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REMARKS ON THE STABILITY OF THE 3–VARIABLE FUNCTIONAL INEQUALITIES OF DRYGAS

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Abstract. We give some remarks on the recent paper [*J. Math. Inequal.* **13** (4) (2019), 1235–1244]. Some misleading conclusions of their stability results are carefully discussed and corrected. Moreover, we reestablish their results with a more general assumption and a stronger conclusion.

1. Introduction and preliminaries

In the theory of functional equations, the following is known as the *stability problem of functional equations*:

> "When is it true that a function which approximately satisfies a functional equation is close to an exact solution of the equation?"

The problem was first asked by Ulam [22] concerning the stability of group homomorphisms. The *additive* or *Cauchy functional equation*

$$f(x+y) = f(x) + f(y), \tag{1.1}$$

seems to be the first functional equation that many authors studied its stability results. By an *additive mapping*, we mean a mapping that satisfies (1.1). Hyers [10] asserted that Ulam's question is affirmative for the additive functional equation (1.1) defined on Banach spaces. Some further generalizations of Hyers' result were investigated in [1, 7, 8, 20].

To study a stability result, some functional inequalities were introduced. For example, Park *et al.* [18] investigated the following functional inequalities in a Banach space:

$$\|f(x)+f(y)+f(z)\| \leqslant \|f(x+y+z)\|;$$
$$\|f(x)+f(y)+f(z)\| \leqslant \left\|2f\left(\frac{x+y+z}{2}\right)\right\|;$$
$$\|f(x)+f(y)+2f(z)\| \leqslant \left\|2f\left(\frac{x+y}{2}+z\right)\right\|.$$

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It is not hard to see that every mapping satisfying one of the above functional inequalities is additive. Moreover, the notion of a ρ-functional inequality is also introduced. Park [16] proved that the following functional inequality is equivalent to the additive functional equation:

$$\left\| f\left(\sum_{i=1}^{k} x_i\right) - \sum_{i=1}^{k} f(x_i) \right\| \leqslant \left\| \rho\left(kf\left(\frac{\sum_{i=1}^{k} x_i}{k}\right) - \sum_{i=1}^{k} f(x_i)\right)\right\|,$$

where $k \geqslant 2$ is a fixed positive integer and $\rho \in \mathbb{C}$ with $|\rho| < 1$. We refer the readers to [5, 9, 17, 19] for more results concerning the stability of functional inequalities.

In 2016, Lu *et al.* [15] introduced the *3-variable Jensen functional equations*:

$$f(x+y+z) + f(x+y-z) = 2f(x) + 2f(y);$$
$$f(x+y+z) - f(x-y-z) = 2f(y) + 2f(z).$$

They also presented the ρ-functional inequality associated with the above two equations and derived some stability results by using the direct method based on the idea of Găvruta [8].

To characterize a quasi-inner product space, Drygas [2] presented the following functional equation (later, known as the *Drygas functional equation*):

$$f(x+y) + f(x-y) = 2f(x) + f(y) + f(-y). \tag{1.2}$$

Every mapping satisfying (1.2) is called a *Drygas mapping*. Given two vector spaces X, Y and a mapping $f : X \to Y$ that satisfies (1.2) for all $x, y \in X$, we have the following observations.

- If f is odd (that is, $f(-x) = -f(x)$ for all $x \in X$), then f is additive.
- If f is even (that is, $f(-x) = f(x)$ for all $x \in X$), then f satisfies the *quadratic functional equation*, that is,

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \quad \text{for all } x, y \in X.$$

Inspired by the observations given above, Ebanks *et al.* [3] proved that a general solution of the Drygas functional equation (1.2) is the sum of a quadratic and an additive mappings. Some stability results of the functional equation (1.2) have been studied in various directions and can be seen for examples in [4, 11, 12, 14] and references therein.

In [21], Sun *et al.* introduced the following *3-variable Drygas functional equation* in a complex Banach space:

$$f(x+y+z) + f(x+y-z) = 2f(x) + 2f(y) + f(z) + f(-z). \tag{1.3}$$

Moreover, functional inequalities for (1.3) and their stability results were investigated. We state their results as follows.

