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ALMOST SURELY STABILITY OF DELAY HYBRID STOCHASTIC SYSTEM DRIVEN BY LÉVY NOISE

CHAO WEI

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Abstract. This study is devoted to investigate the almost sure stability of a class of nonlinear delay hybrid stochastic system driven by Lévy noise. We derive that the system has a unique global solution. Then, we discuss the almost sure stability of the stochastic system. A numerical example is provided to verify the results.

1. Introduction

In nature and production life, random phenomenon exists universally. For example, an error occurring in a scientific experiment and interference received during the transmission of a message. Many random phenomenon especially systems have been described by stochastic differential equations. Furthermore, systems are always influenced by noises. Hence, some authors has modelled the actual systems by stochastic systems ([1, 9, 14, 22]). However, due to the continuity of the Gaussian process, there is no advantage in describing instantaneous perturbation changes. Non-Gaussian Lévy noise can more accurately reflect the objective random disturbances in the system. In the last few years, Lévy noise has been utilized in financial, biological and medical fields ([3, 19, 23]). Wei ([16]) analyzed the consistency and asymptotic distribution of the estimators for CIR model with Lévy noise. Zhou et al. ([21]) discussed the synchronization of stochastic system driven by Lévy noise. Zouine et al. ([25]) investigated stability of highly nonlinear stochastic systems driven by Lévy noise.

With deepening of human production practice, time lags have been noticed in biochemical, population, physics and engineering. It is found that the appearance of this phenomenon may be related to the connection of each sub-component of the system and the characteristics of sub-components. A system with time delay is called delay system because the change of its state is not only dependent on the current state, but also related to the previous state. In the past few decades, many authors investigated the delay system ([2, 10, 13]). Li et al. ([4]) constructed a new slack variable-dependent inequality involving double integrals of system state and derived an improved stability criterion. Qi et al. ([11]) used a new criterion to design controller for delay stochastic

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system with actuator saturation. Zhou et al. ([20]) studied the exponential synchronization for delay stochastic neural networks. During the actual operation of the project, the system may be switched between systems described by the same model with different coefficients due to the influence of component failure and repair and connection mode change of subsystems. Therefore, some authors discussed the stochastic systems with Markovian switching. For example, Liu et al. ([6]) studied the event-based distributed filtering over Markovian switching topologies. Wang et al. ([12]) utilized aperiodically intermittent control to analyze the stabilization of stochastic delayed networks with Markovian switching. Xia et al. ([18]) considered delay-dependent extended dissipative analysis for generalized neural networks with Markovian switching.

In recent years, many authors studied the stability of systems ([7, 15, 17]). Li and Zhu ([5]) analyzed the pth moment exponential stability and almost surely exponential stability of stochastic differential delay equations with Poisson jump. Ma et al. ([8]) studied practical exponential stability of stochastic age-dependent capital system with Lévy noise. Zhu ([24]) discussed pth moment exponential stability problem for a class of stochastic delay differential equations driven by Lévy processes. Since Lévy noise can more accurately reflect the objective random disturbances in the system and stability is one of most important topics in economy and control, it is of great importance to study the stability of stochastic system driven by Lévy noise. Compared with [8] and [24], we discussed the existence of unique global solution and the methods for proving the stability are different. In this paper, the existence and almost sure stability of unique global solution for nonlinear stochastic delay system driven by Lévy noise are investigated by general Itô formula, Hölder inequality, Doob martingale inequality, Chebyshev's inequality and Bolzano-Weierstrass.

The rest of this paper is organized as follows: In Section 2, the delay hybrid stochastic system driven by Lévy noise is introduced. In Section 3, we prove the existence, uniqueness and almost sure stability of the solution. In Section 4, we give a numerical example. In Section 5, the conclusion is provided.

