Journal of Mathematical Inequalities
Volume 17, Number 2 (2023), 671–681
doi:10.7153/jmi-2023-17-44

SHARP INEQUALITIES OF IYENGAR–MADHAVA RAO–NANJUNDIAH TYPE INCLUDING $\cos\left(\frac{x}{\sqrt{3}}+ax^r\right)$

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(*Communicated by L. Mihoković*)

Abstract. In this paper, for $0<x<\frac{\pi}{2}$ and $r>0$, we consider the following Iyengar-Madhava Rao-Nanjundiah type inequality;

$$\cos\left(\frac{x}{\sqrt{3}}+\alpha x^r\right)<\frac{\sin x}{x}<\cos\left(\frac{x}{\sqrt{3}}+\beta x^r\right).$$

Our main theorems shows that α and β depend on $r>0$, and if $0<r<3$ then

$$\beta=\left(\frac{2}{\pi}\right)^r\left(-\frac{\pi}{2\sqrt{3}}+\arccos\frac{2}{\pi}\right)$$

and if $r>4$ then

$$\alpha=\left(\frac{2}{\pi}\right)^r\left(-\frac{\pi}{2\sqrt{3}}+\arccos\frac{2}{\pi}\right).$$

1. Introduction

Mitrinović [2] in p.236, presented that the inequality

$$1\geqslant\cos\frac{x}{2}\geqslant\cos ax\geqslant\frac{\sin x}{x}\geqslant\cos bx\geqslant\sqrt[3]{\cos x}\geqslant\frac{\cos x}{1-\frac{x^2}{3}}\geqslant\cos x \tag{1}$$

holds for $0<x<\frac{\pi}{2}$, where the constants $a=\frac{2}{\pi}\arccos\frac{2}{\pi}\cong 0.560664$ and $b=\frac{1}{\sqrt{3}}\cong 0.57735$ are the best possible. Especially, for $0<x<\frac{\pi}{2}$, the inequality

$$\cos ax>\frac{\sin x}{x}>\cos bx \tag{2}$$

is known as Iyengar-Madhava Rao-Nanjundiah inequality [1]. Recently, Sándor [8] gave a new proof different from Iyengar's one and proved the hyperbolic version. Also, for $0<x<\frac{\pi}{2}$, the inequality

$$\frac{\sin x}{x}>\sqrt[3]{\cos x} \tag{3}$$

Mathematics subject classification (2020): 26D05, 26D07, 26D15.
Keywords and phrases: Inequalities, trigonometric functions, Iyengar-Madhava Rao-Nanjundiah inequality, Adamović-Mitrinović inequality, best possible constant.

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is known as Mitrinović-Adamovic inequality and many known studies of Mitrinović-Adamovic type inequality. Recently, there are the following paper; [2], [3], [6], [7], [9]. On the other hand, little is known about Iyengar-Madhava Rao-Nanjundiah inequality (2). In this paper, we consider the improved Iyengar-Madhava Rao-Nanjundiah inequality and our main theorems are the followings.

THEOREM 1. *For* $0 < x < \frac{\pi}{2}$, *we have*

$$\cos\left(\frac{x}{\sqrt{3}} + \alpha x^3\right) < \frac{\sin x}{x} < \cos\left(\frac{x}{\sqrt{3}} + \beta x^3\right),$$

where the constants $\alpha = -\frac{1}{90\sqrt{3}} \cong -0.006415$ *and* $\beta = \frac{8}{\pi^3}\left(-\frac{\pi}{2\sqrt{3}} + \arccos\frac{2}{\pi}\right) \cong -0.00676262$ *are the best possible.*

THEOREM 2. *For* $0 < x < \frac{\pi}{2}$ *and* $0 < r < 3$, *we have*

$$\cos\left(\frac{x}{\sqrt{3}} + \alpha x^r\right) < \frac{\sin x}{x} < \cos\left(\frac{x}{\sqrt{3}} + \beta_r x^r\right),$$

where the constant $\alpha = 0$ *is the best possible and* $\beta_r = \left(\frac{2}{\pi}\right)^r\left(-\frac{\pi}{2\sqrt{3}} + \arccos\frac{2}{\pi}\right)$.