THEOREM SJPL1. [21, Theorems 2.2 and 2.4] *Suppose that* X *is a complex normed space,* Y *is a complex Banach space, and* $\alpha,\beta \in \mathbb{C}$ *satisfy* $|\alpha|+|\beta|<2$ *and* $|\alpha|<1$. *Suppose that* $f: X \to Y$ *and* $\varphi : X^3 \to [0,\infty)$ *satisfy*

$$\begin{aligned} &\|f(x+y+z)+f(x+y-z)-2f(x)-2f(y)-f(z)-f(-z)\| \\ &\leqslant \|\alpha(f(x+y+z)-f(x)-f(y)-f(z))\| \\ &\quad+\|\beta(f(x+y-z)-f(x)-f(y)-f(-z))\|+\varphi(x,y,z) \end{aligned} \tag{1.4}$$

for all $x,y,z\in X$. *Suppose that* φ *satisfies one of the following conditions:*

(a) $\widetilde{\varphi}(x,y,z) := \sum_{n=0}^{\infty} \frac{1}{2^n}\varphi(2^n x, 2^n y, 2^n z) < \infty$ *for all* $x,y,z \in X$;

(b) $\widetilde{\varphi}(x,y,z) := \sum_{n=1}^{\infty} 2^n \varphi\left(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\right) < \infty$ *for all* $x,y,z \in X$.

Then there exists a unique additive mapping $\mathscr{A}: X\to Y$ *such that*

$$\|\mathscr{A}(x)-f(x)\| \leqslant \frac{\widetilde{\varphi}(x,x,0)}{2(2-|\alpha|-|\beta|)} \quad \text{for all } x\in X.$$

By making a slight modification of (1.4), the following theorem was investigated.

THEOREM SJPL2. [21, Theorems 3.1 and 3.3] *Suppose that* X *is a complex normed space,* Y *is a complex Banach space, and* $\alpha,\beta\in\mathbb{C}$ *satisfy* $|\alpha|+|\beta|<1$. *Suppose that* $f: X\to Y$ *and* $\varphi: X^3 \to [0,\infty)$ *satisfy*

$$\begin{aligned} &\|f(x+y+z)+f(x+y-z)-2f(x)-2f(y)-f(z)-f(-z)\| \\ &\leqslant \|\alpha(f(x+y-z)+f(x-y+z)-2f(x)-f(y)-f(-y)-f(z)-f(-z))\| \\ &\quad+\|\beta(f(x+y+z)-f(x+z)-f(y))\|+\varphi(x,y,z) \end{aligned} \tag{1.5}$$

for all $x,y,z\in X$. *Suppose that* φ *satisfies one of the following conditions:*

(a) $\widetilde{\varphi}(x,y,z) := \sum_{n=0}^{\infty} \frac{1}{4^n}\varphi(2^n x, 2^n y, 2^n z) < \infty$ *for all* $x,y,z \in X$;

(b) $\widetilde{\varphi}(x,y,z) := \sum_{n=1}^{\infty} 4^n \varphi\left(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{4^n}\right) < \infty$ *for all* $x,y,z \in X$.

Then there exists a unique Drygas mapping $\mathscr{D}: X\to Y$ *such that*

$$\|\mathscr{D}(x)-f(x)-f(-x)\| \leqslant \frac{\widetilde{\varphi}(x,0,x)+\widetilde{\varphi}(-x,0,x)}{4(1-|\alpha|)} \quad \text{for all } x\in X.$$

Motivated by these two theorems, we present some further results and refinements. The paper is organized as follows. In Section 2, we discuss Theorem SJPL1. In fact, we show that a mapping satisfying the inequality (1.4) of Theorem SJPL1 is approximately additive and hence the stability result can be obtained directly from the result of Kim [13] in 2005. Moreover, the condition $|\alpha|<1$ is discarded. In Section 3, we improve the conclusion of Theorem SJPL2 with a weaker assumption. The proof of this part is obtained by the application of Forti's result [6].