2. Problem formulation and preliminaries

Let $(\Omega,\mathscr{F},\mathbb{P})$ be a complete probability space equipped with a right continuous and increasing family of σ-algebras $(\{\mathscr{F}_t\}_{t\geqslant 0})$. Denote by $\mathscr{C}^{1,2}(\mathbb{R}^n\times\mathbb{R}_+\times\mathbb{S};\mathbb{R}_+)$ the family of positive real-valued functions $V(x,t,i)$ defined on $\mathbb{R}^n\times\mathbb{R}_+\times\mathbb{S}$ which are continuously twice differentiable in $x\in\mathbb{R}^n$ and once differentiable in $t\in\mathbb{R}_+$. Let $r(t),t\geqslant 0$ be a right-continuous Markov chain on the probability space taking values in a finite state space $\mathbb{S}=\{1,2,\ldots,N\}$ with generator $\Gamma=(\gamma_{ij})_{N\times N}$ given by

$$\mathbb{P}\{r(t+\Delta)=j|r(t)=i\}=\begin{cases}\gamma_{ij}\Delta+o(\Delta) & i\neq j\\ 1+\gamma_{ii}\Delta+o(\Delta) & i=j\end{cases}$$

where $\Delta>0$, $\gamma_{ij}\geqslant 0$ is the transition rate from i to j if $i\neq j$ while $\gamma_{ii}=-\Sigma_{i\neq j}\gamma_{ij}$.

We consider the following nonlinear stochastic system

$$\begin{aligned} dx(t) &= f(x(t),x(t-\tau(t)),t,r(t))dt + g(x(t),x(t-\tau(t)),t,r(t))dW(t) \\ &\quad + \int_Z H(x(t-),x(t-\tau(t)),t,r(t-),\nu)N(dt,d\nu), \end{aligned} \tag{1}$$

where $x(0)=\{x(\theta):-\tau\leqslant\theta\leqslant 0\}=\xi\in\mathscr{C}^b_{\mathscr{F}_0}([-\tau,0);\mathbb{R}^n)$, $\mathscr{C}^b_{\mathscr{F}_0}([-\tau,0);\mathbb{R}^n)$ is bounded random variable set with n-th order vector-valued continuous function, $r(0)=r_0\in\mathbb{S}$, $x(t-)=\lim_{s\downarrow t}x(s)$, $W(t)$ is an m-dimensional standard Brownian motion, $N(t,\nu)$ is an Poisson random measure on $[0,+\infty)\times\mathbb{R}^n$ with compensator $\widetilde{N}(t,\nu)$ which satisfies $\widetilde{N}(t,\nu)=N(dt,d\nu)-\pi(d\nu)dt$, π is a unique stable distribution of Markov chain, $0\leqslant\tau(t)\leqslant\tau$, $\dot{\tau}(t)\leqslant d_\tau<1$, $f:\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}_+\times\mathbb{S}\to\mathbb{R}^n$, $g:\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}_+\times\mathbb{S}\to\mathbb{R}^{n\times m}$, $H:\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}_+\times\mathbb{S}\times\mathbb{R}^n\to\mathbb{R}^n$. It is assumed that $W(t)$, $N(t,\nu)$, and $r(t)$ in system (1) are independent.

Firstly, We provide some assumptions and definition.

ASSUMPTION 1.

$$\sup_{t\geqslant 0, i\in S}\{|f(0,0,t,i)|\vee|g(0,0,t,i)|\}\leqslant K_0,$$

where K_0 is a constant.

ASSUMPTION 2. *$\forall t\geqslant 0$, $|x_1|\vee|x_2|\vee|y_1|\vee|y_2|\leqslant K$ and $i\in S$,*

$$\begin{aligned} &|f(x_1,y_1,t,i)-f(x_2,y_2,t,i)|^2|\vee|g(x_1,y_1,t,i)-g(x_2,y_2,t,i)|^2 \\ &\vee\int_Z|H(x_1,y_1,t,i,\nu)-H(x_2,y_2,t,i,\nu)|^2\pi(d\nu)\leqslant L_K(|x_1-x_2|^2+|y_1-y_2|^2), \end{aligned}$$

where $L_K>0$.

ASSUMPTION 3.

$$\lim_{|x|\to\infty}\inf_{t\geqslant 0, i\in S}V(x,y,t,i)=\infty, \mathscr{L}V(x,y,t,i)\leqslant m(t)-\alpha_1 n_1(x)+\alpha_2 n_2(y)$$

where $V(x,y,t,i)\in\mathscr{C}^{1,2}(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}_+\times\mathbb{S};\mathbb{R}_+)$, $m\in L^1(\mathbb{R}_+;\mathbb{R}_+)$, $n_1,n_2\in\mathscr{C}(\mathbb{R}^n;\mathbb{R}_+)$, $\alpha_1>0$, $\alpha_2>0$.

