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MEAN SQUARE OF QUADRATIC HECKE CHARACTER SUM

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Abstract. Assuming the Generalized Riemann Hypothesis (GRH) and the Riemann Hypothesis (RH), we consider the sum $S(X,Y)$ in the Gaussian field $\mathbb{Q}(i)$, where the sum is taken over all primary primes. The primary tools employed include the Poisson sum and Mellin inversion techniques. The method used in this paper differs significantly from the classical approach which relies on the partial sum method, thereby facilitating a simpler calculation within our proof. Furthermore, we present a valid asymptotic formula that is applicable when both X and Y are of comparable size.

1. Introduction

The prime counting function

$$\pi(X) = \sum_{p \leqslant X} 1$$

is of great interest in the case of rational numbers and is closely linked to the Prime Number Theory. Equivalently, the functions

$$\theta(X) = \sum_{p \leqslant X} \log p \text{ and } \psi(X) = \sum_{n \leqslant X} \Lambda(n)$$

can be studied, where Λ is the Mangoldt function. Additionally, the functions

$$\theta(X,\chi) = \sum_{p \leqslant X} \chi(p) \log p, \pi(X,\chi) = \sum_{p \leqslant X} \chi(p) \text{ and } \psi(X,\chi) = \sum_{n \leqslant X} \Lambda(n)\chi(n)$$

are of interest. Their mean values lead to asymptotic formulas for

$$\theta(X;q,a) = \sum_{\substack{p \leqslant X \\ p \equiv a(q)}} \log p,\ \pi(X;q,a) = \sum_{\substack{p \leqslant X \\ p \equiv a(q)}} \chi(p) \text{ and } \psi(X;q,a) = \sum_{\substack{n \leqslant X \\ n \equiv a(q)}} \Lambda(n)\chi(n),$$

which are all related to the Prime Number Theory for arithmetic progressions. Further introduction on these functions can be found in Chapter 6 and Chapter 11 in [7].

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The present study extends our theory to the Gaussian field $\mathbb{Q}(i)$, with a focus on prime distribution in $\mathbb{Z}[i]$, where $\mathbb{Z}[i]$ is the ring of integers in $\mathbb{Q}(i)$. Let $\left(\frac{\cdot}{\cdot}\right)$ be the Kronecker symbol and $\mu_{\mathbb{Q}(i)}$ be the Möbius function on $\mathbb{Z}[i]$. We consider the sum

$$\sum_{\substack{(d,1+i)=1\\ X\leqslant N(d)\leqslant 2X}} \mu_{\mathbb{Q}(i)}^2(d)\left|\sum_{N(\varpi)\leqslant Y} \log N(\varpi)\left(\frac{(1+i)^5 d}{\varpi}\right)\right|^2.$$

The inner sum is taken over all primary primes in $\mathbb{Z}[i]$ with $N(\varpi)\leqslant Y$. Primary prime in $\mathbb{Z}[i]$ can be interpreted as a prime element with $\varpi\equiv 1\ mod\ (1+i)^3$. This sum is the mean square of $\theta(X,\chi)$, averaging over certain quadratic characters.

The classical method of estimation for this sum is complex. To simplify the process, we introduce smooth, rapidly decaying functions W and Φ, supported on $(0,1)$ and $(1,2)$, respectively, and add them to the above sum. This yields the smooth sum

$$S(X,Y)=\sum_{(d,1+i)=1} \mu_{\mathbb{Q}(i)}^2(d)\left|\sum_{\varpi} \log N(\varpi)\left(\frac{(1+i)^5 d}{\varpi}\right)W\left(\frac{N(\varpi)}{Y}\right)\right|^2 \Phi\left(\frac{N(d)}{X}\right) \tag{1.1}$$

which is more amenable to evaluation using various techniques. Our method in this paper is based on the methodology proposed in [2]. Moreover, by assuming the Generalized Riemann Hypothesis (GRH) (or the Riemann Hypothesis (RH)) for the Hecke L-function (or zeta function), this approach enables us to obtain an asymptotic formula when $X^{1/3}<Y<X^{1-\varepsilon}$. Our main result is presented below.

THEOREM 1.1. *Assume RH and GRH. For any $\varepsilon>0$ and large real number X and Y, we have*

$$\begin{aligned} S(X,Y)=&\frac{2\pi}{3\zeta_{\mathbb{Q}(i)}(2)}XY\log Y-\frac{\pi}{\zeta_{\mathbb{Q}(i)}(2)}XY\\ &+O(X^{1-\varepsilon}Y)+O(XY^{\frac{1}{2}+\varepsilon}\log Y)+O(X^{\frac{5}{4}+\varepsilon}Y^{\frac{1}{4}+\varepsilon}\log Y)\\ &+O(X^{1/2+\varepsilon}Y^{1+\varepsilon}\log Y+Y^{2+\varepsilon}\log Y). \end{aligned}$$

Section 2 presents established findings on Gauss sum, along with estimations for the Mellin transform of W and Φ. Additionally, a modified version of the Poisson sum formula outlined in [2] is introduced.

Section 3 undertakes the computation of the sum. Initially, $S(X,Y)$ is expressed as a sum of multiplicative series and error terms. The Mellin inversion is then applied to each series, resulting in an integral on the vertical line (c). Assuming the GRH (or RH) for the Hecke L-function (or zeta function), the vertical line (c) is shifted to a line (ε) closer to the origin. The main terms of the sum $S(X,Y)$ are determined by the residues, while the error terms are determined by integrals on line (ε).

2. Preliminary

In this section, we introduce functions appearing in (1.1) and present some lemmas.

2.1. Gauss sum

For any $n \equiv 1 \ mod \ (1+i)^3$, we define quadratic Gauss sum

$$g_2(k,n) = \sum_{x \ mod \ n} \left(\frac{x}{n}\right) \widetilde{e}\left(\frac{kx}{n}\right), \tag{2.1}$$

where $\widetilde{e}(z) = e^{2\pi i(\frac{z}{2i} - \frac{\bar{z}}{2i})}$ and $(\frac{\cdot}{\cdot})$ is the Kronecker symbol on $\mathbb{Q}(i)$. The properties of $g_2(k,n)$ are then outlined in Lemma 2.1, which includes three distinct cases. The proof of Lemma 2.1 is available in [2].

LEMMA 2.1. *1.* $g_2(rs,n) = (\frac{\bar{s}}{n}) g_2(r,n)$, $(s,n) = 1$*;*

2. $g_2(k,mn) = g_2(k,m) g_2(k,n)$*,* m,n *are primary and* $(m,n) = 1$*;*
3. *Let* ϖ *be a primary prime in* $\mathbb{Z}[i]$*. Suppose* $\varpi^h || k$*. (If* $k = 0$*, then set* $h = \infty$*.) Then for* $l > 1$*,*

$$g_2(k,\varpi^l) = \begin{cases} 0, & \text{if } l \leqslant h \text{ is odd;} \\ \phi(\varpi^l), & \text{if } l \leqslant h \text{ is even;} \\ -N(\varpi)^{l-1}, & \text{if } l = h+1 \text{ is even;} \\ \left(\frac{ik\varpi^{-h}}{\varpi}\right) N(\varpi)^{l-1/2}, & \text{if } l = h+1 \text{ is odd;} \\ 0, & \text{if } l \geqslant h+2. \end{cases} \tag{2.2}$$

2.2. Poisson sum

The function $\Phi(t)$ is a smooth function with support on the interval $(1,2)$ and satisfies the condition $\Phi(t) = 1$ for $t \in (1+1/U, 2-1/U)$. The Poisson sum formula is provided below.