THEOREM 3. *For* $0 < x < \frac{\pi}{2}$ *and* $r > 4$, *we have*

$$\frac{\sin x}{x} > \cos\left(\frac{x}{\sqrt{3}} + \alpha_r x^r\right),$$

where $\alpha_r = \left(\frac{2}{\pi}\right)^r\left(-\frac{\pi}{2\sqrt{3}} + \arccos\frac{2}{\pi}\right)$ *and the constant* β *such that inequality*

$$\frac{\sin x}{x} < \cos\left(\frac{x}{\sqrt{3}} + \beta x^r\right)$$

holds doesn't exist.

Sándor [8] gave a new proof using the monotonicity of functions and L'Hospital's rule (see e.g. [4], [5]). In this paper, we show the monotonicity of functions using computations by Mathematica software and, like Sándor, prove the our main theorems using the monotonicity of functions and L'Hospital's rule.

2. Proof of Theorem 1

We may show some lemmas required for the proof of Theorem 1.

LEMMA 1. *For* $0 < x < \frac{\pi}{2}$, *we have*

$$\frac{2x}{\sqrt{3}} - 3\arccos\left(\frac{\sin x}{x}\right) - \frac{\cos x - \frac{\sin x}{x}}{\sqrt{1 - \frac{\sin^2 x}{x^2}}} < 0.$$

Proof. We set

$$f(x)=\frac{2x}{\sqrt{3}}-3\arccos\left(\frac{\sin x}{x}\right)-\frac{\cos x-\frac{\sin x}{x}}{\sqrt{1-\frac{\sin^2 x}{x^2}}}$$

and the derivative of $f(x)$ is

$$f'(x)=\frac{2}{\sqrt{3}}+\frac{4x^3\cos x-5x^2\sin x+x^4\sin x-2x\cos x\sin^2 x+3\sin^3 x}{x^4\left(\frac{x^2-\sin^2 x}{x^2}\right)^{\frac{3}{2}}}.$$

For $0<x<\frac{\pi}{2}$ and non-negative integers m and n, we have

$$u(x,2n+1)<\sin x<u(x,2n)\quad\text{and}\quad v(x,2m+1)<\cos x<v(x,2m),$$

where

$$u(x,p)=\sum_{k=0}^{p}\frac{(-1)^k x^{2k+1}}{(2k+1)!}\quad\text{and}\quad v(x,q)=\sum_{k=0}^{q}\frac{(-1)^k x^{2k}}{(2k)!}$$

are truncations of

$$\sin x=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!}\quad\text{and}\quad \cos x=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!}.$$

We have

$$\begin{aligned}f'(x)&<\frac{2}{\sqrt{3}}+\frac{4x^3v(x,4)-5x^2u(x,3)+x^4u(x,4)-2xv(x,5)\left(u(x,3)\right)^2+3\left(u(x,4)\right)^3}{x\left(x^2-\left(u(x,3)\right)^2\right)^{\frac{3}{2}}}\\&=g(x)\end{aligned}$$

and the derivative of $g(x)$ is

$$g'(x)=\frac{x^3h(x)}{622080(8467200-1128960x^2+80640x^4-3444x^6+84x^8-x^{10})^{\frac{5}{2}}}.$$

Here, $h(x)$ is negative as in Appendix A in the final part of this paper, therefore $g(x)$ is strictly decreasing for $0<x<\frac{\pi}{2}$. From $\lim_{x\to+0}g(x)=\frac{2}{\sqrt{3}}-\frac{17698046607360000}{622080(8467200)^{\frac{3}{2}}}=0$, we obtain $g(x)<0$ for $0<x<\frac{\pi}{2}$ and $f(x)$ is strictly decreasing for $0<x<\frac{\pi}{2}$. From L'Hospital's rule, we have

$$\lim_{x\to+0}\sqrt{\frac{x^2-\sin^2 x}{x^4}}=\lim_{x\to+0}\sqrt{\frac{x+\sin x}{x}}\cdot\sqrt{\frac{x-\sin x}{x^3}}=\sqrt{2}\cdot\frac{1}{\sqrt{6}}=\frac{1}{\sqrt{3}}$$