DEFINITION 1. The solution of system (1) is almost surely stable if

$$\mathbb{P}(\lim_{t\to\infty}x(t;\xi,r_0)=0)=1,$$

for any $\xi\in\mathscr{C}^b_{\mathscr{F}_0}([-\tau,0);\mathbb{R}^n)$ and $r_0\in\mathbb{S}$.

Given $V \in \mathscr{C}^{1,2}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}; \mathbb{R}_+)$, we define the operator $\mathscr{L}V$ by

$$\begin{aligned}
&\mathscr{L}V(x,y,t,i)\\
=\ &V_t(x,y,t,i) + V_x(x,y,t,i) f(x,y,t,i)\\
&+\frac{1}{2} trace[g^T(x,y,t,i) V_{xx}(x,y,t,i) g(x,y,t,i)]\\
&+\int_Y \sum_{k=1}^{l} [V(x + H^k(x,t,i,y_k),t,i)\\
&-V(x,t,i)]\nu_k(dy_k) + \sum_{j=1}^{N} \gamma_{ij} V(x,y,t,j).
\end{aligned}$$

3. Main results and proofs

THEOREM 1. *Under Assumptions 1–3, the solution* $\{x(t), t \geqslant 0\}$ *of system* (1) *exists and is unique.*

Proof. For the given initial values x_0 and r_0, it is assumed that $|x_0| \leqslant \rho$. For $k \geqslant \rho$, $k \in \mathbb{N}$, let

$$\begin{aligned}
&f^{(k)}(x,y,t,i) = f\Big(\frac{|x| \wedge k}{|x|} x, \frac{|y| \wedge k}{|y|} y, t, i\Big), \quad g^{(k)}(x,y,t,i) = g\Big(\frac{|x| \wedge k}{|x|} x, \frac{|y| \wedge k}{|y|} y, t, i\Big),\\
&H^{(k)}(x,y,t,i) = H\Big(\frac{|x| \wedge k}{|x|} x, \frac{|y| \wedge k}{|y|} y, t, i\Big),
\end{aligned} \tag{2}$$

where $(\frac{|x|\wedge k}{|x|} x) = 0$ when $x = 0$.

It can be checked that $f^{(k)}$ and $g^{(k)}$ satisfy the existence and uniqueness condition of the solution. Thus, the solution of the following system

$$\begin{aligned}
dx_k(t) = &f^{(k)}(x_k(t), x_k(t-\tau(t)), t, r(t))dt + g^{(k)}(x_k(t), x_k(t-\tau(t)), t, r(t))dW(t) \quad (3)\\
&+ \int_Z H^k(x_k(t-), x_k(t-\tau(t)), t, r(t-), \nu) N(dt, d\nu),
\end{aligned}$$

exists and is unique.

$\forall k \in \mathbb{N}$, let

$$\beta_k = \inf\{t \geqslant 0 : |x_k(t)| \geqslant k\}, \tag{4}$$

where $\inf \phi = \infty$.

When $0 \leqslant t \leqslant \beta_k$, $x_k(t) = x_{k+1}$. Then, there exists a stopping time β such that

$$\beta = \lim_{k \to \infty} \beta_k. \tag{5}$$

When $-\tau \leqslant t < \beta_k$, $x(t) = x_k(t)$. Therefore, when $t \in [-\tau, \beta)$, the solution $x(t)$ of system (1) is unique.

Next, $P\{\beta = \infty\} = 1$ will be proved.