LEMMA 2.2. *For fixed constant* a*, let* $n \in \mathbb{Z}[i]$*,* $n \equiv 1 \ mod \ (1+i)^3$*,* χ *be a quadratic character, then*

$$\sum_{m \in \mathbb{Z}[i]} \chi(m) \Phi\left(\frac{aN(m)}{X}\right) = \frac{X}{aN(n)} \sum_{k \in \mathbb{Z}[i]} g_2(k,n) \widetilde{\Phi}\left(\sqrt{\frac{N(k)X}{aN(n)}}\right), \tag{2.3}$$

where

$$\widetilde{\Phi}(t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \Phi(N(x+yi)) \widetilde{e}(-t(x+yi)) dxdy, \ t \geqslant 0. \tag{2.4}$$

The proof of Lemma 2.2 is available in [2], and a more general case can be found in [8]. We now estimate the function $\widetilde{\Phi}$. (2.4) gives us,

$$\widetilde{\Phi}^{(\mu)}(t) \ll \begin{cases} U^{j-1}|t|^{-j}, \ |t| \geqslant 1; \\ 1, \ |t| < 1. \end{cases} \tag{2.5}$$

Moreover, by rewriting $\widetilde{\Phi}(t)$ using polar coordinates, we obtain,

$$\widetilde{\Phi}(0) = \pi + O\Big(\frac{1}{U}\Big). \tag{2.6}$$

2.3. L-function

The Hecke L-function $L(\chi,s)$ associated with Hecke character χ is defined for $Re(s) > 1$ by the following equation,

$$L(\chi,s) = \sum_{0\neq\mathscr{A}\subset\mathbb{Z}[i]} \frac{\chi(\mathscr{A})}{N(\mathscr{A})^s}, \tag{2.7}$$

where $\mathscr{A}$ denotes integral ideals in $\mathbb{Z}[i]$. Similarly, the Riemann zeta function for $\mathbb{Q}(i)$ is defined by,

$$\zeta_{\mathbb{Q}(i)}(s) = \sum_{0\neq\mathscr{A}\subset\mathbb{Z}[i]} \frac{1}{N(\mathscr{A})^s}. \tag{2.8}$$

We note that a prime ideal $I \subset \mathbb{Z}[i]$ can be uniquely written as $I = (\varpi)$, where ϖ is a primary prime. Therefore, we can take the sums in the above definitions over $n \in \mathbb{Z}[i]$, $n \equiv 1 \ mod \ (1+i)^3$ uniquely, and write $\zeta_{\mathbb{Q}(i)}(s)$ as an Euler product over primary primes. In particular, for $Re(s) > 1$, we have,

$$\frac{1}{\zeta_{\mathbb{Q}(i)}(s)} = \sum_n \frac{\mu_{\mathbb{Q}(i)}(n)}{N(n)^s},$$

where $\mu_{\mathbb{Q}(i)}(n)$ is the Möbius function on $\mathbb{Z}(i)$. This leads to the following trivial estimation,

$$\sum_{\substack{N(n)\leqslant Z \\ (n,1+i)=1 \\ (n,\varpi)=1}} \frac{\mu_{\mathbb{Q}(i)}(n)}{N(n^2)} = \frac{4}{3}\zeta_{\mathbb{Q}(i)}^{-1}(2)\left(1-\frac{1}{N(\varpi^2)}\right)^{-1} + O(Z^{-1}). \tag{2.9}$$

2.4. Mellin inversion

The function $W(t)$ is a rapidly decaying function with support on the interval $(0,1)$. Specifically, $W(t) = 1$ for $t \in (1/U, 1-1/U)$, and $W^{(j)}(t) \ll U^j$ otherwise. The Mellin transform of $W(t)$ is denoted by $\widehat{W}(s)$ and is defined by

$$\widehat{W}(s) = \int_0^{+\infty} W(t)t^{s-1}dt.$$

A crude estimate reveals that $\widehat{W}(1) = \int_0^{+\infty} W(t)dt = 1 + O(U^{-1})$. The Mellin inversion for $W(t)$ is given by the formula

$$W(t) = \frac{1}{2\pi i}\int_{(c)} \widehat{W}(s)t^{-s}ds,$$

where (c) is a vertical line in the complex plane. Moreover, for $Re(s) > 0$ and a positive integer E, integration by parts yields the estimate

$$\widehat{W}(s) \ll \frac{1}{|s|(1+|s|)^E}. \tag{2.10}$$

LEMMA 2.3. *Assume GRH. If k is not a perfect square, for large Y and $\varepsilon > 0$, we have*

$$\sum_{\varpi} \log N(\varpi)\left(\frac{k}{\varpi}\right) W\left(\frac{N(\varpi)}{Y}\right) = O(Y^{\frac{1}{2}+\varepsilon}\log\log(N(k)+2)). \tag{2.11}$$

Proof. First, we have

$$\begin{aligned}&\sum_{\varpi} \log N(\varpi)\left(\frac{k}{\varpi}\right) W\left(\frac{N(\varpi)}{Y}\right)\\ &= \sum_{n\equiv 1} \Lambda(n)\chi_k(n)W\left(\frac{N(n)}{Y}\right) + O\left(\sum_{\substack{N(\varpi^j)<Y^{1+\varepsilon}\\ j\geqslant 2}} \Lambda(\varpi^j)W\left(\frac{N(\varpi^j)}{Y}\right)\right),\end{aligned}$$

where the second sum is taken over all $n \in \mathbb{Z}[i]$, $n \equiv 1 \ mod \ (1+i)^3$ and $\chi_k(\varpi) = \left(\frac{k}{\varpi}\right)$. Λ is Mangoldt function on $\mathbb{Z}[i]$. By Mellin inversion, we have

$$\sum_{n\equiv 1} \Lambda(n)\chi_k(n)W\left(\frac{N(n)}{Y}\right) = \frac{-1}{2\pi i}\int_{(2)} \frac{L'}{L}(\chi_k, s)Y^s\widehat{W}(s)ds.$$

Since k is not a perfect square, there is no pole at $s = 1$. Shifting (2) to $(\frac{1}{2}+\varepsilon)$, we obtain the first error term

$$\int_{(\frac{1}{2}+\varepsilon)} \frac{L'}{L}(\chi_k, s)Y^s\widehat{W}(s)ds.$$

Combined with (2.10) and Theorem 5. 17 in[6], with slight modification, the integration above can be estimated as

$$\sum_{n\equiv 1} \Lambda(n)\chi_k(n)W\left(\frac{N(n)}{Y}\right) = O\left(Y^{\frac{1}{2}+\varepsilon}\log\log(N(k)+2)\right),$$

where the implied constant is absolute. For the second error term, we have

$$\sum_{\substack{N(\varpi^j)<Y^{1+\varepsilon}\\ j\geqslant 2}} \Lambda(\varpi^j)W\left(\frac{N(\varpi^j)}{Y}\right) \ll Y^{\varepsilon} \sum_{N(\varpi)<Y^{\frac{1}{2}+\varepsilon}} \log N(\varpi) \ll Y^{1/2+\varepsilon}. \qquad \square$$

