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NEW QUANTUM INTEGRAL INEQUALITIES VIA m–CONVEX FUNCTIONS OVER FINITE INTERVAL

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Abstract. First we consider a new trapezium identity for twice differentiable functions in quantum integrals. This identity investigates our main results that consist some integral inequalities of trapezium type using the newly introduced q-derivatives for the m-convex functions over finite intervals of real numbers. Some special cases are discussed in details to support our theoretical results.

1. Introduction and preliminaries

The theory of convex functions has a pivotal in the development of many fields of Mathematics. Significant applications of convex functions are found in a variety of applied sciences such as optimization theory, number theory, combinatorics, special means theory, approximation theory and numerical analysis.

Recall that a function $f : J \subseteq \Re \to \Re$ is called convex on J, if

$$f(\alpha a+(1-\alpha)b) \leqslant \alpha f(a)+(1-\alpha)f(b), \tag{1}$$

holds for all $a,b \in J$, $a<b$ and $\alpha \in [0,1]$. We say that f is concave, if the inequality (1) holds in the reverse direction.

The following double inequality:

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2}, \tag{2}$$

is called the Hermite–Hadamard inequality. The indicated inequality gives a simple and useful criteria for a function to be convex. The inequality become an essential tool for estimation of average integral of a convex function, when the integral is not even calculated. For instance, if $f(x)=e^{x^2}$, then we use the double inequality to estimate the average integral. The above useful feature made the researchers to get some nice generalizations of the Hermite–Hadamard inequality by utilizing some general kinds

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of convex functions. So far, many refinements, improvements and generalizations have become the part of literatures, see [15, 16, 17, 21, 22] and the references therein.

The notion of m-convex functions is the marvelous generalization of convex functions and is introduced by Toader [12] as follows: A function f is said to be m-convex, if

$$f(\alpha a+m(1-\alpha)b)\leqslant \alpha f(a)+m(1-\alpha)f(b), \tag{3}$$

holds for some fixed $m\in(0,1]$. Clearly, we recover classical convex function if $m=1$.

The study of quantum calculus (more briefly q-calculus) is the calculus with no limits. The theory has its origin in the 18th century. Famous mathematician Euler was responsible for introducing q as a parameter in Newton's work of infinite series. Jackson worked on the classical definition of the derivative of a function and proved many results in the domain of q-calculus. Jackson got the credit for systematic development of q-calculus. The mentioned branch is considered as an incorporative subject between physics and mathematics. For some current advancement in this field, we refer to [3, 4, 8, 13, 14, 18, 19, 20].

For a real function f, the q-derivative is characterized by

$$\mathbf{D}_q f(x)=\frac{f(qx)-f(x)}{qx-x}, \tag{4}$$

where $q\in(0,1)$. Note that

$$\lim_{q\to 1^-}\mathbf{D}_q f(x)=f'(x).$$

The Jackson integral of a real function f is defined by the following series expansion

$$\int_0^k f(x)d_qx=(1-q)k\sum_{h=0}^{\infty}q^h f(q^h k). \tag{5}$$

The authors in [10, 11], studied the notion of q-derivatives and q-integrals over the finite real interval $[a,b]$ and defined the q_a-derivative and q_a-integrals. For a continuous function $f:[a,b]\to\Re$ and $q\in(0,1)$, the q_a-derivative and q_a-integrals of f at $x\in[a,b]$ are respectively defined and denoted by

$${}_a\mathbf{D}_q f(x)=\frac{f(x)-f(qx+(1-q)a)}{(1-q)(x-a)}, \quad x\neq a, \tag{6}$$

and

$$\int_a^k f(x)_a d_qx=(1-q)(k-a)\sum_{h=0}^{\infty}q^h f(q^h k+(1-q^h)a), \quad k\in[a,b]. \tag{7}$$

If $a=0$, then

$$\int_0^k f(x)_0 d_qx=(1-q)k\sum_{h=0}^{\infty}q^h f(q^h k)=\int_0^k f(x)d_qx. \tag{8}$$

which is the classical q-integral (5).

The authors in [1, 6, 7, 9, 2], have contributed to the ongoing research in q-calculus and have established some interesting results.

The authors in [2], discussed the notion of q-derivatives and q-integrals over the finite real interval $[a,b]$ and defined the q^b-derivative and q^b-integrals. For a continuous function $f:[a,b]\to\Re$ and $q\in(0,1)$, the q^b-derivative and q^b-integrals of f at $x\in[a,b]$ are respectively defined and denoted by

$$ {}^b\mathbf{D}_q f(x) = \frac{f(x) - f(qx+(1-q)b)}{(1-q)(x-b)}, \quad x \neq b, \tag{9} $$

and

$$ \int_k^b f(x)\ {}^b d_q x = (1-q)(b-k)\sum_{h=0}^{\infty} q^h f(q^h k + (1-q^h)b), \quad k\in[a,b]. \tag{10} $$

If $b=0$, then we have a relation with classical Jackson integral (5) as follows:

$$ \int_k^0 f(x)\ {}^0 d_q x = -(1-q)k\sum_{h=0}^{\infty} q^h f(q^h k) = -\int_0^k f(x) d_q x. \tag{11} $$

REMARK 1. It is useful to note that

$$ \lim_{q\to 1^-} {}^b\mathbf{D}_q f(x) = f'(x) = \lim_{q\to 1^-} {}_a\mathbf{D}_q f(x). $$

Here we point out some useful features of q-derivatives and q-integrals.

1. From (4), (6) and (9), we found that

$$ {}_0 d_q x = d_q x = {}^0 d_q x. \tag{12} $$

2. The derivatives ${}^b\mathbf{D}_q f(x)$ and ${}_a\mathbf{D}_q f(x)$ are not same for general functions. Indeed, if $f(x)=x^2$, then

$$ {}^b\mathbf{D}_q f(x) = (1+q)x + (1-q)b $$

and

$$ {}_a\mathbf{D}_q f(x) = (1+q)x + (1-q)a. $$

However,

$$ {}^b\mathbf{D}_q f(x) = f'(x) = {}_a\mathbf{D}_q f(x) $$

provided that $q\to 1^-$.

3. The q-integrals $\int_a^b f(x)\ {}^b d_q x$ and $\int_a^b f(x)\ {}_a d_q x$ are different for general functions. For instance,

$$ \int_a^b x\ {}^b d_q x = \frac{b-a}{1+q}[a+bq] $$

and

$$\int_a^b x\ {}_a d_q x = \frac{b-a}{1+q}[aq+b].$$

Furthermore,

$$\int_a^b x\ {}^b d_q x = \frac{b^2-a^2}{2} = \int_a^b x\ {}_a d_q x$$

subject to the condition that $q \to 1^-$.

In view of the above results and literatures, and following forward with this tendency of the newly introduced q-derivatives and q-integrals, the aim of this paper is to establish a new trapezium identity for twice differentiable functions in quantum integrals. Applying this interesting identity some integral inequalities of trapezium type using the newly introduced q-derivatives for the m-convex functions over finite intervals $[a,b]$ of real numbers a and b. Some special cases will be given in details to support our theoretical results. At the end, a brief conclusion will be provided as well.

2. Some useful results and conventions on q-integrals

Let recall some useful results and conventions used in the indicated domain.

DEFINITION 1. [3] For a fixed real number p,

$$[p] = \frac{1-q^p}{1-q} \tag{13}$$

is called q-analogue of p. In particular,

$$[n] = \frac{1-q^n}{1-q} = q^{n-1}+\ldots+q+1, \tag{14}$$

provided that n is a positive integer.