From Itô formula, for $t \geqslant 0$, one has

$$\begin{aligned}&\mathbb{E}V(x_k(t\wedge\beta_k),t\wedge\beta_k,r(t\wedge\beta_k))\\ =&\ \mathbb{E}V(x_k(0),0,r(0))+\mathbb{E}\int_0^{t\wedge\beta_k}\mathscr{L}^{(k)}V(x_k(s),x_k(s-\tau(s)),s,r(s))ds,\end{aligned}$$

When $0\leqslant s\leqslant t\wedge\beta_k$,

$$\mathscr{L}^{(k)}V(x_k(s),x_k(s-\tau(s)),s,r(s))=\mathscr{L}V(x_k(s),x_k(s-\tau(s)),s,r(s)).$$

Thus, according to Assumption 3, we have

$$\begin{aligned}&\mathbb{E}V(x_k(t\wedge\beta_k),t\wedge\beta_k,r(t\wedge\beta_k))\\ =&\ \mathbb{E}V(x_k(0),0,r(0))+\mathbb{E}\int_0^{t\wedge\beta_k}\mathscr{L}V(x_k(s),x_k(s-\tau(s)),s,r(s))ds\\ \leqslant&\ V(\xi_0,0,r_0)+\mathbb{E}\int_0^t(m(s)-\alpha_1 n_1(x(s))+\alpha_2 n_2(x(s-\tau(s))))ds\\ =&\ V(\xi_0,0,r_0)+\int_0^t m(s)ds+\mathbb{E}\int_0^t[-\alpha_1 n_1(x(s))+\alpha_2 n_2(x(s-\tau(s)))]ds\\ \leqslant&\ V(\xi_0,0,r_0)+\int_0^t m(s)ds-\mathbb{E}\int_0^t\alpha_1 n_1(x(s))ds+\mathbb{E}\int_{-\tau}^{t-\tau(t)}\frac{\alpha_2}{1-d_\tau}n_2(x(s))ds\\ \leqslant&\ V(\xi_0,0,r_0)+\int_0^t m(s)ds-\mathbb{E}\int_0^t\alpha_1 n_1(x(s))ds+\mathbb{E}\int_{-\tau}^0\frac{\alpha_2}{1-d_\tau}n_2(\xi(\theta))d\theta\\ &+\mathbb{E}\int_0^t\frac{\alpha_2}{1-d_\tau}n_2(x(s))ds\\ \leqslant&\ V(\xi_0,0,r_0)+\int_0^t m(s)ds-\mathbb{E}\int_0^t\alpha_1(n_1(x(s))-n_2(x(s)))ds\\ &+\mathbb{E}\int_{-\tau}^0\alpha_1 n_2(\xi(\theta))d\theta\\ \leqslant&\ V(\xi_0,0,r_0)+\int_0^t m(s)ds+\mathbb{E}\int_{-\tau}^0\alpha_1 n_2(\xi(\theta))d\theta.\end{aligned}$$

Since

$$\begin{aligned}&P\{\beta_k\leqslant t\}\inf_{|x|\geqslant k,t\geqslant 0,i\in S}V(x,t,i)\\ \leqslant&\int_{\beta_k\leqslant t}V(x_k(t\wedge\beta_k),t\wedge\beta_k,r(t\wedge\beta_k))dP\\ \leqslant&\ \mathbb{E}V(x_k(t\wedge\beta_k),t\wedge\beta_k,r(t\wedge\beta_k)),\end{aligned}$$

we obtain

$$P\{\beta_k\leqslant t\}\leqslant\frac{V(\xi_0,0,r_0)+\int_0^t m(s)ds+\mathbb{E}\int_{-\tau}^0\alpha_1 n_2(\xi(\theta))d\theta}{\inf_{|x|\geqslant k,t\geqslant 0,i\in S}V(x,t,i)}. \tag{6}$$

When $t \to \infty$, we derive

$$P\{\beta \leqslant t\} = 0. \tag{7}$$

Therefore,

$$P\{\beta = \infty\} = 1. \quad \square \tag{8}$$

THEOREM 2. *Under Assumptions 1–3, $\forall i \in \mathbb{S}$, if there exists function $V \in \mathscr{C}^{1,2}(\mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}; \mathbb{R}_+)$, $m \in L^1(\mathbb{R}_+; \mathbb{R}_+)$, $n_1, n_2 \in \mathscr{C}(\mathbb{R}^n; \mathbb{R}_+)$, $(x,y,t,i) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}$ satisfy*

$$\mathscr{L}V(x,y,t,i) \leqslant m(t) - \alpha_1 n_1(x) + \alpha_2 n_2(y),$$

$$n_1(x) > n_2(x), \quad x \neq 0,$$

$$\lim_{|x|\to\infty} \inf_{t \geqslant 0, i \in S} V(x,t,i) = \infty,$$

the system (1) is almost sure stable.