2. Some remarks on Theorem SJPL1

We begin this section with a counterexample to Theorem SJPL1(a) together with $\alpha = \beta := 0$. A correction of this result is given later in Theorem 2.3(a).

EXAMPLE 2.1. Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(x) := \sqrt{|x|} - 1$ for all $x \in \mathbb{C}$. We also define $\varphi : \mathbb{C}^3 \to [0,\infty)$ by

$$\varphi(x,y,z) := \left|\sqrt{|x+y+z|} + \sqrt{|x+y-z|} - 2\left(\sqrt{|x|} + \sqrt{|y|} + \sqrt{|z|}\right) + 4\right|$$

for all $x,y,z \in \mathbb{C}$. It follows that f and φ satisfy (1.4) for all $x,y,z \in \mathbb{C}$. We also see that the condition (a) is satisfied. In fact,

$$\begin{aligned}\widetilde{\varphi}(x,y,z) &:= \sum_{n=0}^{\infty} \frac{\varphi(2^n x, 2^n y, 2^n z)}{2^n} \\ &= \sum_{n=0}^{\infty} \left| \frac{1}{2^{n/2}} \left(\sqrt{|x+y+z|} + \sqrt{|x+y-z|} - 2\left(\sqrt{|x|} + \sqrt{|y|} + \sqrt{|z|}\right)\right) + \frac{4}{2^n} \right| \\ &< \infty\end{aligned}$$

for all $x,y,z \in \mathbb{C}$. We show that the conclusion of Theorem SJPL1 does not hold. To show this, suppose that there exists an additive mapping $\mathscr{A} : \mathbb{C} \to \mathbb{C}$ such that

$$|\mathscr{A}(x) - f(x)| \leqslant \frac{\widetilde{\varphi}(x,x,0)}{4} = \frac{1}{4}\sum_{n=0}^{\infty} \frac{\varphi(2^n x, 2^n x, 0)}{2^n} \quad \text{for all } x \in \mathbb{C}. \tag{2.1}$$

We first prove that $\mathscr{A} \equiv \mathbf{0}$. For $x \in \mathbb{C}$ and $k \in \mathbb{N}$, we see from the additivity of $\mathscr{A}$ and (2.1) that

$$\left|\mathscr{A}(x) - \frac{f(2^k x)}{2^k}\right| = \left|\frac{\mathscr{A}(2^k x)}{2^k} - \frac{f(2^k x)}{2^k}\right| \leqslant \frac{\widetilde{\varphi}(2^k x, 2^k x, 0)}{4 \cdot 2^k} = \frac{1}{4}\sum_{n=k}^{\infty} \frac{\varphi(2^n x, 2^n x, 0)}{2^n}.$$

It follows that

$$\mathscr{A}(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} = \lim_{n\to\infty} \frac{2^{\frac{n}{2}}\sqrt{|x|} - 1}{2^n} = 0.$$

Thus, $\mathscr{A}(x) = 0$. Hence, (2.1) becomes

$$|f(x)| \leqslant \frac{\widetilde{\varphi}(x,x,0)}{4} \quad \text{for all } x \in \mathbb{C}. \tag{2.2}$$

We can easily see that

$$\varphi(x,x,0) = 2\left|(\sqrt{2} - 2)\sqrt{|x|} + 2\right| \quad \text{for all } x \in \mathbb{C}.$$

By letting $x_0 := (\sqrt{2} + 2)^2$, we see that

$$\varphi(2^n x_0, 2^n x_0, 0) = 2\left| -2 \cdot 2^{n/2} + 2 \right| = 4(2^{n/2} - 1) \quad \text{for all } n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}.$$

It follows that

$$\widetilde{\varphi}(x_0,x_0,0) := \sum_{n=0}^{\infty} \frac{\varphi(2^n x_0, 2^n x_0, 0)}{2^n} = 4\left(\sum_{n=0}^{\infty} \frac{1}{2^{n/2}} - \sum_{n=0}^{\infty} \frac{1}{2^n}\right) = 4\sqrt{2}.$$

One can see that $|f(x_0)| = |\sqrt{2}+1| > \sqrt{2} = \frac{1}{4}\widetilde{\varphi}(x_0,x_0,0)$ which contradicts to (2.2). This shows that the additive mapping $\mathscr{A}$ satisfying (2.1) does not exist. Therefore, Theorem SPJL1(a) is *invalid*.