DEFINITION 2. [3] For any integer s, the q-analogue of $(w-c)^s$ is the polynomial

$$(w-c)_q^s = \begin{cases} 1, & \text{if} \quad s=0, \\ (w-c)(w-qc)\cdots(w-q^{s-1}c), & \text{if} \quad s \geqslant 1. \end{cases} \tag{15}$$

DEFINITION 3. [3] For any $\gamma, \delta > 0$,

$$B_q(\gamma,\delta) = \int_0^1 x^{\gamma-1}(1-qx)_q^{\delta-1} d_q x, \tag{16}$$

is called the q-Beta function. Notice that,

$$B_q(\gamma,1)=\int_0^1 x^{\gamma-1}d_qx=\frac{1}{[\gamma]}, \tag{17}$$

where $[\gamma]$ is the q-analogue of γ. Furthermore, if $q\to 1^-$, then we have the following classical Beta function

$$B(\gamma,\delta)=\int_0^1 x^{\gamma-1}(1-x)^{\delta-1}dx. \tag{18}$$

In [6], the authors introduced some new inequalities of Hadamard type by proving the following identity.

LEMMA 1. *Let $f:[a,b]\to\Re$ be a twice differentiable function on I° (the interior of I) with ${}_a\mathbf{D}_q^2 f$ continuous and integrable on I, where $0<q<1$. Then the following identity holds:*

$$\frac{qf(a)+f(b)}{1+q}-\frac{1}{b-a}\int_a^b f(x)\ {}_ad_qx$$

$$=\frac{q^2(b-a)^2}{1+q}\times\int_0^1 s(1-qs){}_a\mathbf{D}_q^2 f(sb+(1-s)a)\ {}_0d_qs.$$

Now and onward, we adopt the convention d_qx for ${}_0d_qx$ and ${}^0d_qx$ if the lower limit is 0 due to the fact given in Remark 1. Some very similar calculations leads to the following counterpart of Lemma 1.

LEMMA 2. *Let $f:[a,b]\to\Re$ be a twice differentiable function on I° (the interior of I) with ${}^a\mathbf{D}_q^2 f$ continuous and integrable on I, where $0<q<1$. Then the following identity holds:*

$$\frac{f(a)+qf(b)}{1+q}-\frac{1}{b-a}\int_a^b f(x)\ {}^bd_qx$$

$$=\frac{q^2(b-a)^2}{1+q}\times\int_0^1 s(1-qs){}^b\mathbf{D}_q^2 f(sa+(1-s)b)\ d_qs.$$

Now, we have the following new identity from Lemmas 1 and 2.

LEMMA 3. *Let $f:[a,b]\to\Re$ be a twice differentiable function on I° (the interior of I) with ${}_a\mathbf{D}_q^2 f$ and ${}^b\mathbf{D}_q^2 f$ continuous and integrable on I, where $0<q<1$. Then the* *following identity holds:*

$$\frac{f(a)+f(b)}{2}-\frac{1}{2(b-a)}\left[\int_a^b f(x)\ {}_ad_qx+\int_a^b f(x)\ {}^bd_qx\right] \tag{19}$$
$$=\frac{q^2(b-a)^2}{2(1+q)}\int_0^1 s(1-qs)\left[\ {}_a\mathbf{D}_q^2f(sb+(1-s)a)+\ {}^b\mathbf{D}_q^2f(sa+(1-s)b)\right]d_qs.$$

3. Main results

In this section, before we present our main results, let us denote, respectively, $I=[a,b]$ and $I^\circ=(a,b)$.

THEOREM 1. *Let $f:I\to\Re$ be a twice differentiable convex function on I° with ${}_a\mathbf{D}_q^2f$ and ${}^b\mathbf{D}_q^2f$ continuous and integrable on I, where $0<q<1$. If $\left|{}_a\mathbf{D}_q^2f\right|^r$ and $\left|{}^b\mathbf{D}_q^2f\right|^r$ are m-convex functions on I for $r\geqslant 1$, where $m\in(0,1]$ is fixed, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{2(b-a)}\left[\int_a^b f(t)\ {}_ad_qt+\int_a^b f(t)\ {}^bd_qt\right]\right|$$
$$\leqslant\frac{q^2(b-a)^2}{2(1+q)^{2-\frac{1}{r}}}\left\{\left((1-q)\sum_{p=0}^{\infty}q^{3p}(1-q^{p+1})^r\left|{}_a\mathbf{D}_q^2f(b)\right|^r\right.\right.$$
$$\left.+m(1-q)\sum_{p=0}^{\infty}(q^{2p}-q^{3p})^r\left|{}_a\mathbf{D}_q^2f\left(\frac{a}{m}\right)\right|^r\right)^{\frac{1}{r}}$$
$$+\left((1-q)\sum_{p=0}^{\infty}q^{3p}(1-q^{p+1})^r\left|{}^b\mathbf{D}_q^2f(a)\right|^r\right.$$
$$\left.\left.+\ m(1-q)\sum_{p=0}^{\infty}(q^{2p}-q^{3p})^r\left|{}^b\mathbf{D}_q^2f\left(\frac{b}{m}\right)\right|^r\right)^{\frac{1}{r}}\right\}. \tag{20}$$

Proof. By the Lemma 3 and properties of the modulus, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{2(b-a)}\left[\int_a^b f(t)\ {}_ad_qt+\int_a^b f(t)\ {}^bd_qt\right]\right|$$
$$=\frac{q^2(b-a)^2}{2(1+q)}\int_0^1 s(1-qs)\left|\left[\ {}_a\mathbf{D}_q^2f(sb+(1-s)a)+\ {}^b\mathbf{D}_q^2f(sa+(1-s)b)\right]\right|\ d_qs$$

$$\leqslant \frac{q^2(b-a)^2}{2(1+q)}\left[\int_0^1 s(1-qs)\left|\,{}_a\mathbf{D}_q^2 f(sb+(1-s)a)\right|\,d_qs\right.$$
$$\left.+\int_0^1 s(1-qs)\left|\,{}^b\mathbf{D}_q^2 f(sa+(1-s)b)\right|\,d_qs\right]. \tag{21}$$

By m-convexity, application of q-integral (8) and power mean inequality, we get

$$\begin{aligned}
&\int_0^1 s(1-qs)\left|\,{}_a\mathbf{D}_q^2 f(sb+(1-s)a)\right|\,d_qs\\
&\leqslant \left(\int_0^1 s d_qs\right)^{1-\frac{1}{r}}\left(\int_0^1 s(1-qs)^r\left|\,{}_a\mathbf{D}_q^2 f(sb+(1-s)a)\right|^r\,d_qs\right)^{\frac{1}{r}}\\
&\leqslant \left(\int_0^1 s d_qs\right)^{1-\frac{1}{r}}\left(\int_0^1 s(1-qs)^r\left[s\left|{}_a\mathbf{D}_q^2 f(b)\right|^r+m(1-s)\left|{}_aD_q^2 f\left(\frac{a}{m}\right)\right|^r\right]d_qs\right)^{\frac{1}{r}}\\
&=\left(\frac{1}{1+q}\right)^{1-\frac{1}{r}}\left((1-q)\sum_{p=0}^{\infty}q^{3p}(1-q^{p+1})^r\left|{}_a\mathbf{D}_q^2 f(b)\right|^r\right.\\
&\quad\left.+m(1-q)\sum_{p=0}^{\infty}(q^{2p}-q^{3p})^r\left|{}_a\mathbf{D}_q^2 f\left(\frac{a}{m}\right)\right|^r\right)^{\frac{1}{r}}.
\end{aligned} \tag{22}$$

Similarly,

$$\begin{aligned}
&\int_0^1 s(1-qs)\left|\,{}^b\mathbf{D}_q^2 f(sa+(1-s)b)\right|\,d_qs\\
&\leqslant \left(\frac{1}{1+q}\right)^{1-\frac{1}{r}}\left((1-q)\sum_{p=0}^{\infty}q^{3p}(1-q^{p+1})^r\left|{}^b\mathbf{D}_q^2 f(a)\right|^r\right.\\
&\quad\left.+m(1-q)\sum_{p=0}^{\infty}(q^{2p}-q^{3p})^r\left|{}^b\mathbf{D}_q^2 f\left(\frac{b}{m}\right)\right|^r\right)^{\frac{1}{r}}.
\end{aligned} \tag{23}$$