Proof. Since

$$\begin{aligned}
&V(x(t),t,r(t)) \\
&= V(\xi(0),0,r_0) + \int_0^t \mathscr{L}V(x(s),x(s-\tau(s)),s,r(s))ds \\
&\quad + \int_0^t V_x(x(s),s,r(s))g(x(s),x(s-\tau(s),s,r(s))dW(s) \\
&\quad + \int_0^t \int_Z [V((x(s),s,r_0 + H(x(s),x(s-\tau(s)),s,r(s),\nu)) \\
&\quad - V_x(x(s),s,r(s))]\pi(d\nu) \\
&\leqslant V(\xi(0),0,r_0) + \int_0^t m(s)ds - \int_0^t \alpha_1 n_1(x(s))ds \\
&\quad + \int_0^t \alpha_2 n_2(x(s-\tau(s)))ds \\
&\quad + \int_0^t V_x(x(s),s,r(s))g(x(s),x(s-\tau(s),s,r(s))dW(s) \\
&\quad + \int_0^t \int_Z [V((x(s),s,r_0 + H(x(s),x(s-\tau(s)),s,r(s),\nu)) \\
&\quad - V_x(x(s),s,r(s))]\pi(d\nu) \\
&\leqslant V(\xi(0),0,r_0) + \int_0^t m(s)ds - \alpha_1 \int_0^t n_1(x(s))ds + \alpha_2 \int_{-\tau}^0 n_2(x(s))ds \\
&\quad + \int_0^t V_x(x(s),s,r(s))g(x(s),x(s-\tau(s),s,r(s))dW(s) \\
&\quad + \int_0^t \int_Z [V((x(s),s,r_0 + H(x(s),x(s-\tau(s)),s,r(s),\nu)) \\
&\quad - V_x(x(s),s,r(s))]\pi(d\nu).
\end{aligned}$$

As

$$\int_0^\infty m(s)ds < \infty, \tag{9}$$

we have

$$\lim_{t\to\infty}\int_0^t n_1(x(s))ds < \infty \tag{10}$$

and

$$\limsup_{t\to\infty} V(x(t),t,r(t)) < \infty. \tag{11}$$

Then, we obtain

$$\sup_{0\leqslant t<\infty}\ \inf_{|x|\geqslant|x(t)|,0\leqslant t<\infty,i\in S} V(x,t,i) \leqslant \sup_{0\leqslant t<\infty} V(x(t),t,r(t)) < \infty. \tag{12}$$

From Assumption 2, we have

$$\sup_{0\leqslant t<\infty}|x(t)| < \infty. \tag{13}$$

Since $\xi \in \mathscr{C}^b_{\mathscr{F}_0}([-\tau,0);\mathbb{R}^n)$, there exists a positive k_0 and $|\xi| < k_0$. For $k > k_0$, we define a stopping time

$$\eta_k = \inf\{t \geqslant 0 : |x(t)| \geqslant k\}, \tag{14}$$

where $\inf\phi = \infty$.

When $k \to \infty$, it is obvious that $\eta_k \to \infty\ a.s.$

Thus, for any $\varepsilon > 0$, there exists $k_\varepsilon \geqslant k_0$, when $k \geqslant k_\varepsilon$,

$$\mathbb{P}(\eta_k < \infty) \leqslant \varepsilon. \tag{15}$$

According to (9), we have

$$\liminf_{t\to\infty} n_1(x(t)) = 0. \tag{16}$$

Next we will prove that

$$\lim_{t\to\infty} n_1(x(t)) = 0. \tag{17}$$

Suppose (17) dose not hold, we can obtain

$$\mathbb{P}\{\limsup_{t\to\infty} n_1(x(t) > 0\} > 0. \tag{18}$$

Then, there exists the following stopping time sequence:

$$\varsigma_1 = \inf\{t \geqslant 0 : n_1(x(t) \geqslant 2\varepsilon_1\},$$

$$\varsigma_{2j} = \inf\{t \geqslant \varsigma_{2j-1} : n_1(x(t) \leqslant \varepsilon_1\}, \quad j = 1,2,\cdots,$$

$$\varsigma_{2j+1} = \inf\{t \geqslant \varsigma_{2j} : n_1(x(t) \geqslant 2\varepsilon_1\}, \quad j = 1,2,\cdots,$$

and $\varepsilon_0 > 0$, $\varepsilon > \varepsilon_1 > 0$ satisfy

$$\mathbb{P}(\varsigma_{2j} < \infty : j \in Z) \geqslant \varepsilon_0. \tag{19}$$

According to local Lipschitz condition, $\forall k > 0$, there exists $L_k > 0$ satisfy

$$|f(x,y,t,i)| \vee |g(x,y,t,i)| \vee |H(x,y,t,i,\nu)| \leqslant L_k,$$

for any $t \geqslant 0$, $i \in S$ and $|x| \vee |y| \leqslant k$.

According to Hölder inequality and Doob martingale inequality, for any $j \in Z$, when $T < \varsigma_{2j} - \varsigma_{2j-1}$, we obtain

$$\begin{aligned}
&\mathbb{E}[\mathbb{I}_{\{\varsigma_{2j}<\eta_k\}} \sup_{0\leqslant t\leqslant T} |x(\varsigma_{2j-1}+t) - x(\varsigma_{2j-1})|^2] \\
=& \mathbb{E}[\mathbb{I}_{\{\varsigma_{2j}<\eta_k\}} \sup_{0\leqslant t\leqslant T} |\int_{\varsigma_{2j-1}}^{\varsigma_{2j-1}+t} f(x(s),x(s-\tau(s)),s,r(s))ds \\
&+ \int_{\varsigma_{2j-1}}^{\varsigma_{2j-1}+t} g(x(s),x(s-\tau(s)),s,r(s))dW(s) \\
&+ \int_{\varsigma_{2j-1}}^{\varsigma_{2j-1}+t} \int_Z H(x(s-),x(s-\tau(s)),s,r(s-),\nu)N(ds,d\nu)|^2] \\
\leqslant& 4\mathbb{E}[\mathbb{I}_{\{\varsigma_{2j}<\eta_k\}} \sup_{0\leqslant t\leqslant T} |\int_{\varsigma_{2j-1}}^{\varsigma_{2j-1}+t} f(x(s),x(s-\tau(s)),s,r(s))ds|^2] \\
&+16\mathbb{E}[\mathbb{I}_{\{\varsigma_{2j}<\eta_k\}} \sup_{0\leqslant t\leqslant T} \int_{\varsigma_{2j-1}}^{\varsigma_{2j-1}+t} |g(x(s),x(s-\tau(s)),s,r(s))|^2 ds \\
&+4\mathbb{E}[\mathbb{I}_{\{\varsigma_{2j}<\eta_k\}} \sup_{0\leqslant t\leqslant T} \int_{\varsigma_{2j-1}}^{\varsigma_{2j-1}+t} \int_Z |H(x(s-),x(s-\tau(s)),s,r(s-),\nu)|^2 \pi(d\nu)ds \\
\leqslant& 4L_k^2 T(T+5),
\end{aligned}$$

where $\mathbb{I}_A$ is the indicative function of set A.

Since $n_1(x)$ is continuous on $\mathbb{R}^n$, it is uniformly continuous in $\overline{S}_k = \{x \in \mathbb{R}^n : |x| \leqslant k\}$. Then, for any $p > 0$, when $x,y \in \overline{S}_k$ and $|x-y| < c_p$, $|n_1(x) - n_1(y)| < p$. Let $\varepsilon = \frac{\varepsilon_0}{2}$, $k \geqslant k_\varepsilon$ and $p = \varepsilon_1$.