The following proposition shows that: If a mapping f satisfies the functional inequality (1.4), then it is *almost* additive in the following sense.

PROPOSITION 2.2. *Suppose that X is a vector space, Y is a complex normed space, and $\alpha, \beta \in \mathbb{C}$ satisfy $|\alpha| + |\beta| < 2$. If $f : X \to Y$ and $\varphi : X^3 \to [0,\infty)$ satisfy (1.4) for all $x,y,z \in X$. Then the following inequality holds true:*

$$\|f(x+y) - f(x) - f(y)\| \leqslant \frac{\varphi(x,y,0)}{2-|\alpha|-|\beta|} + \frac{2+|\alpha|+|\beta|}{2(2-|\alpha|-|\beta|)^2}\varphi(0,0,0) \tag{2.3}$$

for all $x,y \in X$.

Proof. By letting $x = y = z := 0$ in (1.4), we obtain that

$$\|f(0)\| \leqslant \frac{\varphi(0,0,0)}{2(2-|\alpha|-|\beta|)}.$$

For any $x,y \in X$, we see from (1.4) that

$$\begin{aligned}
&\|2f(x+y) - 2f(x) - 2f(y)\| \\
&\leqslant \|f(x+y+0) + f(x+y-0) - 2f(x) - 2f(y) - f(0) - f(0)\| + 2\|f(0)\| \\
&\leqslant \|\alpha\,(f(x+y+0) - f(x) - f(y) - f(0))\| \\
&\quad + \|\beta(f(x+y-0) - f(x) - f(y) - f(0))\| + \varphi(x,y,0) + 2\|f(0)\| \\
&= (|\alpha|+|\beta|)\|f(x+y) - f(x) - f(y)\| + \varphi(x,y,0) + (2+|\alpha|+|\beta|)\|f(0)\|.
\end{aligned}$$

This proves that (2.3) holds for all $x,y \in X$. □

To give a simpler proof of Theorem SJPL1, we recall the following theorem which is a special case of [13].

THEOREM K. [13] *Suppose that X is a vector space and Y is a Banach space. Suppose that $f : X \to Y$ and $\psi : X^2 \to [0,\infty)$ satisfy*

$$\|f(x+y) - f(x) - f(y)\| \leqslant \psi(x,y) \quad \textit{for all } x,y \in X.$$

If one of the following conditions is satisfied:

(1) $\Psi(x,y) := \sum_{n=0}^{\infty} \frac{1}{2^n}\psi(2^n x, 2^n y) < \infty$ *for all $x,y \in X$;*

(2) $\Psi(x,y) := \sum_{n=1}^{\infty} 2^n \psi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) < \infty$ *for all* $x,y \in X$,

then there exists a unique additive mapping $\mathscr{A} : X \to Y$ *satisfying*

$$\|\mathscr{A}(x) - f(x)\| \leqslant \frac{\Psi(x,x)}{2} \quad \textit{for all } x \in X.$$

With the help of Theorem K and Proposition 2.2, we obtain the following result.

