi||}{b-a}\Big[\frac{(x-a)^2}{1+q_s^2}(1-q_s+{q_s}^2) + \frac{q_s(b-x)^2}{1+q_s^2}\Big] \\
&= \frac{||D_{q_s}\phi||}{b-a}\Big[\frac{(x-a)^2(1-q_s+{q_s}^2)+q_s(b-x)^2}{1+{q_s}^2}\Big] \\
&= \frac{||D_{q_s}\phi||}{(b-a)}\Big[\Big(x - \frac{a(1-q_s+{q_s}^2)+bq_s}{1+{q_s}^2}\Big)^2 + \frac{a^2(1-q_s+{q_s}^2)+b^2q_s}{1+{q_s}^2} \\
&\quad - \Big(\frac{a(1-q_s+{q_s}^2)+bq_s}{1+{q_s}^2}\Big)^2\Big]. \quad \square
\end{aligned}$$

REMARK 2. If we take $q = 1$, then the above inequality reduces to classical Ostrowski inequality

$$\left|\phi(x) - \frac{1}{b-a}\int_a^b \phi(t)dt\right| \leqslant \frac{||D\phi||}{b-a}\Big[\frac{(x-a)^2+(x-b)^2}{2}\Big].$$

LEMMA 1. [q_s-symmetric analogue of Korkine identity] *Let ϕ and $\psi : J \to \mathbb{R}$ be a continuous on J and $0 < q < 1$, then we have*

$$\begin{aligned}
&\frac{1}{2}\int_a^b\int_a^b (\phi(x)-\phi(y))(\psi(x)-\psi(y))d_{q_s}x d_{q_s}y \\
&= (b-a)\int_a^b \phi(x)\psi(x)d_{q_s}x - \Big(\int_a^b \phi(x)d_{q_s}x\Big)\Big(\int_a^b \psi(x)d_{q_s}x\Big).
\end{aligned}$$

Proof. From the Definition 4 of q_s-symmetric integral, we have

$$\begin{aligned}
&\int_a^b\int_a^b(\phi(x)-\phi(y))(\psi(x)-\psi(y))d_{q_s}x\, d_{q_s}y\\
&=\int_a^b\int_a^b\Big[\phi(x)\psi(x)-\phi(x)\psi(y)-\phi(y)\psi(x)+\phi(y)\psi(y)\Big]d_{q_s}xd_{q_s}y\\
&=(b-a)^2(1-q_s^2)\sum_{n=0}^{\infty}q_s^{2n}\phi[q_s^{2n+1}b+(1-q_s^{2n+1}a)]\psi[q_s^{2n+1}b+(1-q_s^{2n+1}a)]\\
&\quad-(b-a)^2(1-q_s^2)^2\sum_{n=0}^{\infty}q_s^{2n}\phi[q_s^{2n+1}b+(1-q_s^{2n+1}a)]\sum_{n=0}^{\infty}q_s^{2n}\psi[q_s^{2n}b+(1-q_s^{2n}a)]\\
&\quad-(b-a)^2(1-q_s^2)^2\sum_{n=0}^{\infty}q_s^{2n}\phi[q_s^{2n+1}b+(1-q_s^{2n+1}a)]\sum_{n=0}^{\infty}q_s^{2n}\psi[q_s^{2n+1}b+(1-q_s^{2n+1}a)]\\
&\quad+(b-a)^2(1-q_s^2)\sum_{n=0}^{\infty}q_s^{2n}\phi[q_s^{2n+1}b+(1-q_s^{2n+1}a)]\psi[q_s^{2n+1}b+(1-q_s^{2n+1}a)]\\
&=2(b-a)\int_a^b\phi(x)\psi(x)d_{q_s}x-2\Big(\int_a^b\phi(x)d_{q_s}x\Big)\Big(\int_a^b\psi(x)d_{q_s}x\Big).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{1}{2}\int_a^b\int_a^b(\phi(x)-\phi(y))(\psi(x)-\psi(y))d_{q_s}xd_{q_s}y\\
&=(b-a)\int_a^b\phi(x)\psi(x)d_{q_s}x-\Big(\int_a^b\phi(x)d_{q_s}x\Big)\Big(\int_a^b\psi(x)d_{q_s}x\Big).\quad\Box
\end{aligned}$$