According to Chebyshev's inequality, we have

$$\begin{aligned}
&\mathbb{P}(\{\eta_k \leqslant \varsigma_{2j}\}) + \mathbb{P}(\{\varsigma_{2j} < \eta_k\} \cap \{\sup_{0\leqslant t\leqslant T} |n_1(x(\varsigma_{2j-1}+t)) - n_1(x(\varsigma_{2j-1}))| \geqslant \varepsilon_1\}) \\
\leqslant& \mathbb{P}(\{\eta_k \leqslant \varsigma_{2j}\} \cap \{\varsigma_{2j} = \infty\}) + \mathbb{P}(\{\eta_k \leqslant \varsigma_{2j}\} \cap \{\varsigma_{2j} < \infty\}) \\
&+\mathbb{P}(\{\varsigma_{2j} < \eta_k\} \cap \{\sup_{0\leqslant t\leqslant T} |x(\varsigma_{2j-1}+t) - x(\varsigma_{2j-1})| \geqslant c_{\varepsilon_1}\}) \\
\leqslant& \frac{4L_k^2 T(T+5)}{c_{\varepsilon_1}^2} + 1 - 2\varepsilon.
\end{aligned}$$

Let $T = T(\varepsilon, \varepsilon_1, k)$ be small enough to satisfy

$$\frac{4L_k^2 T(T+5)}{c_{\varepsilon_1}^2} \leqslant \varepsilon. \tag{20}$$

Then, it can be checked that

$$\mathbb{P}(\{\varsigma_{2j} < \eta_k\} \cap \{\sup_{0 \leqslant t \leqslant T} |n_1(x(\varsigma_{2j-1}+t)) - n_1(x(\varsigma_{2j-1}))| < \varepsilon_1\}) \geqslant \varepsilon. \tag{21}$$

Hence, we obtain

$$\begin{aligned}
&\Sigma_{j=1}^{\infty} T\varepsilon_1\varepsilon = \frac{1}{2}\Sigma_{j=1}^{\infty} T\varepsilon_0\varepsilon_1 = \infty \\
&\leqslant \Sigma_{j=1}^{\infty} T\varepsilon_1 \mathbb{P}(\{\varsigma_{2j} < \eta_k\} \cap \{\sup_{0 \leqslant t \leqslant T} |n_1(x(\varsigma_{2j-1}+t)) - n_1(x(\varsigma_{2j-1}))| < \varepsilon_1\}) \\
&\leqslant \Sigma_{j=1}^{\infty} \varepsilon_1 \mathbb{E}[\mathbb{I}_{\varsigma_{2j}<\eta_k}(\varsigma_{2j} - \varsigma_{2j-1})] \\
&\leqslant \Sigma_{j=1}^{\infty} \varepsilon_1 \mathbb{E}[\mathbb{I}_{\varsigma_{2j}<\eta_k} \int_{\varsigma_{2j-1}}^{\varsigma_{2j-1}+t} n_1(x(t))dt] \\
&\leqslant \mathbb{E}[\int_0^{\infty} n_1(x(t))dt] \\
&< \infty.
\end{aligned}$$

Obviously, the above result is contradictory. Then, there exists $\overline{\Omega} \in \Omega$ such that $\mathbb{P}(\overline{\Omega}) = 1$ and

$$\lim_{t\to\infty} n_1(x(t,\omega)) = 0, \quad \sup_{0\leqslant t<\infty} |x(t,\omega)| < \infty, \quad \forall \omega \in \overline{\Omega}. \tag{22}$$

Therefore, for any given $\omega \in \overline{\Omega}$, $\{x(t,\omega)\}_{t\geqslant 0} \in \mathbb{R}^n$ is bounded. There exists a increasing sequence $\{t_i\}_{i\geqslant 1}$ such that $\{x(t_i,\omega)\}_{i\geqslant 1}$ is convergent. Since $n_1(x) > 0$ as $x \neq 0$, it is known that $n_1(x) = 0$ when $x = 0$.