THEOREM 2.3. *Suppose that* X *is a vector space,* Y *is a complex Banach space, and* $\alpha, \beta \in \mathbb{C}$ *satisfy* $|\alpha| + |\beta| < 2$. *Suppose that* $f : X \to Y$ *and* $\varphi : X^3 \to [0,\infty)$ *satisfy* (1.4) *for all* $x,y,z \in X$. *Then the following statements hold true.*

(1) *If Condition (a) of Theorem SPJL1 is satisfied, then there exists a unique additive mapping* $\mathscr{A} : X \to Y$ *such that*

$$\|\mathscr{A}(x) - f(x)\| \leqslant \frac{\widetilde{\varphi}(x,x,0)}{2(2-|\alpha|-|\beta|)} + \frac{2+|\alpha|+|\beta|}{2(2-|\alpha|-|\beta|)^2}\varphi(0,0,0) \quad \textit{for all } x \in X.$$

(2) *If Condition (b) of Theorem SPJL1 is satisfied, then there exists a unique additive mapping* $\mathscr{A} : X \to Y$ *such that*

$$\|\mathscr{A}(x) - f(x)\| \leqslant \frac{\widetilde{\varphi}(x,x,0)}{2(2-|\alpha|-|\beta|)} \quad \textit{for all } x \in X.$$

Proof. We first define $\psi : X^2 \to [0,\infty)$ by

$$\psi(x,y) := \frac{\varphi(x,y,0)}{2-|\alpha|-|\beta|} + \frac{2+|\alpha|+|\beta|}{2(2-|\alpha|-2|\beta|)^2}\varphi(0,0,0) \quad \text{for all } x,y \in X.$$

Proposition 2.2 shows that f satisfies

$$\|f(x+y) - f(x) - f(y)\| \leqslant \psi(x,y) \quad \text{for all } x,y \in X.$$

We also have the following observations.

- If φ satisfies Theorem SJPL1(a), then $\sum_{n=0}^{\infty} \frac{1}{2^n}\psi(2^n x, 2^n y) < \infty$ for all $x,y \in X$.
- If φ satisfies Theorem SJPL1(b), then we can easily obtain that $\varphi(0,0,0) = 0$. It follows that

$$\sum_{n=1}^{\infty} 2^n \psi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = \frac{\widetilde{\varphi}(x,y,0)}{2-|\alpha|-|\beta|} \quad \text{for all } x,y \in X.$$

Hence, the result follows from Theorem K. □

REMARK 2.4. Theorem 2.3 improves Theorem SJPL1 in the following ways.

(i) The condition $|\alpha| < 1$ of Theorem SJPL1 can be *omitted*.

(ii) Theorem 2.3(a) is a correction of Theorem SJPL1(a).

3. Some comments on Theorem SJPL2

We start this section with the following proposition.

PROPOSITION 3.1. *Suppose that* X *is a vector space,* Y *is a complex normed space, and* $\alpha,\beta \in \mathbb{C}$ *such that* $4|\alpha|+|\beta|<4$. *Suppose that* $f: X \to Y$ *and* $\varphi: X^3 \to [0,\infty)$ *satisfy* $\varphi(0,0,0)=0$ *and* (1.5) *for all* $x,y,z \in X$. *Then the following statements are true.*

(1) *The mapping* f *satisfies*

$$\|f(x)+f(-x)\| \leqslant \frac{1}{2(1-|\alpha|)} \min\{\varphi(x,-x,-x),\varphi(-x,x,x)\}$$

for all $x \in X$.

(2) *The mapping* f *satisfies*

$$\|f(x+y)+f(x-y)-2f(x)\| \leqslant \frac{\varphi(x,0,y)}{1-|\alpha|} + \frac{1+|\alpha|}{2(1-|\alpha|)^2}\varphi(y,-y,-y)$$

for all $x,y \in X$.

Proof. For convenience, we first let $s := |\alpha|$ and $t := |\beta|$. Letting $x=y=z:=0$ in (1.5), we have

$$4\|f(0)\| = \|4f(0)\| \leqslant \alpha\|4f(0)\| + \beta\|f(0)\| = (4s+t)\|f(0)\|.$$

Since $4s+t<4$, one can obtain that $f(0)=0$. Moreover, one can see from our condition that $s<1$ since $4s \leqslant 4s+t<4$.