Inequality (21) leads to the required inequality (20) by utilizing inequalities (22) and (23). □

COROLLARY 1. *If r is a positive integer, then using the facts that*

$$(1-qs)^r \leqslant (1-qs)_q^r,\quad (1-s)(1-qs)^r \leqslant (1-qs)_q^{r+1}, \tag{24}$$

we obtain

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{2(b-a)}\left[\int_a^b f(t)\ {}_ad_qt+\int_a^b f(t)\ {}^bd_qt\right]\right|$$
$$\leqslant \frac{q^2(b-a)^2}{2(1+q)^{2-\frac{1}{r}}}\left\{\left(B_q(3,r+1)\left|{}_a\mathbf{D}_q^2 f(b)\right|^r+mB_q(2,r+2)\left|{}_a\mathbf{D}_q^2 f\left(\frac{a}{m}\right)\right|^r\right)^{\frac{1}{r}}\right.$$
$$\left.+\left(B_q(3,r+1)\left|{}^b\mathbf{D}_q^2 f(a)\right|^r+mB_q(2,r+2)\left|{}^b\mathbf{D}_q^2 f\left(\frac{b}{m}\right)\right|^r\right)^{\frac{1}{r}}\right\}. \tag{25}$$

REMARK 2. If $q\to 1^-$, then we have the following inequality for m-convex functions:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\ dt\right|$$
$$\leqslant \frac{(b-a)^2}{2^{3-\frac{1}{r}}}\left\{\left(\frac{2\left|f''(b)\right|^r+m(r+1)\left|f''\left(\frac{a}{m}\right)\right|^r}{(r+1)(r+2)(r+3)}\right)^{\frac{1}{r}}\right.$$
$$\left.+\left(\frac{2\left|f''(a)\right|^r+m(r+1)\left|f''\left(\frac{b}{m}\right)\right|^r}{(r+1)(r+2)(r+3)}\right)^{\frac{1}{r}}\right\}. \tag{26}$$

THEOREM 2. *Let $f: I\to\Re$ be a twice differentiable convex function on I° with ${}_a\mathbf{D}_q^2 f$ and ${}^b\mathbf{D}_q^2 f$ continuous and integrable on I, where $0<q<1$. If $\left|{}_a\mathbf{D}_q^2 f\right|^r$ and $\left|{}^b\mathbf{D}_q^2 f\right|^r$ are m-convex functions on I, where $r,u>1$, $\frac{1}{u}+\frac{1}{r}=1$ and $m\in(0,1]$ is fixed, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{2(b-a)}\left[\int_a^b f(t)\ {}_ad_qt+\int_a^b f(t)\ {}^bd_qt\right]\right|$$
$$\leqslant \frac{q^2(b-a)^2}{2(1+q)}\left((1-q)\sum_{s=0}^{\infty}q^{2s}(1-q^{s+1})^u\right)^{\frac{1}{u}}$$
$$\times\left\{\left(\frac{(1+q)\left|{}_a\mathbf{D}_q^2 f(b)\right|^r+mq^2\left|{}_a\mathbf{D}_q^2 f\left(\frac{a}{m}\right)\right|^r}{(q+1)(1+q+q^2)}\right)^{\frac{1}{r}}\right.$$
$$\left.+\left(\frac{(1+q)\left|{}^b\mathbf{D}_q^2 f(a)\right|^r+mq^2\left|{}^b\mathbf{D}_q^2 f\left(\frac{b}{m}\right)\right|^r}{(q+1)(1+q+q^2)}\right)^{\frac{1}{r}}\right\}. \tag{27}$$

Proof. The Hölder's inequality, m-convexity and the application of q-integral leads to

$$\begin{aligned}
&\int_0^1 s(1-qs)\left|\,{}_a\mathbf{D}_q^2 f(sb+(1-s)a)\right|\,d_qs \\
&\leqslant \left(\int_0^1 s(1-qs)^u d_qs\right)^{\frac{1}{u}}\left(\int_0^1 s\left|\,{}_a\mathbf{D}_q^2 f(sb+(1-s)a)\right|^r\,d_qs\right)^{\frac{1}{r}} \\
&\leqslant \left((1-q)\sum_{s=0}^{\infty} q^{2s}(1-q^{s+1})^u\right)^{\frac{1}{u}} \\
&\quad\times\left(\int_0^1 s\left[s\left|{}_a\mathbf{D}_q^2 f(b)\right|^r + m(1-s)\left|{}_a\mathbf{D}_q^2 f\left(\frac{a}{m}\right)\right|^r\right]d_qs\right)^{\frac{1}{r}} \\
&= \left((1-q)\sum_{s=0}^{\infty} q^{2s}(1-q^{s+1})^u\right)^{\frac{1}{u}}\left(\frac{(1+q)\left|{}_a\mathbf{D}_q^2 f(b)\right|^r + mq^2\left|{}_a\mathbf{D}_q^2 f\left(\frac{a}{m}\right)\right|^r}{(q+1)(1+q+q^2)}\right)^{\frac{1}{r}}. \qquad (28)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_0^1 s(1-qs)\left|\,{}^b\mathbf{D}_q^2 f(sa+(1-s)b)\right|\,d_qs \\
&\leqslant \left((1-q)\sum_{s=0}^{\infty} q^{2s}(1-q^{s+1})^u\right)^{\frac{1}{u}}\left(\frac{(1+q)\left|{}^b\mathbf{D}_q^2 f(a)\right|^r + mq^2\left|{}^b\mathbf{D}_q^2 f\left(\frac{b}{m}\right)\right|^r}{(q+1)(1+q+q^2)}\right)^{\frac{1}{r}}. \qquad (29)
\end{aligned}$$

Inequality (27) is then obtained by inequalities (21), (28) and (29). □

COROLLARY 2. *If p is a positive integer, then using the fact that*

$$(1-qs)^p \leqslant (1-qs)_q^p, \qquad (30)$$

we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{2(b-a)}\left[\int_a^b f(t)\,{}_ad_qt + \int_a^b f(t)\,{}^bd_qt\right]\right| \\
&\leqslant \frac{q^2(b-a)^2}{2(1+q)}\left(B_q(2,u+1)\right)^{\frac{1}{u}}\left\{\left(\frac{(1+q)\left|{}_a\mathbf{D}_q^2 f(b)\right|^r + mq^2\left|{}_a\mathbf{D}_q^2 f\left(\frac{a}{m}\right)\right|^r}{(1+q+q^2)(1+q)}\right)^{\frac{1}{r}}\right. \\
&\quad\left.+\left(\frac{(1+q)\left|{}^b\mathbf{D}_q^2 f(a)\right|^r + mq^2\left|{}^b\mathbf{D}_q^2 f\left(\frac{b}{m}\right)\right|^r}{(1+q+q^2)(1+q)}\right)^{\frac{1}{r}}\right\}. \qquad (31)
\end{aligned}$$

REMARK 3. If $q \to 1^-$, then we get the following inequality for m-convex functions:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right|$$

$$\leqslant \frac{(b-a)^2}{4}\left(\frac{1}{(u+1)(u+2)}\right)^{\frac{1}{u}}$$

$$\times\left\{\left(\frac{2\left|f''(b)\right|^r + m\left|f''\left(\frac{a}{m}\right)\right|^r}{6}\right)^{\frac{1}{r}} + \left(\frac{2\left|f''(a)\right|^r + m\left|f''\left(\frac{b}{m}\right)\right|^r}{6}\right)^{\frac{1}{r}}\right\}. \tag{32}$$