and

$$\lim_{x\to+0}\frac{x\cos x-\sin x}{x^2}=\lim_{x\to+0}\frac{-x\sin x}{2x}=0.$$

Hence, we can get

$$\lim_{x\to+0}\frac{\cos x-\frac{\sin x}{x}}{\sqrt{1-\frac{\sin^2 x}{x^2}}}=\lim_{x\to+0}\frac{\cos x-\frac{\sin x}{x}}{x\sqrt{\frac{x^2-\sin^2 x}{x^4}}}=\lim_{x\to+0}\frac{1}{\sqrt{\frac{x^2-\sin^2 x}{x^4}}}\cdot\frac{x\cos x-\sin x}{x^2}$$
$$=\sqrt{3}\cdot 0=0$$

and $\lim_{x\to+0} f(x)=0$ and $f(x)<0$ for $0<x<\frac{\pi}{2}$. □

LEMMA 2. *We have*

$$\lim_{x\to+0}\frac{-\sqrt{3}x\cos x+\sqrt{3}\sin x-x\sqrt{x^2-\sin^2 x}}{x^5}=-\frac{1}{30\sqrt{3}}.$$

Proof. Since we have $u(x,3)<\sin x<u(x,2)$ and $v(x,3)<\cos x<v(x,2)$ for $0<x<\frac{\pi}{2}$, we have

$$\frac{-\sqrt{3}xv(x,2)+\sqrt{3}u(x,3)-x\sqrt{x^2-(u(x,3))^2}}{x^5}$$
$$<\frac{-\sqrt{3}x\cos x+\sqrt{3}\sin x-x\sqrt{x^2-\sin^2 x}}{x^5}$$
$$<\frac{-\sqrt{3}xv(x,3)+\sqrt{3}u(x,2)-x\sqrt{x^2-(u(x,2))^2}}{x^5}$$

and

$$\frac{1680\sqrt{3}-168\sqrt{3}x^2-\sqrt{3}x^4-\sqrt{8467200-1128960x^2+80640x^4-3444x^6+84x^8-x^{10}}}{5040x^2}$$
$$<\frac{-\sqrt{3}x\cos x+\sqrt{3}\sin x-x\sqrt{x^2-\sin^2 x}}{x^5}$$
$$<\frac{240\sqrt{3}-24\sqrt{3}x^2+\sqrt{3}x^4-6\sqrt{4800-640x^2+40x^4-x^6}}{720x^2}.$$

Since we have

$$\lim_{x\to+0}\frac{1680\sqrt{3}-168\sqrt{3}x^2-\sqrt{3}x^4-\sqrt{8467200-1128960x^2+80640x^4-3444x^6+84x^8-x^{10}}}{5040x^2}$$
$$=\lim_{x\to+0}\frac{-336\sqrt{3}x-4\sqrt{3}x^3-\frac{-2257920x+322560x^3-20664x^5+672x^7-10x^9}{2\sqrt{8467200-1128960x^2+80640x^4-3444x^6+84x^8-x^{10}}}}{10080x}$$
$$=\frac{-336\sqrt{3}-\frac{-2257920}{2\sqrt{8467200}}}{10080}=-\frac{1}{30\sqrt{3}}$$

and

$$\lim_{x\to+0}\frac{240\sqrt{3}-24\sqrt{3}x^2+\sqrt{3}x^4-6\sqrt{4800-640x^2+40x^4-x^6}}{720x^2}$$
$$=\lim_{x\to+0}\frac{-48\sqrt{3}x+4\sqrt{3}x^3-\frac{3(-1280x+160x^3-6x^5)}{\sqrt{4800-640x^2+40x^4-x^6}}}{1440x}=\frac{-48\sqrt{3}-\frac{-3\cdot1280}{\sqrt{4800}}}{1440}=-\frac{1}{30\sqrt{3}},$$

we can get

$$\lim_{x\to+0}\frac{-\sqrt{3}x\cos x+\sqrt{3}\sin x-x\sqrt{x^2-\sin^2 x}}{x^5}=-\frac{1}{30\sqrt{3}}.\quad\square$$