3. Explict formula

For arbitrary positive constant Z, we have

$$\mu^2_{\mathbb{Q}(i)}(d) = \sum_{\substack{l^2|d\\ N(l)\leqslant Z}} \mu_{\mathbb{Q}(i)}(l) + \sum_{\substack{l^2|d\\ N(l)>Z}} \mu_{\mathbb{Q}(i)}(l). \tag{3.1}$$

Then, replacing d by dl^2, we have

$$S(X,Y) = S_M + S_R, \tag{3.2}$$

where

$$S_M = \sum_{\varpi_1,\varpi_2} \log N(\varpi_1)\log N(\varpi_2) W\left(\frac{N(\varpi_1)}{Y}\right) W\left(\frac{N(\varpi_2)}{Y}\right) \sum_{\substack{N(l)<Z\\ (l,1+i)\\ l\equiv 1 mod(1+i)^3}} \mu_{\mathbb{Q}(i)}(l)\left(\frac{l^2}{\varpi_1\varpi_2}\right)$$
$$\times \sum_{(d,1+i)=1}\left(\frac{d}{\varpi_1\varpi_2}\right)\Phi\left(\frac{N(dl^2)}{X}\right) \tag{3.3}$$

and

$$S_R = \sum_{\substack{l^2|d\\ N(l)>Z\\ l\equiv 1 mod(1+i)^3}} \mu_{\mathbb{Q}(i)}(l) \sum_{(d,1+i)=1}\left|\sum_{\varpi}\log N(\varpi)\left(\frac{(1+i)^5 dl^2}{\varpi}\right)W\left(\frac{N(\varpi)}{Y}\right)\right|^2 \Phi\left(\frac{N(dl^2)}{X}\right). \tag{3.4}$$

The parameter Z should be chosen appropriately to ensure that S_R is small.

3.1. The error term S_R

Lemma 2.3 gives the inner sum

$$\sum_{\varpi}\log N(\varpi)\left(\frac{(1+i)^5 dl^2}{\varpi}\right)W\left(\frac{N(\varpi)}{Y}\right) = O(Y^{1/2+\varepsilon}\log\log N(dl^2)).$$

Then

$$S_R \ll \sum_{\substack{N(l)>Z\\ l\equiv 1}} \mu_{\mathbb{Q}(i)}(l) \sum_{X/N(l^2)<N(d)<2X/N(l^2)} \left(Y^{1/2+\varepsilon}\log\log N(dl^2)\right)^2$$
$$\ll \frac{XY^{1+\varepsilon}(\log\log X)^2}{Z}. \tag{3.5}$$

3.2. Rewrite S_M

The Poisson sum formula in Lemma 2.2 gives

$$\sum_{\substack{m\in\mathbb{Z}[i]\\(m,1+i)=1}}\chi(m)\Phi\left(\frac{aN(m)}{X}\right)=\frac{X}{2aN(n)}\chi(1+i)\sum_{k\in\mathbb{Z}[i]}(-1)^{N(k)}g(k,n)\widetilde{W}\left(\sqrt{\frac{N(k)X}{2aN(n)}}\right). \tag{3.6}$$

Then we have

$$S_M=\frac{X}{2}\sum_{\varpi_1,\varpi_2}\frac{\log N(\varpi_1)\log N(\varpi_2)}{N(\varpi_1\varpi_2)}W\left(\frac{N(\varpi_1)}{Y}\right)W\left(\frac{N(\varpi_2)}{Y}\right)\sum_{\substack{N(l)\leqslant Z\\(l,1+i)=1\\(l,\varpi_1\varpi_2)=1\\l\equiv 1 mod (1+i)^3}}\frac{\mu_{\mathbb{Q}(i)}(l)}{N(l^2)}$$
$$\times\sum_k(-1)^{N(k)}g(k,\varpi_1\varpi_2)\widetilde{\Phi}\left(\sqrt{\frac{N(k)X}{2N(l^2\varpi_1\varpi_2)}}\right). \tag{3.7}$$

S_M can be expressed as the sum of three individual multiplicative series.

$$S_M=\frac{X}{2}\left(\sum_{\varpi_1,\varpi_2}\frac{\log N(\varpi_1)\log N(\varpi_2)}{N(\varpi_1\varpi_2)}W\left(\frac{N(\varpi_1)}{Y}\right)W\left(\frac{N(\varpi_2)}{Y}\right)\sum_{\substack{N(l)\leqslant Z\\(l,1+i)=1\\(l,\varpi_1\varpi_2)=1}}\frac{\mu_{\mathbb{Q}(i)}(l)}{N(l^2)}\right. \tag{3.8}$$
$$\times\sum_k(-1)^{N(k)}g(k,\varpi_1\varpi_2)\widetilde{\Phi}\left(\sqrt{\frac{N(k)X}{2N(l^2\varpi_1\varpi_2)}}\right)$$
$$+\sum_{\varpi}\frac{\log^2 N(\varpi)}{N^2(\varpi)}W\left(\frac{N(\varpi)}{Y}\right)^2\sum_{\substack{N(l)\leqslant Z\\(l,1+i)=1\\(l,\varpi)=1}}\frac{\mu_{\mathbb{Q}(i)}(l)}{N(l^2)}\sum_k\widetilde{\Phi}\left(\sqrt{\frac{N(k)X}{2N(l^2\varpi^2)}}\right)g(k,\varpi^2)$$
$$-\sum_{\varpi}\frac{\log^2 N(\varpi)}{N^2(\varpi)}W\left(\frac{N(\varpi)}{Y}\right)^2$$
$$\left.\times\sum_{\substack{N(l)\leqslant Z\\(l,1+i)=1\\(l,\varpi)=1}}\frac{\mu_{\mathbb{Q}(i)}(l)}{N(l^2)}\sum_k(-1)^{N(k)}\widetilde{\Phi}\left(\sqrt{\frac{N(k)X}{2N(l^2\varpi^2)}}\right)g^2(k,\varpi)\right)$$
$$=\frac{X}{2}(S_1+S_2-S_3).$$

3.3. The first main term in S_2

To deal with S_2, we take

$$S_2 = \sum_{\varpi} \frac{\log^2 N(\varpi)}{N^2(\varpi)} W\left(\frac{N(\varpi)}{Y}\right)^2 \sum_{\substack{N(l)\leqslant Z\\(l,1+i)=1\\(l,\varpi)=1}} \frac{\mu_{\mathbb{Q}(i)}(l)}{N(l^2)} \sum_k \widetilde{\Phi}\left(\sqrt{\frac{N(k)X}{2N(l^2\varpi^2)}}\right) g(k,\varpi^2)$$
$$= S_{2,0} + \sum_{k\neq 0} S_{2,k},$$

where

$$S_{2,0} = \widetilde{\Phi}(0) \sum_{\varpi} \frac{\log^2 N(\varpi)}{N^2(\varpi)} W\left(\frac{N(\varpi)}{Y}\right)^2 g(0,\varpi^2) \sum_{\substack{N(l)\leqslant Z\\(l,1+i)=1\\(l,\varpi)=1}} \frac{\mu_{\mathbb{Q}(i)}(l)}{N(l^2)}, \tag{3.9}$$

and

$$\sum_{k\neq 0} S_{2,k} = \sum_{\varpi} \frac{\log^2 N(\varpi)}{N^2(\varpi)} W\left(\frac{N(\varpi)}{Y}\right)^2 \sum_{\substack{N(l)\leqslant Z\\(l,1+i)=1\\(l,\varpi)=1}} \frac{\mu_{\mathbb{Q}(i)}(l)}{N(l^2)} \sum_{k\neq 0} \widetilde{\Phi}\left(\sqrt{\frac{N(k)X}{2N(l^2\varpi^2)}}\right) g(k,\varpi^2). \tag{3.10}$$