THEOREM 3. *Let $f : I \to \Re$ be a twice differentiable convex function on I° with ${}_a\mathbf{D}_q^2 f$ and ${}^b\mathbf{D}_q^2 f$ continuous and integrable on I, where $0 < q < 1$. If $\left|{}_a\mathbf{D}_q^2 f\right|^r$ and $\left|{}^b\mathbf{D}_q^2 f\right|^r$ are m-convex functions on I for $r \geqslant 1$, where $m \in (0,1]$ is fixed, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{2(b-a)}\left[\int_a^b f(t)\ {}_ad_qt + \int_a^b f(t)\ {}^bd_qt\right]\right|$$

$$\leqslant \frac{q^2(b-a)^2}{2(1+q)}\left\{\left((1-q)\sum_{s=0}^{\infty}(q^s)^{r+2}(1-q^{s+1})^r\left|{}_a\mathbf{D}_q^2 f(b)\right|^r\right.\right.$$

$$\left. + m(1-q)\sum_{s=0}^{\infty}(q^s)^{r+1}(1-q^{s+1})^r(1-q^s)\left|{}_a\mathbf{D}_q^2 f\left(\frac{a}{m}\right)\right|^r\right)^{\frac{1}{r}}$$

$$+\left((1-q)\sum_{s=0}^{\infty}(q^s)^{r+2}(1-q^{s+1})^r\left|{}^b\mathbf{D}_q^2 f(a)\right|^r\right.$$

$$\left.\left. + m(1-q)\sum_{s=0}^{\infty}(q^s)^{r+1}(1-q^{s+1})^r(1-q^s)\left|{}^b\mathbf{D}_q^2 f\left(\frac{b}{m}\right)\right|^r\right)^{\frac{1}{r}}\right\}. \tag{33}$$

Proof. Applying the power mean inequality, m-convexity and the q-integral (8), we get

$$\int_0^1 s(1-qs)\left|\,{}_a\mathbf{D}_q^2 f(sb+(1-s)a)\right|\,d_qs$$

$$\leqslant \left(\int_0^1 1\,d_qs\right)^{1-\frac{1}{r}}\left(\int_0^1 s^r(1-qt)^r\left|\,{}_a\mathbf{D}_q^2 f(sb+(1-s)a)\right|^r\,d_qs\right)^{\frac{1}{r}}$$

$$\leqslant \left(\int_0^1 s^{r+1}(1-qt)^r\left|{}_a\mathbf{D}_q^2 f(b)\right|^r d_qt + m\int_0^1 s^r(1-qt)^r(1-s)\left|{}_a\mathbf{D}_q^2 f\left(\frac{a}{m}\right)\right|^r d_qs\right)^{\frac{1}{r}}$$

$$
= \left((1-q)\sum_{s=0}^{\infty}(q^s)^{r+2}(1-q^{s+1})^r \left|{}_a\mathbf{D}_q^2 f(b)\right|^r \right.
$$

$$
\left. + m(1-q)\sum_{s=0}^{\infty}(q^s)^{r+1}(1-q^{s+1})^r(1-q^s)\left|{}_a\mathbf{D}_q^2 f\left(\frac{a}{m}\right)\right|^r \right)^{\frac{1}{r}}. \tag{34}
$$

Similarly,

$$
\int_0^1 s(1-qs)\left|{}^b\mathbf{D}_q^2 f(sa+(1-s)b)\right| d_q s
$$

$$
\leqslant \left((1-q)\sum_{s=0}^{\infty}(q^s)^{r+2}(1-q^{s+1})^r \left|{}^b\mathbf{D}_q^2 f(a)\right|^r \right.
$$

$$
\left. + m(1-q)\sum_{s=0}^{\infty}(q^s)^{r+1}(1-q^{s+1})^r(1-q^s)\left|{}^b\mathbf{D}_q^2 f\left(\frac{b}{m}\right)\right|^r \right)^{\frac{1}{r}}. \tag{35}
$$

Inequality (33) is then obtained by inequalities (21), (34) and (35). □

COROLLARY 3. *If r is a positive integer, then using the inequalities (24) we obtain*

$$
\left|\frac{f(a)+f(b)}{2} - \frac{1}{2(b-a)}\left[\int_a^b f(t)\ {}_a d_q t + \int_a^b f(t)\ {}^b d_q t\right]\right|
$$

$$
\leqslant \frac{q^2(b-a)^2}{2(1+q)}\left\{\left(B_q(r+2,r+1)\left|{}_a\mathbf{D}_q^2 f(b)\right|^r + mB_q(r+1,r+2)\left|{}_a\mathbf{D}_q^2 f\left(\frac{a}{m}\right)\right|^r\right)^{\frac{1}{r}}\right.
$$

$$
\left. + \left(B_q(r+2,r+1)\left|{}^b\mathbf{D}_q^2 f(a)\right|^r + mB_q(r+1,r+2)\left|{}^b\mathbf{D}_q^2 f\left(\frac{b}{m}\right)\right|^r\right)^{\frac{1}{r}}\right\}. \tag{36}
$$

REMARK 4. If $q \to 1^-$, then we have the following inequality for m-convex functions:

$$
\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\ dt\right|
$$

$$
\leqslant \frac{(b-a)^2}{4}\left\{\left(B(r+2,r+1)\left|f''(b)\right|^r + mB(r+1,r+2)\left|f''\left(\frac{a}{m}\right)\right|^r\right)^{\frac{1}{r}}\right.
$$

$$
\left. + \left(B(r+2,r+1)\left|f''(a)\right|^r + mB(r+1,r+2)\left|f''\left(\frac{b}{m}\right)\right|^r\right)^{\frac{1}{r}}\right\}. \tag{37}
$$

THEOREM 4. *Let $f : I \to \Re$ be a twice differentiable convex function on I° with ${}_a\mathbf{D}_q^2 f$ and ${}^b\mathbf{D}_q^2 f$ continuous and integrable on I, where $0 < q < 1$. If $\left|{}_a\mathbf{D}_q^2 f\right|^r$ and $\left|{}^b\mathbf{D}_q^2 f\right|^r$ are m-convex functions on I, where $r, u > 1$, $\frac{1}{u} + \frac{1}{r} = 1$ and $m \in (0,1]$ is fixed, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{2(b-a)}\left[\int_a^b f(t)\ {}_a d_q t + \int_a^b f(t)\ {}^b d_q t\right]\right|$$
$$\leqslant \frac{q^2(b-a)^2}{2(1+q)}\left((1-q)\sum_{s=0}^{\infty}(q^s)^{u+1}(1-q^{s+1})^u\right)^{\frac{1}{u}}$$
$$\times\left\{\left(\frac{\left|{}_a\mathbf{D}_q^2 f(b)\right|^r + mq\left|{}_a\mathbf{D}_q^2 f\left(\frac{a}{m}\right)\right|^r}{1+q}\right)^{\frac{1}{r}} + \left(\frac{\left|{}^b\mathbf{D}_q^2 f(a)\right|^r + mq\left|{}^b\mathbf{D}_q^2 f\left(\frac{b}{m}\right)\right|^r}{1+q}\right)^{\frac{1}{r}}\right\}. \tag{38}$$

Proof. By Hölder's inequality, m-convexity and the q-integral (8), we have

$$\int_0^1 s(1-qs)\left|\ {}_a\mathbf{D}_q^2 f(sb+(1-s)a)\right|\ d_q s$$
$$\leqslant \left(\int_0^1 s^u(1-qs)^u d_q s\right)^{\frac{1}{u}}\left(\int_0^1 \left|\ {}_a\mathbf{D}_q^2 f(sb+(1-s)a)\right|^r\ d_q s\right)^{\frac{1}{r}}$$
$$\leqslant \left(\int_0^1 s^u(1-qs)^u d_q s\right)^{\frac{1}{u}}\left(\left|{}_a\mathbf{D}_q^2 f(b)\right|^r \int_0^1 s d_q s + m\left|{}_a\mathbf{D}_q^2 f\left(\frac{a}{m}\right)\right|^r \int_0^1 (1-s) d_q s\right)^{\frac{1}{r}}$$
$$= \left((1-q)\sum_{s=0}^{\infty}(q^s)^{u+1}(1-q^{s+1})^u\right)^{\frac{1}{u}}\left(\frac{\left|{}_a\mathbf{D}_q^2 f(b)\right|^r + mq\left|{}_a\mathbf{D}_q^2 f\left(\frac{a}{m}\right)\right|^r}{1+q}\right)^{\frac{1}{r}}. \tag{39}$$