LEMMA 3. *We have*

$$\lim_{x\to+0}\left(\frac{\arccos\left(\frac{\sin x}{x}\right)}{x^3}-\frac{1}{\sqrt{3}x^2}\right)=-\frac{1}{90\sqrt{3}}.$$

Proof. From Lemma 2 and L'Hospital's rule, we have

$$\lim_{x\to+0}\left(\frac{\arccos\left(\frac{\sin x}{x}\right)}{x^3}-\frac{1}{\sqrt{3}x^2}\right)=\lim_{x\to+0}\frac{\sqrt{3}\arccos\left(\frac{\sin x}{x}\right)-x}{\sqrt{3}x^3}$$
$$=\lim_{x\to+0}\frac{-1-\frac{\sqrt{3}\left(\frac{\cos x}{x}-\frac{\sin x}{x^2}\right)}{\sqrt{1-\frac{\sin^2 x}{x^2}}}}{3\sqrt{3}x^2}=\lim_{x\to+0}\frac{-\sqrt{3}x\cos x+\sqrt{3}\sin x-x\sqrt{x^2-\sin^2 x}}{3\sqrt{3}x^5\sqrt{\frac{x^2-\sin^2 x}{x^4}}}$$
$$=\lim_{x\to+0}\frac{1}{3\sqrt{3}\sqrt{\frac{x^2-\sin^2 x}{x^4}}}\cdot\frac{-\sqrt{3}x\cos x+\sqrt{3}\sin x-x\sqrt{x^2-\sin^2 x}}{x^5}$$
$$=\frac{1}{3}\cdot\left(-\frac{1}{30\sqrt{3}}\right)=-\frac{1}{90\sqrt{3}}.\quad\square$$

Proof of Theorem 1. We consider the equation

$$\frac{x}{\sqrt{3}}+ax^3=\arccos\left(\frac{\sin x}{x}\right)$$

and we have

$$a=\frac{\arccos\left(\frac{\sin x}{x}\right)}{x^3}-\frac{1}{\sqrt{3}x^2}=f(x).$$

The derivative of $f(x)$ is

$$f'(x)=\frac{1}{x^4}\left(\frac{2x}{\sqrt{3}}-3\arccos\left(\frac{\sin x}{x}\right)-\frac{\cos x-\frac{\sin x}{x}}{\sqrt{1-\frac{\sin^2 x}{x^2}}}\right).$$

By Lemma 1, we have $f'(x) < 0$ for $0 < x < \frac{\pi}{2}$. Since $f(x)$ is strictly decreasing for $0 < x < \frac{\pi}{2}$ and by Lemma 3, we have

$$\lim_{x \to -\frac{\pi}{2}} f(x) < f(x) < \lim_{x \to +0} f(x)$$

and

$$\frac{8}{\pi^3}\left(-\frac{\pi}{2\sqrt{3}} + \arccos\frac{2}{\pi}\right) < f(x) < -\frac{1}{90\sqrt{3}}$$

for $0 < x < \frac{\pi}{2}$. Hence, the proof of Theorem 1 is complete. □

3. Proof of Theorems 2 and 3

We may show some lemmas required for the proof of Theorems 2 and 3.

LEMMA 4. *For $0 < x < \frac{\pi}{2}$, we have*

$$\frac{3x}{\sqrt{3}} - 4\arccos\left(\frac{\sin x}{x}\right) - \frac{\cos x - \frac{\sin x}{x}}{\sqrt{1 - \frac{\sin^2 x}{x^2}}} > 0.$$

Proof. We set

$$f(x) = \frac{3x}{\sqrt{3}} - 4\arccos\left(\frac{\sin x}{x}\right) - \frac{\cos x - \frac{\sin x}{x}}{\sqrt{1 - \frac{\sin^2 x}{x^2}}}$$

and the derivative of $f(x)$ is

$$\begin{aligned} f'(x) &= \sqrt{3} + \frac{5x^3\cos x - 6x^2\sin x + x^4\sin x - 3x\cos x\sin^2 x + 4\sin^3 x}{x^4\left(\frac{x^2-\sin^2 x}{x^2}\right)^{\frac{3}{2}}} \\ &> \sqrt{3} + \frac{5x^3 v(x,3) - 6x^2 u(x,4) + x^4 u(x,3) - 3xv(x,4)(u(x,2))^2 + 4(u(x,3))^3}{x^4\left(x^2 - \frac{(u(x,2))^2}{x^2}\right)^{\frac{3}{2}}} \\ &= j(x) \end{aligned}$$