Combined with (2.9) and $g(0,\varpi^2) = \varphi(\varpi^2)$, $S_{2,0}$ becomes

$$S_{2,0} = \frac{4}{3} \frac{\widetilde{\Phi}(0)}{\zeta_{\mathbb{Q}(i)}(2)} \sum_{\varpi} \log^2 N(\varpi) W\left(\frac{N(\varpi)}{Y}\right)^2 + O\left(\sum_{\varpi} \frac{\log^2 N(\varpi)}{N(\varpi)} W\left(\frac{N(\varpi)}{Y}\right)^2\right)$$
$$+ O\left(\frac{\sum_{\varpi} \log^2 N(\varpi) W\left(\frac{N(\varpi)}{Y}\right)^2}{Z}\right). \tag{3.11}$$

We first estimate the sum $\sum_{\varpi} \log^2 N(\varpi) W\left(\frac{N(\varpi)}{Y}\right)^2$. As we can see, in this part, $W(t)$ is replaced by $W(t)^2$. For $Re(s) > 1$, we note that

$$\left(\frac{\zeta'_{\mathbb{Q}(i)}}{\zeta_{\mathbb{Q}(i)}}(s)\right)' = \sum_{\varpi} \frac{\log N(\varpi)^2}{N(\varpi)^s} \frac{1}{(1-N(\varpi)^{-s})^2}. \tag{3.12}$$

By Mellin inversion, we have

$$W\left(\frac{N(n)}{Y}\right)^2 = \frac{1}{2\pi i} \int_{(c)} \left(\frac{Y}{N(n)}\right)^s \widehat{W^2}(s) ds, \tag{3.13}$$

where

$$\widehat{W^2}(s) = \int_0^{\infty} W(t)^2 t^{s-1} ds \tag{3.14}$$

is the Mellin transform for $W(t)^2$ and c is a suitable constant. Without lose of generality, we may take $c=2$. Meanwhile, we have

$$\widehat{W^2}'(s)=\int_0^{+\infty}W^2(t)t^{s-1}\log t\,dt.$$

Then we have

$$\begin{aligned}
&\sum_{\varpi}\log^2 N(\varpi)W\left(\frac{N(\varpi)}{Y}\right)^2\\
=&\frac{1}{2\pi i}\int_{(2)}\sum_{\varpi}\frac{\log^2 N(\varpi)}{N(\varpi)^s}\frac{1}{(1-N(\varpi)^{-s})^2}Y^s\widehat{W^2}(s)ds\\
&+\frac{1}{2\pi i}\int_{(2)}\sum_{\varpi}\frac{\log^2 N(\varpi)}{N(\varpi)^{2s}}\frac{N(\varpi)^{-s}-2}{(1-N(\varpi)^{-s})^2}Y^s\widehat{W^2}(s)ds\\
=&\frac{1}{2\pi i}\int_{(2)}\left(\frac{\zeta'_{\mathbb{Q}(i)}}{\zeta_{\mathbb{Q}(i)}}(s)\right)'\widehat{W^2}(s)Y^sds+O\Big(\sum_{\varpi}\log^2 N(\varpi)W\Big(\frac{N(\varpi^2)}{Y}\Big)\Big).
\end{aligned}\tag{3.15}$$

Similar with Lemma 2.3, we obtain the following lemma.

LEMMA 3.1. *Assume RH. For arbitrary* $\varepsilon>0$, *we have*

$$\sum_{\varpi}\log^2 N(\varpi)W\left(\frac{N(\varpi)}{Y}\right)^2=Y\log Y\widehat{W^2}(1)+Y\widehat{W^2}'(1)+O(Y^{1/2+\varepsilon}\log Y).\tag{3.16}$$

Proof. We estimate the first term in (3.15) by integral by part and shifting line (2) to line $(\frac{1}{2}+\varepsilon)$. Assuming RH, $\frac{\zeta'_{\mathbb{Q}(i)}}{\zeta_{\mathbb{Q}(i)}}(s)$ has a pole at $s=1$ with residue -1. Then, using (2.10) and (3.14), we have

$$\begin{aligned}
\frac{\log Y}{2\pi i}\int_{(2)}\frac{\zeta'_{\mathbb{Q}(i)}}{\zeta_{\mathbb{Q}(i)}}(s)Y^s\widehat{W^2}(s)ds&=-Y\log Y\widehat{W^2}(1)+\frac{1}{2\pi i}\int_{(1/2+\varepsilon)}\frac{\zeta'_{\mathbb{Q}(i)}}{\zeta_{\mathbb{Q}(i)}}(s)Y^s\widehat{W^2}(s)ds\\
&=-Y\log Y\widehat{W^2}(1)+O(Y^{1/2+\varepsilon}).
\end{aligned}$$

Similarly, for the second term, we have

$$\int_{(2)}\frac{\zeta'_{\mathbb{Q}(i)}}{\zeta_{\mathbb{Q}(i)}}(s)(\widehat{W^2}(s))'Y^sds=-Y(\widehat{W^2}(1))'+O(Y^{1/2+\varepsilon}).$$

And the error term in (3.15) is given by

$$\ll Y^{1/2}\log Y+Y^{1/4}\log^3 Y\ll Y^{1/2}\log Y.\quad\square$$

The second term in $S_{2,0}$ can be obtained by Lemma 3.1 with slight modification. In fact, we have

$$\sum_{\varpi}\frac{\log^2 N(\varpi)}{N(\varpi)}W\left(\frac{N(\varpi)}{Y}\right)^2\ll Y^{\varepsilon}\log Y.$$

As a result, we have

$$S_{2,0} = \frac{4}{3}\frac{\widetilde{\Phi}(0)}{\zeta_{\mathbb{Q}(i)}(2)}(Y\log Y\widehat{W^2}(1)+Y\widehat{W^2}'(1))+O(Y^{1/2+\varepsilon}\log Y)+O\Big(\frac{Y\log Y}{Z}\Big). \tag{3.17}$$

Next, we estimate $\sum_{k\neq 0} S_{2,k}$. Recalling Lemma 2.1, we can react $\sum_{k\neq 0} S_{2,k}$ as

$$\begin{aligned}
&\sum_{k\neq 0} S_{2,k}\\
&= \sum_{\varpi}\frac{\log^2 N(\varpi)}{N(\varpi^2)}W\left(\frac{N(\varpi)}{Y}\right)^2 \sum_{\substack{N(l)\leqslant Z\\(l,1+i)=1\\(l,\varpi)=1}}\frac{\mu_{\mathbb{Q}(i)}(l)}{N(l^2)}\\
&\quad\times\sum_{\substack{k\neq 0\\(k,\varpi)=1}}(-1)^{N(k)}\widetilde{\Phi}\left(\sqrt{\frac{N(k)X}{2N(l^2\varpi)}}\right)(-N(\varpi))\\
&\quad+\sum_{\varpi}\frac{\log^2 N(\varpi)}{N(\varpi^2)}W\left(\frac{N(\varpi)}{Y}\right)^2 \sum_{\substack{N(l)\leqslant Z\\(l,1+i)=1\\(l,\varpi)=1}}\frac{\mu_{\mathbb{Q}(i)}(l)}{N(l^2)}\sum_{k\neq 0}(-1)^{N(k)}\widetilde{\Phi}\left(\sqrt{\frac{N(k)X}{2N(l^2)}}\right)\varphi(\varpi^2).
\end{aligned} \tag{3.18}$$