The proof is complete. □

4. Example

Let $W(t)$ be a one-dimensional Brownian motion, The character measure π of Poisson jump satisfies $\pi(d\nu) = \lambda\phi(d\nu)$, where $\lambda = 2$ is the intensity of Poisson distribution and ϕ is the probability intensity of the standard normal distributed variable ν, $r(t) \in \mathbb{S} = \{1,2\}$ and $\Gamma = (\gamma_{ij})_{2\times 2} =$

$$\begin{pmatrix} -0.8 & 0.8 \\ 0.5 & -0.5 \end{pmatrix}$$

Consider the nonlinear delay hybrid stochastic system driven by Lévy noises as follows:

$$\begin{aligned}
dx(t) &= f(x(t),x(t-\tau(t)),t,r(t))dt + g(x(t),x(t-\tau(t)),t,r(t))dW(t) \\
&\quad + \int_Z H(x(t-),x(t-\tau(t)),t,r(t-),\nu)N(dt,d\nu),
\end{aligned}$$

where

$$f(x,y,t,1) = -3x^{\frac{1}{3}} + 3y^{\frac{2}{3}},$$
$$g(x,y,t,1) = -x^{\frac{2}{3}} + y^{\frac{2}{3}},$$
$$f(x,y,t,2) = 2(1+t)^{-\frac{1}{3}} - 2x^{\frac{1}{3}},$$
$$g(x,y,t,2) = 2x^{\frac{2}{3}}\cos(t) + \frac{3}{2}y^{\frac{2}{3}}\sin(t),$$
$$H(x,y,t,1,\nu) = -2x^{\frac{1}{3}} + 2y^{\frac{2}{3}},$$
$$H(x,y,t,2,\nu) = 3x^{\frac{1}{3}} + y^{\frac{2}{3}},$$

where $\tau(t) = 0.5 + 0.5\sin(t)$.

Let $V(x,i) = x^2$. Then, we obtain

$$\mathscr{L}V(x,y,t,1) \leqslant -9x^{\frac{4}{3}} + 5y^{\frac{4}{3}},$$
$$\mathscr{L}V(x,y,t,2) \leqslant 4x(1+t)^{-\frac{1}{3}} - 4x^{\frac{4}{3}} + \frac{9}{4}y^{\frac{4}{3}}.$$

Since for any $\kappa > 0$,

$$4x(1+t)^{-\frac{1}{3}} = \left(\frac{4}{3}\kappa x^{\frac{4}{3}}\right)^{\frac{3}{4}}\left(4\left(\frac{\kappa}{3}\right)^{-3}(1+t)^{-\frac{4}{3}}\right)^{\frac{1}{4}} \leqslant \kappa x^{\frac{4}{3}} + \left(\frac{\kappa}{3}\right)^{-3}(1+t)^{-\frac{4}{3}}.$$

Thus, for all $t \geqslant 0$, $i \in S$, it is easy to check that

$$\mathscr{L}V(x,y,t,i) \leqslant \left(\frac{\kappa}{3}\right)^{-3}(1+t)^{-\frac{4}{3}} - (4-\kappa)x^{\frac{4}{3}} + 5y^{\frac{4}{3}}.$$

Let $\xi_0 = 3$, $r_0 = 1$. Figure 1 verify the results.

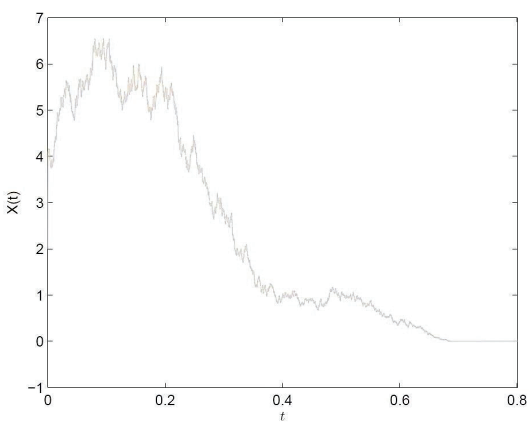

Figure 1: *State trajectory*

5. Conclusion

In this paper, we have analyzed the almost surely stability of nonlinear delay hybrid stochastic system driven by Lévy noise. The existence and uniqueness of the solution for nonlinear stochastic delay system has been discussed by general Itô formula. The almost sure stability of the solution has been studied by Hölder inequality, Doob martingale inequality, Chebyshev's inequality and Bolzano-Weierstrass. Further research topics will include stability of nonlinear delay stochastic system driven by fractional Lévy noise.

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Chao Wei
School of Mathematics and Statistics
Anyang Normal University
Anyang 455000, China
e-mail: chaowei0806@aliyun.com