Similarly,

$$\int_0^1 s(1-qs)\left|\ {}^b\mathbf{D}_q^2 f(sa+(1-s)b)\right|\ d_q s$$
$$\leqslant \left((1-q)\sum_{s=0}^{\infty}(q^s)^{u+1}(1-q^{s+1})^u\right)^{\frac{1}{u}}\left(\frac{\left|{}^b\mathbf{D}_q^2 f(a)\right|^r + mq\left|{}^b\mathbf{D}_q^2 f\left(\frac{b}{m}\right)\right|^r}{1+q}\right)^{\frac{1}{r}}. \tag{40}$$

Inequality (38) is then obtained by inequalities (21), (39) and (40). □

COROLLARY 4. *If u is a positive integer, then using the inequality (30) we get*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{2(b-a)}\left[\int_a^b f(t)\ {}_a d_q t+\int_a^b f(t)\ {}^b d_q t\right]\right|$$
$$\leqslant \frac{q^2(b-a)^2}{2(1+q)}\left(B_q(u+1,u+1)\right)^{\frac{1}{u}}\left\{\left(\frac{\left|{}_a\mathbf{D}_q^2 f(b)\right|^r+mq\left|{}_a\mathbf{D}_q^2 f\left(\frac{a}{m}\right)\right|^r}{1+q}\right)^{\frac{1}{r}}\right.$$
$$\left.+\left(\frac{\left|{}^b\mathbf{D}_q^2 f(a)\right|^r+mq\left|{}^b\mathbf{D}_q^2 f\left(\frac{b}{m}\right)\right|^r}{1+q}\right)^{\frac{1}{r}}\right\}. \tag{41}$$

REMARK 5. Considering the integral,

$$B(u+1,u+1)=\int_0^1 s^u(1-s)^u ds, \tag{42}$$

and taking the following facts into account,

$$2^{2s-1}\Gamma(s)\Gamma\left(s+\frac{1}{2}\right)=\Gamma\left(\frac{1}{2}\right)\Gamma(2s),$$

and

$$B(s,t)=\frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)},$$

then

$$B(u+1,u+1)=\frac{2^{-1-2u}\Gamma(u+1)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(u+\frac{3}{2}\right)},$$

where $\Gamma(\cdot)$ is the Gamma function.

If $q\to 1^-$, then we have the following inequality for m-convex functions from inequality (38):

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|$$
$$\leqslant \frac{(b-a)^2}{32}\left(\frac{\sqrt{\pi}\Gamma(u+1)}{\Gamma\left(u+\frac{3}{2}\right)}\right)^{\frac{1}{u}}$$
$$\times\left\{\left(\left|f''(b)\right|^r+m\left|f''\left(\frac{a}{m}\right)\right|^r\right)^{\frac{1}{r}}+\left(\left|f''(a)\right|^r+m\left|f''\left(\frac{b}{m}\right)\right|^r\right)^{\frac{1}{r}}\right\}. \tag{43}$$

THEOREM 5. *Let $f:I\to\Re$ be a twice differentiable convex function on I° with ${}_a\mathbf{D}_q^2 f$ and ${}^b\mathbf{D}_q^2 f$ continuous and integrable on I, where $0<q<1$. If $\left|{}_a\mathbf{D}_q^2 f\right|^r$ and*

$\left|{}^{b}\mathbf{D}_q^2 f\right|^r$ *are* m*-convex functions on* I*, where* $r,u>1$*,* $\frac{1}{u}+\frac{1}{r}=1$ *and* $m\in(0,1]$ *is fixed, then*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{2(b-a)}\left[\int_a^b f(t)\ {}_a d_q t+\int_a^b f(t)\ {}^b d_q t\right]\right|\\&\leqslant \frac{q^2(b-a)^2}{2(1+q)}\left(\frac{1}{[u+1]}\right)^{\frac{1}{u}}\left\{\left((1-q)\sum_{s=0}^{\infty}q^{2s}(1-q^{s+1})^r\left|{}_a\mathbf{D}_q^2 f(b)\right|^r\right.\right.\\&\quad\left.+m(1-q)\sum_{s=0}^{\infty}(q^s-q^{2s})(1-q^{s+1})^r\left|{}_a\mathbf{D}_q^2 f\left(\frac{a}{m}\right)\right|^r\right)^{\frac{1}{r}}\\&\quad+\left((1-q)\sum_{s=0}^{\infty}q^{2s}(1-q^{s+1})^r\left|{}^b\mathbf{D}_q^2 f(a)\right|^r\right.\\&\quad\left.\left.+m(1-q)\sum_{s=0}^{\infty}(q^s-q^{2s})(1-q^{s+1})^r\left|{}^b\mathbf{D}_q^2 f\left(\frac{b}{m}\right)\right|^r\right)^{\frac{1}{r}}\right\}.\end{aligned}\tag{44}$$

Proof. By using Hölder's inequality, m-convexity and the application of q-integral, we have

$$\begin{aligned}&\int_0^1 s(1-qs)\left|{}_a\mathbf{D}_q^2 f(sb+(1-s)a)\right|d_qs\\&\leqslant\left(\int_0^1 s^u d_qs\right)^{\frac{1}{u}}\left(\int_0^1(1-qs)^r\left|{}_a\mathbf{D}_q^2 f(sb+(1-s)a)\right|^r d_qs\right)^{\frac{1}{r}}\\&\leqslant\left(\frac{1-q}{1-q^{u+1}}\right)^{\frac{1}{u}}\left(\left|{}_a\mathbf{D}_q^2 f(b)\right|^r\int_0^1 s(1-qs)^r d_qs\right.\\&\quad\left.+m\left|{}_a\mathbf{D}_q^2 f\left(\frac{a}{m}\right)\right|^r\int_0^1(1-s)(1-qs)^r d_qs\right)^{\frac{1}{r}}\\&=\left(\frac{1}{[u+1]}\right)^{\frac{1}{u}}\left((1-q)\sum_{s=0}^{\infty}q^{2s}(1-q^{s+1})^r\left|{}_a\mathbf{D}_q^2 f(b)\right|^r\right.\\&\quad\left.+m(1-q)\sum_{s=0}^{\infty}(q^s-q^{2s})(1-q^{s+1})^r\left|{}_a\mathbf{D}_q^2 f\left(\frac{a}{m}\right)\right|^r\right)^{\frac{1}{r}}.\end{aligned}\tag{45}$$

Similarly,

$$\begin{aligned}
&\int_0^1 s(1-qs)\left|\,{}^b\mathbf{D}_q^2 f(sa+(1-s)b)\right| d_qs \\
&\leqslant \left(\frac{1}{[u+1]}\right)^{\frac{1}{u}} \Bigg((1-q)\sum_{s=0}^{\infty} q^{2s}(1-q^{s+1})^r \left|{}^b\mathbf{D}_q^2 f(a)\right|^r \\
&\quad + m(1-q)\sum_{s=0}^{\infty}(q^s-q^{2s})(1-q^{s+1})^r \left|{}^b\mathbf{D}_q^2 f\left(\frac{b}{m}\right)\right|^r\Bigg)^{\frac{1}{r}}. \qquad (46)
\end{aligned}$$