and the derivative of $j(x)$ is

$$j'(x) = \frac{xk(x)}{148176(4800 - 640x^2 + 40x^4 - x^6)^{\frac{5}{2}}}.$$

Here, $k(x)$ is positive as in Appendix A in the final part of this paper, therefore $j(x)$ is strictly increasing for $0 < x < \frac{\pi}{2}$. From $\lim_{x \to +0} j(x) = \sqrt{3} - \frac{85349376000}{148176(4800)^{\frac{3}{2}}} = 0$, we obtain $j(x) > 0$ for $0 < x < \frac{\pi}{2}$ and $f(x)$ is strictly increasing for $0 < x < \frac{\pi}{2}$. From $\lim_{x\to+0} \frac{\cos x - \frac{\sin x}{x}}{\sqrt{1-\frac{\sin^2 x}{x^2}}} = 0$ (see the proof of Lemma 1), we can get $\lim_{x\to+0} f(x) = 0$ and $f(x) > 0$ for $0 < x < \frac{\pi}{2}$. □

LEMMA 5. *We have*

$$\lim_{x\to+0}\left(\frac{\arccos\left(\frac{\sin x}{x}\right)}{x^r} - \frac{x^{1-r}}{\sqrt{3}}\right) = 0$$

for $0 < r < 3$.

Proof. We set

$$f(x,r) = \frac{\arccos\left(\frac{\sin x}{x}\right)}{x^r} - \frac{x^{1-r}}{\sqrt{3}}.$$

From $\cos\left(\frac{x}{\sqrt{3}}\right) < \frac{\sin x}{x}$ for $0 < x < \frac{\pi}{2}$, we have $\arccos\left(\frac{\sin x}{x}\right) < \frac{x}{\sqrt{3}}$ and

$$f(x,r) = \frac{\arccos\left(\frac{\sin x}{x}\right)}{x^r} - \frac{x^{1-r}}{\sqrt{3}} < \frac{x}{\sqrt{3}x^r} - \frac{x^{1-r}}{\sqrt{3}} = 0$$

for $0 < x < \frac{\pi}{2}$ and $r > 0$. Also, by Theorem 1, we have

$$\frac{8}{\pi^3}\left(-\frac{\pi}{2\sqrt{3}} + \arccos\frac{2}{\pi}\right) < \frac{\arccos\left(\frac{\sin x}{x}\right)}{x^3} - \frac{x^{-2}}{\sqrt{3}}$$

and

$$x^{3-r}\cdot\frac{8}{\pi^3}\left(-\frac{\pi}{2\sqrt{3}} + \arccos\frac{2}{\pi}\right) < f(x,r)$$

for $0 < x < \frac{\pi}{2}$ and $0 < r < 3$. From $\lim_{x\to+0} \frac{8x^{3-r}}{\pi^3}\left(-\frac{\pi}{2\sqrt{3}} + \arccos\frac{2}{\pi}\right) = 0$ for $0 < r < 3$, we can get $\lim_{x\to+0} f(x,r) = 0$ for $0 < x < \frac{\pi}{2}$ and $0 < r < 3$. □

LEMMA 6. *We have*

$$\lim_{x\to+0}\left(\frac{\arccos\left(\frac{\sin x}{x}\right)}{x^r} - \frac{x^{1-r}}{\sqrt{3}}\right) = -\infty$$

for $r > 4$.