LEMMA 2. q_s-symmetric analogue of Cauchy-Bunyakovsky-Schwarz integral inequality for double integrals *Let ϕ and ψ: $J\to\mathbb{R}$ be a continuous on J and* $0<q<1$, *then we have*

$$\begin{aligned}
&\Big|\int_a^b\int_a^b\phi(x,y)\psi(x,y)d_{q_s}x\, d_{q_s}y\Big|\\
&\leqslant\Big[\int_a^b\int_a^b\phi^2(x,y)d_{q_s}x\, d_{q_s}y\Big]^{\frac{1}{2}}\Big[\int_a^b\int_a^b\psi^2(x,y)d_{q_s}x\, d_{q_s}y\Big]^{\frac{1}{2}}.
\end{aligned}$$

Proof. From the Definition 4 of q_s-symmetric integral, we have the double q_s-integral on J as

$$\begin{aligned}
&\Big[\int_a^b\int_a^b\phi(x,y)\psi(x,y)d_{q_s}x\, d_{q_s}y\Big]^2\\
&=\Big[(1-q_s^2)^2(b-a)^2\sum_{n=0}^{\infty}\sum_{i=0}^{\infty}q_s^{2n+2i}\phi[q_s^{2n+1}b+(1-q_s^{2n+1})a,\ q_s^{2i+1}b+(1-q_s^{2i+1})a]\\
&\quad\times\psi[q_s^{2n+1}b+(1-q_s^{2n+1})a,\ q_s^{2i+1}b+(1-q_s^{2i+1})a]\Big]^2.
\end{aligned}$$

Applying the discrete Cauchy-Schwarz inequality, we get

$$\leqslant (1-q_s^2)^2(b-a)^2\sum_{n=0}^{\infty}\sum_{i=0}^{\infty} q_s^{2n+2i}\phi[q_s^{2n+1}b+(1-q_s^{2n+1})a,\ q_s^{2i+1}b+(1-q_s^{2i+1})a]$$
$$\times(1-q_s^2)^2(b-a)^2\sum_{n=0}^{\infty}\sum_{i=0}^{\infty} q_s^{2n+2i}\psi[q_s^{2n+1}b+(1-q_s^{2n+1})a,\ q_s^{2i+1}b+(1-q_s^{2i+1})a]$$
$$\leqslant \Big[\int_a^b\int_a^b \phi^2(x,y)d_{q_s}x\, d_{q_s}y\Big]\Big[\int_a^b\int_a^b \psi^2(x,y)d_{q_s}x\, d_{q_s}y\Big].$$

Hence,

$$\Big|\int_a^b\int_a^b \phi(x,y)\psi(x,y)d_{q_s}x\, d_{q_s}y\Big|$$
$$\leqslant \Big[\int_a^b\int_a^b \phi^2(x,y)d_{q_s}x\, d_{q_s}y\Big]^{\frac{1}{2}}\Big[\int_a^b\int_a^b \psi^2(x,y)d_{q_s}x\, d_{q_s}y\Big]^{\frac{1}{2}}. \quad \square$$

REMARK 3. If $q_s=1$ then both the above lemma are reduced to the usual Korkine identity and Cauchy-Bunyakovsky-Schwarz integral inequality respectively. For more details see [10, 11]

THEOREM 6. [q_s-symmetric analogue of *Grüss-Čebyšev* integral inequality] *Let ϕ and $\psi : J=[a,b]\to\mathbb{R}$ be L_1,L_2-Lipschitzian continuous function on $[a,b]$ so that*

$$|\phi(x)-\phi(y)|\leqslant L_1|x-y|;\quad |\psi(x)-\psi(y)|\leqslant L_2|x-y|$$

for all $x,y\in[a,b]$. Then, we have the following inequality

$$\Big|\frac{1}{b-a}\int_a^b \phi(x)\psi(x)d_{q_s}x-\Big(\frac{1}{b-a}\int_a^b \phi(x)d_{q_s}x\Big)\Big(\frac{1}{b-a}\int_a^b \psi(x)d_{q_s}x\Big)\Big|$$
$$\leqslant \frac{L_1L_2\, q_s^4(b-a)^2}{(1+q_s^2+q_s^4)(1+q_s^2)^2}.$$

Proof. From the Lipschitzian continuous function property, we set

$$|(\phi(x)-\phi(y))\ (\psi(x)-\psi(y))|\leqslant L_1L_2(x-y)^2$$

Now, the double q_s integral on both side of above inequality, we get

$$\int_a^b\int_a^b |(\phi(x)-\phi(y))\ (\psi(x)-\psi(y))|d_{q_s}xd_{q_s}y\leqslant L_1L_2\int_a^b\int_a^b (x-y)^2 d_{q_s}x\, d_{q_s}y$$
$$=L_1L_2\int_a^b\int_a^b (x^2-2xy+y^2)d_{q_s}x\, d_{q_s}y$$
$$=L_1L_2\Big[2(b-a)\int_a^b x^2 d_{q_s}x-2\Big(\int_a^b xd_{q_s}\Big)^2\Big]. \qquad (*)$$