Inequality (44) is then obtained by inequalities (21), (45) and (46). □

COROLLARY 5. *If r is a positive integer, then using the facts that*

$$(1-s)^{r+1} \leqslant (1-qs)_q^{r+1}, \quad (1-s)^r \leqslant (1-qs)_q^r, \qquad (47)$$

we get

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{2(b-a)}\left[\int_a^b f(t)\ {}_ad_qt + \int_a^b f(t)\ {}^bd_qt\right]\right| \\
&\leqslant \frac{q^2(b-a)^2}{2(1+q)}\left(\frac{1}{[u+1]}\right)^{\frac{1}{u}} \\
&\quad \times\Bigg\{\left(B_q(2,r+1)\left|{}_a\mathbf{D}_q^2 f(b)\right|^r + mB_q(1,r+2)\left|{}_a\mathbf{D}_q^2 f\left(\frac{a}{m}\right)\right|^r\right)^{\frac{1}{r}} \\
&\quad + \left(B_q(2,r+1)\left|{}^b\mathbf{D}_q^2 f(a)\right|^r + mB_q(1,r+2)\left|{}^b\mathbf{D}_q^2 f\left(\frac{b}{m}\right)\right|^r\right)^{\frac{1}{r}}\Bigg\}. \qquad (48)
\end{aligned}$$

REMARK 6. If $q \to 1^-$, then we obtain the following inequality for m-convex functions:

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
&\leqslant \frac{(b-a)^2}{4}\left(\frac{1}{u+1}\right)^{\frac{1}{u}}\Bigg\{\left(\frac{|f''(b)|^r + m(r+1)\left|f''\left(\frac{a}{m}\right)\right|^r}{(r+1)(r+2)}\right)^{\frac{1}{r}} \\
&\quad + \left(\frac{|f''(a)|^r + m(r+1)\left|f''\left(\frac{b}{m}\right)\right|^r}{(r+1)(r+2)}\right)^{\frac{1}{r}}\Bigg\}. \qquad (49)
\end{aligned}$$

THEOREM 6. *Let $f : I \to \Re$ be a twice differentiable convex function on I° with ${}_a\mathbf{D}_q^2 f$ and ${}^b\mathbf{D}_q^2 f$ continuous and integrable on I, where $0 < q < 1$. If $\left|{}_a\mathbf{D}_q^2 f\right|^r$ and $\left|{}^b\mathbf{D}_q^2 f\right|^r$ are m-convex functions on I, where $r,u > 1$, $\frac{1}{u} + \frac{1}{r} = 1$ and $m \in (0,1]$ is fixed, then*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{2(b-a)}\left[\int_a^b f(t)\ {}_a d_q t + \int_a^b f(t)\ {}^b d_q t\right]\right| \\
&\leqslant \frac{q^2(b-a)^2}{2(1+q)}\left((1-q)\sum_{s=0}^{\infty} q^s(1-q^{s+1})^u\right)^{\frac{1}{u}} \\
&\times\left\{\left(\frac{[r+1]\left|{}_a\mathbf{D}_q^2 f(b)\right|^r + mq^{r+1}\left|{}_a\mathbf{D}_q^2 f\left(\frac{a}{m}\right)\right|^r}{[r+1][r+2]}\right)^{\frac{1}{r}}\right. \\
&\left.+\left(\frac{[r+1]\left|{}^b\mathbf{D}_q^2 f(a)\right|^r + mq^{r+1}\left|{}^b\mathbf{D}_q^2 f\left(\frac{b}{m}\right)\right|^r}{[r+1][r+2]}\right)^{\frac{1}{r}}\right\}. \qquad (50)
\end{aligned}$$

Proof. By utilizing m-convexity, the Hölder's inequality and the q-integral, we obtain

$$\begin{aligned}
&\int_0^1 s(1-qs)\left|\ {}_a\mathbf{D}_q^2 f(sb+(1-s)a)\right|\ d_q s \\
&\leqslant \left(\int_0^1 (1-s)^u d_q s\right)^{\frac{1}{u}}\left(\int_0^1 s^r\left|\ {}_a\mathbf{D}_q^2 f(sb+(1-s)a)\right|^r\ d_q s\right)^{\frac{1}{r}} \\
&\leqslant \left((1-q)\sum_{s=0}^{\infty} q^s(1-q^{s+1})^u\right)^{\frac{1}{u}} \\
&\times\left(\left|{}_a\mathbf{D}_q^2 f(b)\right|^r\int_0^1 s^{r+1} d_q s + m\left|{}_a\mathbf{D}_q^2 f\left(\frac{a}{m}\right)\right|^r\int_0^1 s^r(1-s) d_q s\right)^{\frac{1}{r}} \\
&= \left((1-q)\sum_{s=0}^{\infty} q^s(1-q^{s+1})^u\right)^{\frac{1}{u}}\left(\frac{[r+1]\left|{}_a\mathbf{D}_q^2 f(b)\right|^r + mq^{r+1}\left|{}_a\mathbf{D}_q^2 f\left(\frac{a}{m}\right)\right|^r}{[r+1][r+2]}\right)^{\frac{1}{r}}. \qquad (51)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_0^1 s(1-qs)\left|\ {}^b\mathbf{D}_q^2 f(sa+(1-s)b)\right|\ d_q s \\
&\leqslant \left((1-q)\sum_{s=0}^{\infty} q^s(1-q^{s+1})^u\right)^{\frac{1}{u}}\left(\frac{[r+1]\left|{}^b\mathbf{D}_q^2 f(a)\right|^r + mq^{r+1}\left|{}^b\mathbf{D}_q^2 f\left(\frac{b}{m}\right)\right|^r}{[r+1][r+2]}\right)^{\frac{1}{r}}. \qquad (52)
\end{aligned}$$

Inequality (50) is then obtained by inequalities (21), (51) and (52). □

COROLLARY 6. *If u is a positive integer, then using the inequality* (30) *we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{2(b-a)}\left[\int_a^b f(t)\,{}_ad_qt+\int_a^b f(t)\,{}^bd_qt\right]\right| \leqslant \frac{q^2(b-a)^2}{2(1+q)}\left(B_q(1,u+1)\right)^{\frac{1}{u}} \times\left\{\left(\frac{[r+1]\left|{}_a\mathbf{D}_q^2 f(b)\right|^r+mq^{r+1}\left|{}_a\mathbf{D}_q^2 f\left(\frac{a}{m}\right)\right|^r}{[r+1][r+2]}\right)^{\frac{1}{r}} +\left(\frac{[r+1]\left|{}^b\mathbf{D}_q^2 f(a)\right|^r+mq^{r+1}\left|{}^b\mathbf{D}_q^2 f\left(\frac{b}{m}\right)\right|^r}{[r+1][r+2]}\right)^{\frac{1}{r}}\right\}. \tag{53}$$

REMARK 7. If $q\to 1^-$, then we get the following inequality for m-convex functions:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leqslant \frac{(b-a)^2}{4}\left(\frac{1}{u+1}\right)^{\frac{1}{u}}\left\{\left(\frac{(r+1)\left|f''(b)\right|^r+m\left|f''\left(\frac{a}{m}\right)\right|^r}{(r+1)(r+2)}\right)^{\frac{1}{r}} +\left(\frac{(r+1)\left|f''(a)\right|^r+m\left|f''\left(\frac{b}{m}\right)\right|^r}{(r+1)(r+2)}\right)^{\frac{1}{r}}\right\}. \tag{54}$$

THEOREM 7. *Let $f:I\to\Re$ be a twice differentiable convex function on I° with ${}_a\mathbf{D}_q^2 f$ and ${}^b\mathbf{D}_q^2 f$ continuous and integrable on I, where $0<q<1$. If $\left|{}_a\mathbf{D}_q^2 f\right|$ and $\left|{}^b\mathbf{D}_q^2 f\right|$ are m-convex functions on I, where $m\in(0,1]$ is fixed, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{2(b-a)}\left[\int_a^b f(t)\,{}_ad_qt+\int_a^b f(t)\,{}^bd_qt\right]\right| \leqslant \frac{q^2(b-a)^2}{2(1+q)} \times\left\{\frac{\left|{}_a\mathbf{D}_q^2 f(b)\right|^r+\left|{}^b\mathbf{D}_q^2 f(a)\right|+mq^2\left[\left|{}_a\mathbf{D}_q^2 f\left(\frac{a}{m}\right)\right|+\left|{}^b\mathbf{D}_q^2 f\left(\frac{b}{m}\right)\right|\right]}{(1+q+q^2)(1+q+q^2+q^3)}\right\}. \tag{55}$$