Using (2.5) with $j=3$, we have

$$\begin{aligned}
&\sum_{k\neq 0} S_{2,k}\\
&\ll \sum_{\varpi}\frac{\log^2 N(\varpi)}{N(\varpi)}W\left(\frac{N(\varpi)}{Y}\right)^2 \sum_{\substack{N(l)\leqslant Z\\(l,1+i)=1\\(l,\varpi)=1}}\frac{1}{N(l^2)}\left(\sum_{\substack{\frac{N(k)X}{2N(l^2\varpi)}<1\\k\neq 0}}1+\sum_{\frac{N(k)X}{2N(l^2\varpi)}>1}\frac{N(l^3)N^{\frac{3}{2}}(\varpi)}{N(k)^{\frac{3}{2}}X^{\frac{3}{2}}}U^2\right)\\
&\quad+\sum_{\varpi}\log^2 N(\varpi)W\left(\frac{N(\varpi)}{Y}\right)^2 \sum_{\substack{N(l)\leqslant Z\\(l,1+i)=1\\(l,\varpi)=1}}\frac{1}{N(l^2)}\left(\sum_{\substack{\frac{N(k)X}{2N(l^2)}<1\\k\neq 0}}1+\sum_{\frac{N(k)X}{2N(l^2)}>1}\frac{N(l^3)}{N(k)^{\frac{3}{2}}X^{\frac{3}{2}}}U^2\right)\\
&\ll \frac{U^2Z}{X}\sum_{\varpi}\log^2 N(\varpi)W\left(\frac{N(\varpi)}{Y}\right)^2.
\end{aligned}$$

Then Lemma 3.1 gives

$$\sum_{k\neq 0} S_{2,k} = O\Big(\frac{U^2Z}{X}Y\log Y\Big). \tag{3.19}$$

3.4. The error term S_3

By Lemma 2.1, S_3 can be written as

$$S_3 = \sum_{\varpi} \frac{\log^2 N(\varpi)}{N(\varpi)} W\left(\frac{N(\varpi)}{Y}\right)^2 \sum_{\substack{N(l)\leqslant Z \\ (l,1+i)=1 \\ (l,\varpi)=1}} \frac{\mu_{\mathbb{Q}(i)}(l)}{N(l^2)} \sum_{\substack{k\neq 0 \\ (k,\varpi)=1}} (-1)^{N(k)} \widetilde{\Phi}\left(\sqrt{\frac{N(k)X}{2N^2(l\varpi)}}\right).$$

Note that

$$\begin{aligned}
&\sum_{\substack{k\neq 0 \\ (k,\varpi)=1}} (-1)^{N(k)} \widetilde{\Phi}\left(\sqrt{\frac{N(k)X}{2N^2(l\varpi)}}\right) \\
&= \sum_{k} (-1)^{N(k)} \widetilde{\Phi}\left(\sqrt{\frac{N(k)X}{2N^2(l\varpi)}}\right) - \sum_{k} (-1)^{N(k)} \widetilde{\Phi}\left(\sqrt{\frac{N(k)X}{2N(l^2\varpi)}}\right).
\end{aligned}$$

For the first term, taking Fourier inversion, we have

$$\begin{aligned}
&\sum_{k} (-1)^{N(k)} \widetilde{\Phi}\left(\sqrt{\frac{N(k)X}{2N^2(l\varpi)}}\right) \\
&= 2\sum_{k} \widetilde{\Phi}\left(\sqrt{\frac{N(k)X}{N^2(l\varpi)}}\right) - \sum_{k} \widetilde{\Phi}\left(\sqrt{\frac{N(k)X}{2N^2(l\varpi)}}\right) \\
&= \frac{2N(l^2\varpi^2)}{X}\left(\sum_{k} \Phi\left(\frac{N(k)N(l^2\varpi^2)}{X}\right) - \sum_{k} \Phi\left(\frac{2N(k)N(l^2\varpi^2)}{X}\right)\right) \\
&\ll \frac{N(l^2\varpi^2)}{X}\left(\sum_{\frac{N(k)N(l^2\varpi^2)}{X}<1} 1 - \sum_{\frac{2N(k)N(l^2\varpi^2)}{X}<1} 1\right) \\
&\ll 1.
\end{aligned}$$

Similarly, we have the second term $\ll 1$. Then

$$S_3 \ll \sum_{\varpi} \frac{\log^2 N(\varpi)}{N(\varpi)} W\left(\frac{N(\varpi)}{Y}\right)^2 \sum_{\substack{N(l)\leqslant Z \\ (l,1+i)=1 \\ (l,\varpi)=1}} \frac{1}{N(l^2)} \ll Y^{\varepsilon} \log Y.$$

3.5. The second main term from S_1

We note that the term corresponding to $k=0$ in S_1 is 0. We therefore consider the sum taken over $k \neq 0$.

Applying Lemma 2.1, we have

$$S_1 = \sum_{\varpi_1,\varpi_2} \frac{\log N(\varpi_1)\log N(\varpi_2)}{N^{1/2}(\varpi_1\varpi_2)} W\left(\frac{N(\varpi_1)}{Y}\right) W\left(\frac{N(\varpi_2)}{Y}\right) \sum_{\substack{N(l)<Z\\(1,1+i)=1\\(1,\varpi_1\varpi_2)=1}} \frac{\mu_{\mathbb{Q}(i)}(l)}{N^2(l)} \times \sum_{\substack{k\neq 0\\(k,\varpi_1\varpi_2)=1}} (-1)^{N(k)} \left(\frac{ik}{\varpi_1\varpi_2}\right) \widetilde{\Phi}\left(\sqrt{\frac{N(k)X}{2N(l^2\varpi_1\varpi_2)}}\right). \tag{3.20}$$

Note that $N(ik)=N(k)$, $(ik,\varpi_1\varpi_2)=(k,\varpi_1\varpi_2)$, then S_1 can be written as

$$S_1 = \sum_{\varpi_1,\varpi_2} \frac{\log N(\varpi_1)\log N(\varpi_2)}{N^{1/2}(\varpi_1\varpi_2)} W\left(\frac{N(\varpi_1)}{Y}\right) W\left(\frac{N(\varpi_2)}{Y}\right) \sum_{\substack{N(l)<Z\\(1,1+i)=1\\(1,\varpi_1\varpi_2)=1}} \frac{\mu_{\mathbb{Q}(i)}(l)}{N^2(l)} \times \sum_{\substack{k\neq 0\\(k,\varpi_1\varpi_2)=1}} (-1)^{N(k)} \left(\frac{k}{\varpi_1\varpi_2}\right) \widetilde{\Phi}\left(\sqrt{\frac{N(k)X}{2N(l^2\varpi_1\varpi_2)}}\right). \tag{3.21}$$