Now, we prove that (1) holds. Let $(x,y,z) := (x,-x,-x)$ where $x \in X$. It follows from (1.5) that

$$\begin{aligned}
&\|f(-x)+f(x)-2f(x)-2f(-x)-f(-x)-f(x)\| \\
&\leqslant s\|f(x)+f(x)-2f(x)-f(-x)-f(x)-f(-x)-f(x)\| \\
&\quad + t\|f(-x)-f(0)-f(-x)\| + \varphi(x,-x,-x)
\end{aligned}$$

for all $x \in X$. So, we have

$$\|f(x)+f(-x)\| \leqslant \frac{1}{2(1-s)}\varphi(x,-x,-x)$$

for all $x \in X$. By replacing x by $-x$, we obtain that

$$\|f(x)+f(-x)\| \leqslant \frac{1}{2(1-s)}\varphi(-x,x,x)$$

for all $x \in X$. Next, we prove that (2) holds true. For $x,y,z \in X$, we see from (1.5) that

$$\begin{aligned}
&\|f(x+y+z)+f(x+y-z)-2f(x)-2f(y)\| \\
&\leqslant \|f(x+y+z)+f(x+y-z)-2f(x)-2f(y)-f(z)-f(-z)\|+\|f(z)+f(-z)\| \\
&\leqslant s\|f(x+y-z)+f(x-y+z)-2f(x)-f(y)-f(-y)-f(z)-f(-z)\| \\
&\quad +t\|f(x+y+z)-f(x+z)-f(y)\|+\varphi(x,y,z)+\|f(z)+f(-z)\| \\
&\leqslant s\|f(x+y-z)+f(x-y+z)-2f(x)\|+s\|f(y)+f(-y)\| \\
&\quad +(1+s)\|f(z)+f(-z)\|+t\|f(x+y+z)-f(x+z)-f(y)\|+\varphi(x,y,z).
\end{aligned}$$

So, we have

$$\begin{aligned}
&\|f(x+y+z)+f(x+y-z)-2f(x)-2f(y)\| \\
&\leqslant s\|f(x+y-z)+f(x-y+z)-2f(x)\|+s\|f(y)+f(-y)\| \\
&\quad +(1+s)\|f(z)+f(-z)\|+t\|f(x+y+z)-f(x+z)-f(y)\|+\varphi(x,y,z)
\end{aligned} \tag{3.1}$$

for all $x,y,z \in X$. We see by letting $y := 0$ in (3.1) that

$$\begin{aligned}
&\|f(x+z)+f(x-z)-2f(x)\| \\
&\leqslant s\|f(x-z)+f(x+z)-2f(x)\|+(s+1)\|f(z)+f(-z)\|+\varphi(x,0,z)
\end{aligned}$$

for all $x,z \in X$. It follows from (1) that

$$\begin{aligned}
\|f(x+z)+f(x-z)-2f(x)\| &\leqslant \frac{1}{1-s}\left((1+s)\|f(z)+f(-z)\|+\varphi(x,0,z)\right) \\
&\leqslant \frac{1+s}{2(1-s)^2}\varphi(z,-z,-z)+\frac{1}{1-s}\varphi(x,0,z)
\end{aligned}$$

for all $x,z \in X$. $\square$

REMARK 3.2. We see from Proposition 3.1(1) that if f satisfies (1.5) for all $x,y,z \in X$ then its *even part* f_e (where $f_e(x) := \frac{1}{2}(f(x)+f(-x))$ for all $x \in X$) is approximately zero.

The following example shows that our two estimates in Proposition 3.1 are *sharp* where $\alpha = \beta := 0$.

EXAMPLE 3.3. We consider the following two examples.