Proof. By m-convexity and the application of q-integral, we get

$$\begin{aligned}
&\int_0^1 s(1-qs)\left|\, {}_a\mathbf{D}_q^2 f(sb+(1-s)a)\right| \, d_qs \\
&\leqslant \left|{}_a\mathbf{D}_q^2 f(b)\right| \int_0^1 s^2(1-qs)d_qs + m\left|{}_a\mathbf{D}_q^2 f\left(\frac{a}{m}\right)\right| \int_0^1 s(1-s)(1-qs)d_qs \\
&= \frac{\left|{}_a\mathbf{D}_q^2 f(b)\right| + mq^2\left|{}_a\mathbf{D}_q^2 f\left(\frac{a}{m}\right)\right|}{(1+q+q^2)(1+q+q^2+q^3)}. \qquad (56)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_0^1 s(1-qs)\left|\, {}^b\mathbf{D}_q^2 f(sa+(1-s)b)\right| \, d_qs \\
&\leqslant \frac{\left|{}^b\mathbf{D}_q^2 f(a)\right| + mq^2\left|{}^b\mathbf{D}_q^2 f\left(\frac{b}{m}\right)\right|}{(1+q+q^2)(1+q+q^2+q^3)}. \qquad (57)
\end{aligned}$$

Inequality (55) is then obtained by inequalities (21), (56) and (57). □

REMARK 8. If $q \to 1^-$, then we have the following inequality for m-convex functions:

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\, dt\right| \\
&\leqslant \frac{(b-a)^2}{48}\left\{\left|f''(b)\right| + \left|f''(a)\right| + m\left[\left|f''\left(\frac{a}{m}\right)\right| + \left|f''\left(\frac{b}{m}\right)\right|\right]\right\}. \qquad (58)
\end{aligned}$$

THEOREM 8. *Let $f : I \to \Re$ be a twice differentiable convex function on I° with ${}_a\mathbf{D}_q^2 f$ and ${}^b\mathbf{D}_q^2 f$ continuous and integrable on I, where $0 < q < 1$. If $\left|{}_a\mathbf{D}_q^2 f\right|^r$ and $\left|{}^b\mathbf{D}_q^2 f\right|^r$ are m-convex functions on I for $r \geqslant 1$, where $m \in (0,1]$ is fixed, then*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{2(b-a)}\left[\int_a^b f(t)\, {}_ad_qt + \int_a^b f(t)\, {}^bd_qt\right]\right| \\
&\leqslant \frac{q^2(b-a)^2}{2(1+q)^{2-\frac{1}{r}}(1+q+q^2)}\left\{\left(\frac{\left|{}_a\mathbf{D}_q^2 f(b)\right|^r + mq^2\left|{}_a\mathbf{D}_q^2 f\left(\frac{a}{m}\right)\right|^r}{1+q+q^2+q^3}\right)^{\frac{1}{r}}\right. \\
&\quad \left. + \left(\frac{\left|{}^b\mathbf{D}_q^2 f(a)\right|^r + mq^2\left|{}^b\mathbf{D}_q^2 f\left(\frac{b}{m}\right)\right|^r}{1+q+q^2+q^3}\right)^{\frac{1}{r}}\right\}. \qquad (59)
\end{aligned}$$

Proof. By using power mean inequality, m-convexity and the application of q-integral, we have

$$\begin{aligned}
&\int_0^1 s(1-qs)\left|\,{}_a\mathbf{D}_q^2 f(sb+(1-s)a)\right|\,d_qs \\
&\leqslant \left(\int_0^1 s(1-qs)d_qs\right)^{1-\frac{1}{r}}\left(\int_0^1 s(1-qs)\left|\,{}_a\mathbf{D}_q^2 f(sb+(1-s)a)\right|^r\,d_qs\right)^{\frac{1}{r}} \\
&\leqslant \left(\frac{1}{(1+q)(1+q+q^2)}\right)^{1-\frac{1}{r}} \\
&\quad\times\left(\left|{}_a\mathbf{D}_q^2 f(b)\right|^r\int_0^1 s^2(1-qs)d_qs+m\left|{}_a\mathbf{D}_q^2 f\left(\frac{a}{m}\right)\right|^r\int_0^1 s(1-s)(1-qs)d_qs\right)^{\frac{1}{r}} \\
&=\left(\frac{1}{(1+q)(1+q+q^2)}\right)^{1-\frac{1}{r}}\left(\frac{\left|{}_a\mathbf{D}_q^2 f(b)\right|^r+mq^2\left|{}_a\mathbf{D}_q^2 f\left(\frac{a}{m}\right)\right|^r}{(1+q+q^2)(1+q+q^2+q^3)}\right)^{\frac{1}{r}}. \qquad (60)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_0^1 s(1-qs)\left|\,{}^b\mathbf{D}_q^2 f(sa+(1-s)b)\right|\,d_qs \\
&\leqslant \left(\frac{1}{(1+q)(1+q+q^2)}\right)^{1-\frac{1}{r}}\left(\frac{\left|{}^b\mathbf{D}_q^2 f(a)\right|^r+mq^2\left|{}^b\mathbf{D}_q^2 f\left(\frac{b}{m}\right)\right|^r}{(1+q+q^2)(1+q+q^2+q^3)}\right)^{\frac{1}{r}}. \qquad (61)
\end{aligned}$$

Inequality (59) is then obtained by inequalities (21), (60) and (61). □

REMARK 9. If $q\to 1^-$, then we have the following inequality for m-convex functions:

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
&\leqslant \frac{(b-a)^2}{3\cdot 2^{3+\frac{1}{r}}} \\
&\quad\times\left\{\left(\left|f''(b)\right|^r+m\left|f''\left(\frac{a}{m}\right)\right|^r\right)^{\frac{1}{r}}+\left(\left|f''(a)\right|^r+m\left|f''\left(\frac{b}{m}\right)\right|^r\right)^{\frac{1}{r}}\right\}. \qquad (62)
\end{aligned}$$

THEOREM 9. *Let $f:I\to\Re$ be a twice differentiable convex function on I° with ${}_a\mathbf{D}_q^2 f$ and ${}^b\mathbf{D}_q^2 f$ continuous and integrable on I, where $0<q<1$. If $\left|{}_a\mathbf{D}_q^2 f\right|^r$ and*

$\left|{}^{b}\mathbf{D}_q^2 f\right|^r$ *are* m*-convex functions on* I *for* $r \geqslant 1$*, where* $m \in (0,1]$ *is fixed, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{2(b-a)}\left[\int_a^b f(t)\ {}_a d_q t + \int_a^b f(t)\ {}^b d_q t\right]\right|$$
$$\leqslant \frac{q^2(b-a)^2}{2(1+q)}\left(\frac{1}{1+q}\right)^{1-\frac{1}{r}}$$
$$\times\left\{\left(\frac{[r+1]\left|{}_a\mathbf{D}_q^2 f(b)\right|^r + mq^{r+1}(1+q)\left|{}_a\mathbf{D}_q^2 f\left(\frac{a}{m}\right)\right|^r}{[r+1][r+2][r+3]}\right)^{\frac{1}{r}}\right.$$
$$\left. + \left(\frac{[r+1]\left|{}^b\mathbf{D}_q^2 f(a)\right|^r + mq^{r+1}(1+q)\left|{}^b\mathbf{D}_q^2 f\left(\frac{b}{m}\right)\right|^r}{[r+1][r+2][r+3]}\right)^{\frac{1}{r}}\right\}. \tag{63}$$