Hence we have

$$S_1 = S_{1,1} + S_{1,2},$$

where

$$S_{1,1} = \frac{1}{2}\sum_{\varpi_1,\varpi_2} \frac{\log N(\varpi_1)\log N(\varpi_2)}{N^{1/2}(\varpi_1\varpi_2)} W\left(\frac{N(\varpi_1)}{Y}\right) W\left(\frac{N(\varpi_2)}{Y}\right) \sum_{\substack{N(l)<Z\\(1,1+i)=1\\(1,\varpi_1\varpi_2)=1}} \frac{\mu_{\mathbb{Q}(i)}(l)}{N^2(l)} \times \sum_{\substack{(k,\varpi_1\varpi_2)=1\\k\neq 0}} (-1)^{N(k)} \widetilde{\Phi}\left(\sqrt{\frac{N(k^2)X}{2N(l^2\varpi_1\varpi_2)}}\right)$$

and

$$S_{1,2} = \sum_{\substack{N(l)<Z\\(1,1+i)=1}} \frac{\mu_{\mathbb{Q}(i)}(l)}{N^2(l)} \sum_{\substack{k\neq 0\\k\neq k_1^2}} (-1)^{N(k)} \sum_{(l,\varpi_2)=1} \frac{\log N(\varpi_2)}{N^{1/2}(\varpi_2)} W\left(\frac{N(\varpi_2)}{Y}\right)\left(\frac{k}{\varpi_2}\right) \times \sum_{(\varpi_1,l)=1} W\left(\frac{N(\varpi_1)}{Y}\right)\left(\frac{k}{\varpi_1}\right) \widetilde{\Phi}\left(\sqrt{\frac{N(k)X}{2N(l^2\varpi_1\varpi_2)}}\right).$$

First, we focus on $S_{1,1}$ who will contribute the second main term. To do so, we express $S_{1,1}$ as

$$
\begin{aligned}
2S_{1,1} = & \sum_{\varpi_1,\varpi_2} \frac{\log N(\varpi_1)\log N(\varpi_2)}{N^{1/2}(\varpi_1\varpi_2)} W\left(\frac{N(\varpi_1)}{Y}\right) W\left(\frac{N(\varpi_2)}{Y}\right) \\
& \times \sum_{\substack{N(l)<Z \\ (1,1+i)=1 \\ (1,\varpi_1\varpi_2)=1}} \frac{\mu_{\mathbb{Q}(i)}(l)}{N^2(l)} \sum_{k\neq 0} (-1)^{N(k)} \widetilde{\Phi}\left(\sqrt{\frac{N(k^2)X}{2N(l^2\varpi_1\varpi_2)}}\right) \\
& - \sum_{\varpi_1,\varpi_2} \frac{\log N(\varpi_1)\log N(\varpi_2)}{N^{1/2}(\varpi_1\varpi_2)} W\left(\frac{N(\varpi_1)}{Y}\right) W\left(\frac{N(\varpi_2)}{Y}\right) \\
& \times \sum_{\substack{N(l)<Z \\ (1,1+i)=1 \\ (1,\varpi_1\varpi_2)=1}} \frac{\mu_{\mathbb{Q}(i)}(l)}{N^2(l)} \sum_{\substack{\varpi_1|k \\ k\neq 0 \\ (k,\varpi_2)=1}} (-1)^{N(k)} \widetilde{\Phi}\left(\sqrt{\frac{N(k^2)X}{2N(l^2\varpi_1\varpi_2)}}\right) \\
& - \sum_{\varpi_1,\varpi_2} \frac{\log N(\varpi_1)\log N(\varpi_2)}{N^{1/2}(\varpi_1\varpi_2)} W\left(\frac{N(\varpi_1)}{Y}\right) W\left(\frac{N(\varpi_2)}{Y}\right) \\
& \times \sum_{\substack{N(l)<Z \\ (1,1+i)=1 \\ (1,\varpi_1\varpi_2)=1}} \frac{\mu_{\mathbb{Q}(i)}(l)}{N^2(l)} \sum_{\substack{(k,\varpi_1)=1 \\ k\neq 0 \\ \varpi_2|k}} (-1)^{N(k)} \widetilde{\Phi}\left(\sqrt{\frac{N(k^2)X}{2N(l^2\varpi_1\varpi_2)}}\right) \\
& - \sum_{\varpi_1,\varpi_2} \frac{\log N(\varpi_1)\log N(\varpi_2)}{N^{1/2}(\varpi_1\varpi_2)} W\left(\frac{N(\varpi_1)}{Y}\right) W\left(\frac{N(\varpi_2)}{Y}\right) \\
& \times \sum_{\substack{N(l)<Z \\ (1,1+i)=1 \\ (1,\varpi_1\varpi_2)=1}} \frac{\mu_{\mathbb{Q}(i)}(l)}{N^2(l)} \sum_{\substack{\varpi_1\varpi_2|k \\ k\neq 0}} (-1)^{N(k)} \widetilde{\Phi}\left(\sqrt{\frac{N(k^2)X}{2N(l^2\varpi_1\varpi_2)}}\right) \\
= & M - R,
\end{aligned}
$$

where

$$R = R_{\varpi_1} + R_{\varpi_2} + R_{\varpi_1\varpi_2}.$$

To separate the main term form M, we need the Lemma 3.5 in [2].

LEMMA 3.2.

$$\sum_{k\neq 0} (-1)^{N(k)} \tilde{\Phi}\left(\frac{N(k)}{y}\right) = -\widetilde{\Phi}(0) + O\Big(\frac{U^2}{y^{1/2}}\Big). \tag{3.22}$$

As a result, we have

$$M=-\widetilde{\Phi}(0)\sum_{\varpi_1,\varpi_2}\frac{\log N(\varpi_1)\log N(\varpi_2)}{N^{1/2}(\varpi_1\varpi_2)}W\left(\frac{N(\varpi_1)}{Y}\right)W\left(\frac{N(\varpi_2)}{Y}\right)\sum_{\substack{N(l)<Z\\(1,1+i)=1\\(1,\varpi_1\varpi_2)=1}}\frac{\mu_{\mathbb{Q}(i)}(l)}{N^2(l)}$$
$$+O\Bigg(U^2X^{1/4}\sum_{\varpi_1,\varpi_2}\frac{\log N(\varpi_1)\log N(\varpi_2)}{N^{3/4}(\varpi_1\varpi_2)}$$
$$\times W\left(\frac{N(\varpi_1)}{Y}\right)W\left(\frac{N(\varpi_2)}{Y}\right)\sum_{\substack{N(l)<Z\\(1,1+i)=1\\(1,\varpi_1\varpi_2)=1}}\frac{1}{N^{5/2}(l)}\Bigg). \tag{3.23}$$

The condition $(l,\varpi_1\varpi_2)=1$ may imply $\varpi_1\neq\varpi_2$, so the sum taken over ϖ_1, ϖ_2 should be

$$\left(\sum_{\varpi}\frac{\log N(\varpi)}{N^{1/2}(\varpi)}(1-N^{-2}(\varpi))^{-1}W\left(\frac{N(\varpi)}{Y}\right)\right)^2$$
$$-\sum_{\varpi}\frac{\log^2 N(\varpi)}{N(\varpi)}(1-N^{-2}(\varpi))^{-2}W\left(\frac{N(\varpi)}{Y}\right)^2$$
$$+\sum_{\varpi}\frac{\log^2 N(\varpi)}{N(\varpi)}(1-N^{-2}(\varpi))^{-1}W\left(\frac{N(\varpi)}{Y}\right)^2.$$