Proof. We set

$$f(x,r) = \frac{\arccos\left(\frac{\sin x}{x}\right)}{x^r} - \frac{x^{1-r}}{\sqrt{3}}$$

and the derivative of $f(x,r)$ by r is

$$\frac{\partial f(x,r)}{\partial r} = \frac{x^{-r}\left(x - \sqrt{3}\arccos\left(\frac{\sin x}{x}\right)\right)\ln x}{\sqrt{3}}.$$

From $\cos\left(\frac{x}{\sqrt{3}}\right) < \frac{\sin x}{x}$ for $0 < x < \frac{\pi}{2}$, we have $\arccos\left(\frac{\sin x}{x}\right) < \frac{x}{\sqrt{3}}$ and $\frac{\partial f(x,r)}{\partial r} < 0$ for $0 < x < 1$ and $r > 0$. Therefore, for $0 < x < 1$, $\frac{\partial f(x,r)}{\partial r} < 0$ and $f(x,r)$ is strictly decreasing for $r > 0$. Hence, we have

$$f(x,r) < f(x,4) = \frac{\arccos\left(\frac{\sin x}{x}\right)}{x^4} - \frac{x^{-3}}{\sqrt{3}}$$

for $0 < x < 1$ and $r > 4$. By Theorem 1, we have

$$\frac{\arccos\left(\frac{\sin x}{x}\right)}{x^3} - \frac{x^{-2}}{\sqrt{3}} < -\frac{1}{90\sqrt{3}} \quad \text{and} \quad f(x,r) < f(x,4) < -\frac{1}{90\sqrt{3}x}.$$

From $\lim_{x\to+0} -\dfrac{1}{90\sqrt{3}x} = -\infty$, we obtain $\lim_{x\to+0} f(x,r) = -\infty$ for $r > 4$. □

We consider the equation

$$\frac{x}{\sqrt{3}} + ax^r = \arccos\left(\frac{\sin x}{x}\right)$$

and we have

$$a = \frac{\arccos\left(\frac{\sin x}{x}\right)}{x^r} - \frac{x^{1-r}}{\sqrt{3}} = f(x,r).$$

The derivative of $f(x,r)$ by x is

$$\frac{\partial f(x,r)}{\partial x} = x^{-1-r}\left(\frac{x(r-1)}{\sqrt{3}} - r\arccos\left(\frac{\sin x}{x}\right) - \frac{\cos x - x^{-1}\sin x}{\sqrt{1-\frac{\sin^2 x}{x^2}}}\right) = x^{-1-r}g(x,r)$$

and the derivative of $g(x,r)$ by r is

$$\frac{\partial g(x,r)}{\partial r} = \frac{x}{\sqrt{3}} - \arccos\left(\frac{\sin x}{x}\right).$$

From $\cos\left(\frac{x}{\sqrt{3}}\right) < \frac{\sin x}{x}$ for $0 < x < \frac{\pi}{2}$, we have $\frac{\partial g(x,r)}{\partial r} > 0$ for $0 < x < \frac{\pi}{2}$ and $r > 0$. Hence, $g(x,r)$ is strictly increasing for $r > 0$.

Proof of Theorem 2. By Lemma 1, we have $g(x,r) < g(x,3) < 0$ for $0 < x < \frac{\pi}{2}$ and $0 < r < 3$. Thus, $f(x,r)$ is strictly decreasing for $0 < x < \frac{\pi}{2}$ and we have

$$\lim_{x\to-\frac{\pi}{2}} f(x,r) < f(x,r) < \lim_{x\to+0} f(x,r)$$

for $0<x<\frac{\pi}{2}$ and $0<r<3$. By Lemma 5, we can get

$$\left(\frac{2}{\pi}\right)^r\left(-\frac{\pi}{2\sqrt{3}}+\arccos\frac{2}{\pi}\right)<f(x,r)<0$$

for $0<x<\frac{\pi}{2}$ and $0<r<3$. Hence, the proof of Theorem 2 is complete. □

Proof of Theorem 3. By Lemma 4, we have $g(x,r)>g(x,4)>0$ for $0<x<\frac{\pi}{2}$ and $r>4$. Since $f(x,r)$ is strictly increasing for $0<x<\frac{\pi}{2}$, we have

$$\lim_{x\to+0} f(x,r)<f(x,r)<\lim_{x\to-\frac{\pi}{2}} f(x,r)$$

for $0<x<\frac{\pi}{2}$ and $r>4$. By Lemma 6, we can get

$$-\infty<f(x,r)<\left(\frac{2}{\pi}\right)^r\left(-\frac{\pi}{2\sqrt{3}}+\arccos\frac{2}{\pi}\right).$$

for $0<x<\frac{\pi}{2}$ and $r>4$. Hence, the proof of Theorem 3 is complete. □

4. Appendix A

Computations in this paper were made using Mathematica software.