From Example 1, we get q_s-anti-derivative or integral of $(x-a)^\alpha$ as follows:

$$\int (x-a)^\alpha d_{q_s}x = \frac{q_s^{-1}-q_s}{q_s^{-\alpha-1}-q_s^{\alpha+1}}(x-a)^{\alpha+1}$$

With the use of the above integral, we evaluate the following

$$\begin{aligned}\int_a^b x^2 d_{q_s}x &= \int_a^b (x-a+a)^2 d_{q_s}x\\ &= \frac{q_s^2}{(1+q_s^2+q_s^4)}(b-a)^3 + a^2(b-a) + \frac{2aq_s}{1+q_s^2}(b-a)^2\\ \int_a^b x\, d_{q_s}x &= \int_a^b (x-a+a) d_{q_s}x\\ &= \frac{q_s(1-q_s^2}{1-q_s^4}(b-a)^2 + a(b-a).\end{aligned}$$

Substituting these values of the integral in equation $(*)$, we get

$$\begin{aligned}&\int_a^b\int_a^b |(\phi(x)-\phi(y))(\psi(x)-\psi(y))| d_{q_s}x d_{q_s}y\\ &\leqslant 2L_1L_2\Big[(b-a)\Big(\frac{q_s^2}{(1+q_s^2+q_s^4)}(b-a)^3 + a^2(b-a) + \frac{2aq_s}{1+q_s^2}(b-a)^2\Big)\\ &\quad -\Big(\frac{q_s}{1+q_s^2}(b-a)^2 + a(b-a)\Big)^2\Big]\\ &= \frac{2L_1\, L_2\, q_s^4(b-a)^4}{(1+q_s^2+q_s^4)(1+q_s^2)^2}\end{aligned}$$

$$\frac{1}{2}\int_a^b\int_a^b |(\phi(x)-\phi(y))\ (\psi(x)-\psi(y))| d_{q_s}x\, d_{q_s}y \leqslant \frac{L_1\, L_2\, q_s^4(b-a)^4}{(1+q_s^2+q_s^4)(1+q_s^2)^2}.$$

Using q_s-Korkine identity

$$\begin{aligned}&\Big|(b-a)\int_a^b \phi(x)\psi(x) d_{q_s}x - \Big(\int_a^b \phi(x) d_{q_s}x\Big)\Big(\int_a^b \psi(x) d_{q_s}x\Big)\Big|\\ &= \frac{1}{2}\Big|\int_a^b\int_a^b (\phi(x)-\phi(y))\ (\psi(x)-\psi(y)) d_{q_s}x\, d_{q_s}y\Big|\\ &\leqslant \frac{L_1\, L_2\, q_s^4(b-a)^4}{(1+q_s^2+q_s^4)(1+q_s^2)^2}.\end{aligned}$$

On dividing throughout by $(b-a)^2$, we get

$$\begin{aligned}&\Big|\frac{1}{(b-a)}\int_a^b \phi(x)\psi(x) d_{q_s}x - \Big(\frac{1}{b-a}\int_a^b \phi(x) d_{q_s}x\Big)\Big(\frac{1}{b-a}\int_a^b \psi(x) d_{q_s}x\Big)\Big|\\ &\leqslant \frac{L_1\, L_2\, q_s^4(b-a)^2}{(1+q_s^2+q_s^4)(1+q_s^2)^2}. \quad \square\end{aligned}$$

REMARK 4. If $q_s = 1$ the the above inequality reduces to the Classical integral *Grüss-Čebyšev* as

$$\left|\frac{1}{(b-a)}\int_a^b \phi(x)\psi(x)dx - \left(\frac{1}{b-a}\int_a^b \phi(x)dx\right)\left(\frac{1}{b-a}\int_a^b \psi(x)dx\right)\right| \leqslant \frac{L_1\, L_2\, (b-a)^2}{12}.$$

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Mohd Bilal
Department of Mathematics
Umm Al Qura University
Makkah, Saudi Arabia
e-mail: mohd7bilal@gmail.com
mbghaffar@uqu.edu.sa

Akhlad Iqbal
Department of Mathematics
Aligarh Muslim University
Aligarh-202001, UP, India
e-mail: akhlad6star@gmail.com

Sachin Rastogi
Department of Mathematics, Hindu College
M.J.P. Rohilkhand University
Bareilly-243003, UP, India
e-mail: sachin.sachras@gmail.com