By (2.9) and Lemma 3.1, we have the first term

$$=-\frac{\tilde{\Phi}(0)}{\zeta_{\mathbb{Q}(i)}(2)}Y\widehat{W}\left(\frac{1}{2}\right)^2+O(Y^{1/2+\varepsilon})+O\Big(\frac{Y}{Z}\Big)$$

and the second term

$$\ll U^2X^{1/4}Y^{\frac{1}{2}+\varepsilon}.$$

Hence

$$M=-\frac{4\widetilde{\Phi}(0)}{3\zeta_{\mathbb{Q}(i)}(2)}\widehat{W}^2\left(\frac{1}{2}\right)Y+O\Big(Y^{1/2+\varepsilon}+\frac{Y}{Z}+U^2X^{1/4}Y^{\frac{1}{2}+\varepsilon}\Big). \tag{3.24}$$

(2.5) gives

$$R_{\varpi_1}\ll\sum_{\varpi_1,\varpi_2}\frac{\log N(\varpi_1)\log N(\varpi_2)}{N^{1/2}(\varpi_1\varpi_2)}W\left(\frac{N(\varpi_1)}{Y}\right)W\left(\frac{N(\varpi_2)}{Y}\right)$$
$$\times\sum_{\substack{N(l)<Z\\(1,1+i)=1\\(1,\varpi_1\varpi_2)=1}}\frac{1}{N^2(l)}\sum_{k\neq 0}\widetilde{\Phi}\left(\sqrt{\frac{N(\varpi_1)N(k^2)X}{2N(l^2\varpi_2)}}\right)$$

$$\ll \prod_{i=1}^{2}\sum_{\varpi_i}\frac{\log N(\varpi_i)}{N^{1/2}(\varpi_i)}W\left(\frac{N(\varpi_i)}{Y}\right)$$
$$\times \sum_{\substack{N(l)\leqslant Z\\(l,1+i)=1\\(l,\varpi_1\varpi_2)=1}}\frac{1}{N^2(l)}\left(\sum_{N(k^2)<\frac{N(l^2\varpi_2)}{XN(\varpi_1)}}1+\sum_{N(k^2)>\frac{N(l^2\varpi_2)}{XN(\varpi_1)}}\frac{UN(l^2\varpi_2)}{N(k^2\varpi_1)X}\right)$$
$$\ll \frac{U\log Z}{X^{1/2}}\sum_{\varpi_1}\frac{\log N(\varpi_1)}{N(\varpi_1)}W\left(\frac{N(\varpi_1)}{Y}\right)\sum_{\varpi_2}\log N(\varpi_2)W\left(\frac{N(\varpi_2)}{Y}\right)$$
$$\ll \frac{U\log Z}{X^{1/2}}Y^{1+\varepsilon}. \tag{3.25}$$

Also, we have $R_{\varpi_2} \ll \frac{U\log Z}{X^{1/2}}Y^{1+\varepsilon}$. Similarly,

$$R_{\varpi_1\varpi_2} \ll \frac{Z}{X}. \tag{3.26}$$

Finally, we obtain

$$R = O\Big(\frac{U\log Z}{X^{1/2}}Y^{1+\varepsilon}\Big). \tag{3.27}$$

REMARK 3.1. *In fact, to find out the main part of each error term, we only need to consider the residue of the integral of form* $\frac{1}{2\pi i}\int_{(c)}\frac{\zeta'_{\mathbb{Q}(i)}}{\zeta_{\mathbb{Q}(i)}}(s+t)Y^s\widehat{W}(s)ds$, *where* t *is the power of* $\frac{1}{N(\varpi)}$ *in* $\sum_{\varpi}\frac{\log N(\varpi)}{N^t(\varpi)}W\left(\frac{N(\varpi)}{Y}\right)$.

3.6. The error term $S_{1,2}$

It is a bit complicated to obtain the last error term $S_{1,2}$. We begin with Mellin inversion who gives

$$W\left(\frac{N(\varpi_1)}{Y}\right)\widetilde{\Phi}\left(\sqrt{\frac{N(k)X}{N(l^2\varpi_1\varpi_2)}}\right)=\frac{1}{2\pi i}\int_{(c)}\left(\frac{Y}{N(\varpi_1)}\right)^s\widetilde{f}(s,k,\varpi_2)ds, \tag{3.28}$$

where

$$\widetilde{f}(s,k,\varpi_2)=\int_0^{+\infty}W(t)\widetilde{\Phi}\left(\sqrt{\frac{N(k)X}{2YN(l^2\varpi_2 t)}}\right)t^{s-1}dt. \tag{3.29}$$

For $Re(s)>0$ and integers $D>0,\ E>0$, the function $\widetilde{f}$ has upper bound

$$\widetilde{f}(s,k,\varpi_2)\ll(1+|s|)^{-D}\left(1+\sqrt{\frac{N(k)X}{2YN(l^2\varpi_2)}}\right)^{-E+D}U^{E-1}. \tag{3.30}$$

$$
\begin{aligned}
S_{1,2} = &\sum_{\substack{k\neq 0\\ k\neq k_1^2}} (-1)^{N(k)} \sum_{\substack{N(l)<Z\\ (1,1+i)=1}} \frac{\mu_{\mathbb{Q}(i)}(l)}{N^2(l)} \sum_{\varpi_2} \frac{\log N(\varpi_2)}{N^{1/2}(\varpi_2)} W\left(\frac{N(\varpi_2)}{Y}\right)\left(\frac{k}{\varpi_2}\right)\\
&\times \sum_{\varpi_1} \frac{\log N(\varpi_1)}{N^{1/2}(\varpi_1)} W\left(\frac{N(\varpi_1)}{Y}\right)\left(\frac{k}{\varpi_1}\right)\widetilde{\Phi}\left(\sqrt{\frac{N(k)X}{2N(l^2\varpi_1\varpi_2)}}\right)\\
&+O\Bigg(\sum_{\substack{N(l)<Z\\ (1,1+i)=1}} \frac{1}{N^2(l)} \sum_{\varpi_1\varpi_2|l} \frac{\log N(\varpi_1)\log N(\varpi_2)}{N^{1/2}(\varpi_1\varpi_2)} W\left(\frac{N(\varpi_2)}{Y}\right) W\left(\frac{N(\varpi_1)}{Y}\right)\\
&\times \sum_k \widetilde{\Phi}\left(\sqrt{\frac{N(k)X}{2N(l^2\varpi_1\varpi_2)}}\right)\Bigg).
\end{aligned}
$$

Taking Fourier inversion, we have the error term $\ll 1$ immediately. Then we deal with the inner term taken over ϖ_1.