Proof. Applying the power mean inequality, m-convexity and the q-integral operator we have

$$\int_0^1 s(1-qs)\left|\ {}_a\mathbf{D}_q^2 f(sb+(1-s)a)\right|\ d_q s$$
$$\leqslant \left(\int_0^1 (1-qs)d_q s\right)^{1-\frac{1}{r}}\left(\int_0^1 s^r(1-qs)\left|\ {}_a\mathbf{D}_q^2 f(sb+(1-s)a)\right|^r\ d_q s\right)^{\frac{1}{r}}$$
$$\leqslant \left(\frac{1}{1+q}\right)^{1-\frac{1}{r}}$$
$$\times\left(\left|{}_a\mathbf{D}_q^2 f(b)\right|^r \int_0^1 s^{r+1}(1-qs)d_q s + m\left|{}_a\mathbf{D}_q^2 f\left(\frac{a}{m}\right)\right|^r \int_0^1 s^r(1-s)(1-qs)^r d_q s\right)^{\frac{1}{r}}$$
$$= \left(\frac{1}{1+q}\right)^{1-\frac{1}{r}}\left(\frac{[r+1]\left|{}_a\mathbf{D}_q^2 f(b)\right|^r + mq^{r+1}(1+q)\left|{}_a\mathbf{D}_q^2 f\left(\frac{a}{m}\right)\right|^r}{[r+1][r+2][r+3]}\right)^{\frac{1}{r}}. \tag{64}$$

Similarly,

$$\int_0^1 s(1-qs)\left|\ {}^b\mathbf{D}_q^2 f(sa+(1-s)b)\right|\ d_q s$$
$$\leqslant \left(\frac{1}{1+q}\right)^{1-\frac{1}{r}}\left(\frac{[r+1]\left|{}^b\mathbf{D}_q^2 f(a)\right|^r + mq^{r+1}(1+q)\left|{}^b\mathbf{D}_q^2 f\left(\frac{b}{m}\right)\right|^r}{[r+1][r+2][r+3]}\right)^{\frac{1}{r}}. \tag{65}$$

Inequality (63) is then obtained by inequalities (21), (64) and (65). $\square$

REMARK 10. If $q \to 1^-$, then we have the following inequality for m-convex functions:

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt \right|$$
$$\leqslant \frac{(b-a)^2}{8}\left(\frac{2}{(r+1)(r+2)(r+3)}\right)^{\frac{1}{r}}$$
$$\times\left\{\left((r+1)\left|f''(b)\right|^r + 2m\left|f''\left(\frac{a}{m}\right)\right|^r\right)^{\frac{1}{r}}\right.$$
$$\left. + \left((r+1)\left|f''(a)\right|^r + 2m\left|f''\left(\frac{b}{m}\right)\right|^r\right)^{\frac{1}{r}}\right\}. \tag{66}$$

THEOREM 10. *Let $f : I \to \Re$ be a twice differentiable convex function on I° with ${}_a\mathbf{D}_q^2 f$ and ${}^b\mathbf{D}_q^2 f$ continuous and integrable on I, where $0 < q < 1$. If $\left|{}_a\mathbf{D}_q^2 f\right|^r$ and $\left|{}^b\mathbf{D}_q^2 f\right|^r$ are m-convex functions on I, where $r, u > 1$, $\frac{1}{u}+\frac{1}{r} = 1$ and $m \in (0,1]$ is fixed, then*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{2(b-a)}\left[\int_a^b f(t)\ {}_ad_qt + \int_a^b f(t)\ {}^bd_qt\right] \right|$$
$$\leqslant \frac{q^2(b-a)^2}{2(1+q)}\left(\frac{1}{[u+1]}\right)^{\frac{1}{u}}\left\{\left((1-q)\sum_{s=0}^{\infty} q^{2s}(1-q^{s+1})^r \left|{}_a\mathbf{D}_q^2 f(b)\right|^r\right.\right.$$
$$\left. + m(1-q)\sum_{s=0}^{\infty}(q^s - q^{2s})(1-q^{s+1})^r\left|{}_a\mathbf{D}_q^2 f\left(\frac{a}{m}\right)\right|^r\right)^{\frac{1}{r}}$$
$$+\left((1-q)\sum_{s=0}^{\infty} q^{2s}(1-q^{s+1})^r \left|{}^b\mathbf{D}_q^2 f(a)\right|^r\right.$$
$$\left.\left. + m(1-q)\sum_{s=0}^{\infty}(q^s - q^{2s})(1-q^{s+1})^r\left|{}^b\mathbf{D}_q^2 f\left(\frac{b}{m}\right)\right|^r\right)^{\frac{1}{r}}\right\}. \tag{67}$$

Proof. Applying Hölder's inequality, m-convexity and the q-integral operator we obtain

$$\int_0^1 s(1-qs)\left|\ {}_a\mathbf{D}_q^2 f(sb+(1-s)a)\right|\ d_qs$$
$$\leqslant \left(\int_0^1 s^u(1-qs)d_qs\right)^{\frac{1}{u}}\left(\int_0^1 (1-qs)\left|\ {}_a\mathbf{D}_q^2 f(sb+(1-s)a)\right|^r\ d_qs\right)^{\frac{1}{r}}$$

$$\leqslant \left(\int_0^1 s^u (1-qs)_q^1 \, d_q s\right)^{\frac{1}{u}}$$

$$\times \left(\left|{}_a\mathbf{D}_q^2 f(b)\right|^r \int_0^1 s(1-qs) d_q s + m \left|{}_a\mathbf{D}_q^2 f\left(\frac{a}{m}\right)\right|^r \int_0^1 (1-s)(1-qs) d_q s \right)^{\frac{1}{r}}$$

$$= (B_q(u+1,2))^{\frac{1}{u}} \left(\frac{\left|{}_a\mathbf{D}_q^2 f(b)\right|^r + m(q+q^2)\left|{}_a\mathbf{D}_q^2 f\left(\frac{a}{m}\right)\right|^r}{(1+q)(1+q+q^2)} \right)^{\frac{1}{r}}. \tag{68}$$

Similarly,

$$\int_0^1 s(1-qs) \left| {}^b\mathbf{D}_q^2 f(sa+(1-s)b) \right| \, d_q s$$

$$\leqslant (B_q(u+1,2))^{\frac{1}{u}} \left(\frac{\left|{}^b\mathbf{D}_q^2 f(a)\right|^r + m(q+q^2)\left|{}^b\mathbf{D}_q^2 f\left(\frac{b}{m}\right)\right|^r}{(1+q)(1+q+q^2)} \right)^{\frac{1}{r}}. \tag{69}$$

Inequality (67) is then obtained by inequalities (21), (68) and (69). □

REMARK 11. If $q \to 1^-$, then we have the following inequality for m-convex functions:

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) \, dt \right|$$

$$\leqslant \frac{(b-a)^2}{4} \left(\frac{1}{(u+1)(u+2)} \right)^{\frac{1}{u}}$$

$$\times \left\{ \left(\frac{|f''(b)|^r + 2m\left|f''\left(\frac{a}{m}\right)\right|^r}{6} \right)^{\frac{1}{r}} + \left(\frac{|f''(a)|^r + 2m\left|f''\left(\frac{b}{m}\right)\right|^r}{6} \right)^{\frac{1}{r}} \right\}. \tag{70}$$

REMARK 12. Taking $m=1$ in our above theorems, we can obtain some interesting results for convex functions. We omit their proofs and the details are left to the interested reader.

4. Conclusion

In this study, we established some new Hermite–Hadamard inequalities for twice q-differentiable m-convex functions. For the desired results, we utilize two kinds of q-integrals defined on the finite interval of real numbers. We developed different estimates for a trapezium type inequality. To the best of our knowledge these results are new in the literature. Since the class of convex (m-convex) functions has many applications in many mathematical areas, they can be applied to obtain several results in convex analysis, special functions, quantum mechanics, related optimization theory, mathematical inequalities and may stimulate further research in different areas of pure and applied sciences.

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