$$\begin{aligned}g(x)=&\frac{2}{\sqrt{3}}+\frac{1}{622080(8467200-1128960x^2+80640x^4-3444x^6+84x^8-x^{10})^{\frac{3}{2}}}\\&\times\Big(-17698046607360000+3539609321472000x^2-389910089318400x^4\\&+28531077120000x^6-1412654100480x^8+48718817280x^{10}\\&-1159401600x^{12}+18418752x^{14}-177552x^{16}+648x^{18}+5x^{20}\Big).\end{aligned}$$

$$\begin{aligned}h(x)=&-6466866230329344000000+19935787620394598400 0x^2\\&-2626390116532224 0000x^4+214627529591193600 0x^6\\&-127829057745715200x^8+6144522998906880x^{10}\\&-250109274562560x^{12}+8492502412800x^{14}-226881527040x^{16}\\&+4427664192x^{18}-55679328x^{20}+314724x^{22}+1416x^{24}-25x^{26}\\<&-6466866230329344000000+199357876203945984000\left(\frac{158}{100}\right)^2\\&-262639011653222400000x^4+21462752959119360000x^4\left(\frac{158}{100}\right)^2\\&-127829057745715200x^8+6144522998906880x^8\left(\frac{158}{100}\right)^2\end{aligned}$$

$$
\begin{aligned}
&-250109274562560x^{12}+8492502412800x^{12}\left(\frac{158}{100}\right)^2-226881527040x^{16}\\
&+4427664192x^{16}\left(\frac{158}{100}\right)^2-55679328x^{20}+314724x^{20}\left(\frac{158}{100}\right)^2\\
&+1416x^{20}\left(\frac{158}{100}\right)^4-25x^{26}\\
=&\ \frac{1}{781250}\Big(-116413766310471598080000000\\
&-16332765247349752320000000x^4-8788271135253442560000 0x^8\\
&-1788348371400360000000x^{12}-16861586418054000 0x^{16}\\
&-42878770684413x^{20}-19531250x^{26}\Big)<0\,.
\end{aligned}
$$

$$
\begin{aligned}
j(x)=&\ \sqrt{3}+\frac{1}{148176(4800-640x^2+40x^4-x^6)^{\frac{3}{2}}}\\
&\times\Big(-85349376000+17713382400x^2-1844579520x^4+106686720x^6\\
&-3625776x^8+64512x^{10}-315x^{12}-8x^{14}\Big)\,.
\end{aligned}
$$

$$
\begin{aligned}
k(x)=&\ 6177669120000-3595511808000x^2+650822860800x^4-72508262400x^6\\
&+5476161600x^8-277159680x^{10}+7972944x^{12}-83792x^{14}-1615x^{16}+40x^{18}\\
>&\ 6177669120000-3595511808000x^2+650822860800x^4\\
&-72508262400x^4\left(\frac{158}{100}\right)^2+5476161600x^8-277159680x^8\left(\frac{158}{100}\right)^2\\
&+7972944x^{12}-83792x^{12}\left(\frac{158}{100}\right)^2-1615x^{12}\left(\frac{158}{100}\right)^4+40x^{18}\\
=&\ 6177669120000-3595511808000x^2+\frac{11745330863616x^4}{25}\\
&+\frac{598032521856x^8}{125}+\frac{9692126187837x^{12}}{1250000}+40x^{18}\\
>&\ 6177669120000-3595511808000x^2+\frac{11745330863616x^4}{25}\\
=&\ -\frac{946064525328384 0000}{13486243}+\frac{11745330863616}{25}\left(x^2-\frac{464450000}{121376187}\right)^2\\
=&\ -\frac{9460645253283840000}{13486243}+\frac{11745330863616}{25}\left(\left(\frac{158}{100}\right)^2-\frac{464450000}{121376187}\right)^2\\
=&\ \frac{1266753993852697056}{9765625}>0\,.
\end{aligned}
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(Received March 31, 2022)

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