$$
\begin{aligned}
&\sum_{\varpi_1} \frac{\log N(\varpi_1)}{N^{1/2}(\varpi_1)} \left(\frac{k}{\varpi_1}\right) W\left(\frac{N(\varpi_1)}{Y}\right)\widetilde{\Phi}\left(\sqrt{\frac{N(k)X}{2N(l^2\varpi_1\varpi_2)}}\right)\\
=&\sum_n \frac{\Lambda(n)}{N^{1/2}(n)}\left(\frac{k}{n}\right) W\left(\frac{N(n)}{Y}\right)\widetilde{\Phi}\left(\sqrt{\frac{N(k)X}{2N(l^2 n\varpi_2)}}\right)\\
&+O\left(\sum_{\substack{N(\varpi^j)<Y\\ j\geqslant 2}} \frac{\Lambda(\varpi)}{N(\varpi^{j/2})} W\left(\frac{N(\varpi^{j/2})}{Y}\right)\widetilde{\Phi}\left(\sqrt{\frac{N(k)X}{2N(l^2\varpi^j\varpi_2)}}\right)\right)\\
=&\frac{-1}{2\pi i}\int_{(c)} \frac{L'}{L}\left(s+\frac{1}{2},\chi_k\right) Y^s \widetilde{f}(s,k,\varpi_2)\,ds\\
&+O\left(\sum_{\substack{N(\varpi^j)<Y\\ j\geqslant 2}} \frac{\Lambda(\varpi^j)}{N^{1/2}(\varpi^j)} W\left(\frac{N(\varpi^j)}{Y}\right)\widetilde{\Phi}\left(\sqrt{\frac{N(k)X}{2N(l^2\varpi^j\varpi_2)}}\right)\right).
\end{aligned}
$$

For the sum taken over ϖ_2, we have

$$
\begin{aligned}
&\sum_{\varpi_2} \frac{\log N(\varpi_2)}{N^{1/2}(\varpi_2)} \chi_k(\varpi_2) W\left(\frac{N(\varpi_2)}{Y}\right)\\
=&\frac{-1}{2\pi i}\int_{(c)} \frac{L'}{L}(s+1/2,\chi_k) Y^s \widetilde{W}(s)\,ds + \sum_{\substack{N(\varpi_2^j)<Y\\ j\geqslant 2}} \frac{\Lambda(\varpi_2^j)}{N^{1/2}(\varpi_2^j)} W\left(\frac{N(\varpi_2^j)}{Y}\right).
\end{aligned}
$$

Then $S_{1,2}$ becomes

$$
S_{1,2} = R_1 + R_2 + R_3 + O(1),
$$

where

$$R_1 = \sum_{\substack{k\neq 0\\ k\neq k_1^2}} (-1)^{N(k)} \sum_{\substack{N(l)<Z\\ (1,1+i)=1}} \frac{\mu_{\mathbb{Q}(i)}(l)}{N^2(l)} \left(\frac{-1}{2\pi i}\right)^2 \int_{(c)} \frac{L'}{L}(s+1/2,\chi_k) Y^s \widetilde{W}(s) ds$$

$$\times \int_{(c)} \frac{L'}{L}(s+\frac{1}{2},\chi_k) Y^s \widetilde{f}(s,k,\varpi_2) ds,$$

$$R_2 = \sum_{\substack{k\neq 0\\ k\neq k_1^2}} (-1)^{N(k)} \sum_{\substack{N(l)<Z\\ (1,1+i)=1}} \frac{\mu_{\mathbb{Q}(i)}(l)}{N^2(l)} \sum_{\substack{N(\varpi_2^j)<Y\\ j\geqslant 2}} \frac{\Lambda(\varpi_2^j)}{N^{1/2}(\varpi_2^j)} W\left(\frac{N(\varpi_2^j)}{Y}\right)$$

$$\times \frac{-1}{2\pi i} \int_{(c)} \frac{L'}{L}(s+\frac{1}{2},\chi_k) Y^s \widetilde{f}(s,k,\varpi_2) ds,$$

$$R_3 = \sum_{\substack{k\neq 0\\ k\neq k_1^2}} (-1)^{N(k)} \sum_{\substack{N(l)<Z\\ (1,1+i)=1}} \frac{\mu_{\mathbb{Q}(i)}(l)}{N^2(l)} \frac{-1}{2\pi i} \int_{(c)} \frac{L'}{L}(s+1/2,\chi_k) Y^s \widetilde{W}(s) ds$$

$$\times \sum_{\substack{N(\varpi^j)<Y\\ j\geqslant 2}} \frac{\Lambda(\varpi^j)}{N^{1/2}(\varpi^j)} W\left(\frac{N(\varpi^j)}{Y}\right) \widetilde{\Phi}\left(\sqrt{\frac{N(k)X}{2N(l^2\varpi^j\varpi_2)}}\right)$$

and

$$R_4 = \sum_{\substack{k\neq 0\\ k\neq k_1^2}} (-1)^{N(k)} \sum_{\substack{N(l)<Z\\ (1,1+i)=1}} \frac{\mu_{\mathbb{Q}(i)}(l)}{N^2(l)} \sum_{\substack{N(\varpi_2^j)<Y\\ j\geqslant 2}} \frac{\Lambda(\varpi_2^j)}{N^{1/2}(\varpi_2^j)} W\left(\frac{N(\varpi_2^j)}{Y}\right)$$

$$\times \sum_{\substack{N(\varpi^j)<Y\\ j\geqslant 2}} \frac{\Lambda(\varpi^j)}{N^{1/2}(\varpi^j)} W\left(\frac{N(\varpi^j)}{Y}\right) \widetilde{\Phi}\left(\sqrt{\frac{N(k)X}{2N(l^2\varpi^j\varpi_2)}}\right).$$

To deal with R_1, taking $c=2$ and $\widetilde{\Phi} \ll 1$ uniformly, shifting (2) to (ε), since k is not a perfect square, there is no pole between (2) and (ε). Similar with Lemma 2.3, combined with Theorem 5.17 in [6] who gives a bound of logarithmic derivative of $L(s,\chi)$, we have

$$R_1 \ll \sum_{N(l)<Z} \frac{1}{N^2(l)} \sum_{N(k)<\frac{Y^2N^2(l)}{X}} \left(\frac{-1}{2\pi i}\int_{(c)} \frac{L'}{L}(s+\frac{1}{2},\chi) Y^s \widetilde{W}(s) ds\right)^2 \ll \frac{Y^{2+\varepsilon}Z}{X}.$$

For R_i, $i=2,3,4$, by similar method, we have

$$R_i \ll \frac{Y^{2+\varepsilon}Z}{X^{\varepsilon}}.$$

Finally, we have

$$S_{1,2} = O\left(\frac{Y^{2+\varepsilon}Z}{X^{\varepsilon}}\right). \tag{3.31}$$

3.7. Conclusion

Combining (3.5), (3.27), (3.17), (3.19) and (3.31), we have

$$
\begin{aligned}
S(X,Y) = {} & \frac{X}{2}\frac{4}{3}\frac{\widetilde{\Phi}(0)}{\zeta_{\mathbb{Q}(i)}(2)}\left(Y\log Y\widehat{W^2}(1)+Y\widehat{W^2}'(1)\right)-\frac{X}{4}\frac{4\widetilde{\Phi}(0)}{3\zeta_{\mathbb{Q}(i)}(2)}\widehat{W}^2\left(\frac{1}{2}\right)Y \\
& +O\left(XY^{1/2+\varepsilon}\log Y+\frac{XY^{1+\varepsilon}\log Y}{Z}\right) \\
& +O\left(XY^{1/2+\varepsilon}+\frac{XY}{Z}+U^2X^{5/4}Y^{1/2+\varepsilon}+U\log ZX^{1/2}Y^{1+\varepsilon}\right) \\
& +O(X^{-\varepsilon}Y^{2+\varepsilon}Z)
\end{aligned}
$$

Taking $U=X^{\varepsilon}Y^{\varepsilon}\log Y$, $Z=X^{\varepsilon}Y^{\varepsilon}\log X$, we completely prove the Theorem 1.1.

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