(i) Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(x) := \sqrt{|x|}$ for all $x \in \mathbb{C}$. For any $x,y,z \in \mathbb{C}$, we see that

$$\begin{aligned}
&|f(x+y+z)+f(x+y-z)-2f(x)-2f(y)-f(z)-f(-z)| \\
&=\left|\sqrt{|x+y+z|}+\sqrt{|x+y-z|}-2\left(\sqrt{|x|}+\sqrt{|y|}+\sqrt{|z|}\right)\right| =: \varphi(x,y,z).
\end{aligned}$$

This shows that f satsifies (1.5) for all $x,y,z\in\mathbb{C}$ with $\alpha=\beta:=0$. We also see that

$$\varphi(x,-x,-x)=\varphi(-x,x,x)=4\sqrt{|x|}\quad \text{for all } x\in\mathbb{C}.$$

Proposition 3.1(1) asserts that

$$|f(x)+f(-x)|=2\sqrt{|x|}=\frac{1}{2}\min\{\varphi(x,-x,-x),\varphi(-x,x,x)\}\quad \text{for all } x\in\mathbb{C}.$$

(ii) Let $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(u):=\sqrt[3]{u}$ for all $u\in\mathbb{R}$. Let $u,v,w\in\mathbb{R}$ be given. We see that

$$\begin{aligned}&|f(u+v+w)+f(u+v-w)-2f(u)-2f(v)-f(w)-f(-w)|\\&=\left|\sqrt[3]{u+v+w}+\sqrt[3]{u+v-w}-2\sqrt[3]{u}-2\sqrt[3]{v}\right|=:\varphi(u,v,w),\end{aligned}$$

where $\varphi:\mathbb{R}^3\to[0,\infty)$. This shows that f satisfies (1.5) for all $u,v,w\in\mathbb{R}$ with $\alpha=\beta:=0$. It can be seen that

$$\varphi(u,0,v)=\left|\sqrt[3]{u+v}+\sqrt[3]{u-v}-2\sqrt[3]{u}\right|\quad\text{and}\quad\varphi(v,-v,-v)=0$$

Hence, we have

$$\begin{aligned}|f(u+v)+f(u-v)-2f(u)|&=\left|\sqrt[3]{u+v}+\sqrt[3]{u-v}-2\sqrt[3]{u}\right|\\&=\varphi(u,0,v)+\frac{1}{2}\varphi(v,-v,-v).\end{aligned}$$

Now, we define $g:\mathbb{C}\to\mathbb{R}$ and $\widetilde{\varphi}:\mathbb{C}^3\to[0,\infty)$ by

$$\begin{aligned}g(x)&:=\text{ the real part of } x\quad \text{for all } x\in\mathbb{C};\\ \widetilde{\varphi}(x,y,z)&:=\varphi(g(x),g(y),g(z))\quad \text{for all } x,y,z\in\mathbb{C}.\end{aligned}$$

So, we obtain the desired example by defining $\widetilde{f}:=f\circ g$.

The following result is a direct consequence of Proposition 3.1 by letting $\varphi\equiv 0$.

COROLLARY 3.4. *Suppose that X is a vector space, Y is a complex normed space, and $\alpha,\beta\in\mathbb{C}$ satisfy $4|\alpha|+|\beta|<4$. Then a mapping $f:X\to Y$ satisfies the functional inequality (1.5) for all $x,y,z\in X$ if and only if f is additive.*

Proof. If f is additive, then (1.5) holds for all $x,y,z\in X$.

Conversely, suppose that f satisfies (1.5) for all $x,y,z\in X$. Proposition 3.1(2) shows that

$$f(x+y)+f(x-y)=2f(x)\quad \text{for all } x,y\in X.$$

To show that f is additive, let $x,y\in X$ be given. We can easily see that

$$2f(x+y)=f((x+y)+(x-y))+f((x+y)-(x-y))=f(2x)+f(2y)=2(f(x)+f(y)).$$

Hence, f is additive. □

REMARK 3.5. According to Corollary 3.4, we have the following observations.

(1) Our assumption $4|\alpha|+|\beta|<4$ is more general than their original assumption $|\alpha|+|\beta|<1$. Moreover, this is a *strict* generalization. In fact, let $\alpha:=\frac{1}{4}$ and $\beta:=2$. Then $4|\alpha|+|\beta|=3<4$ but $|\alpha|+|\beta|=\frac{9}{4}>1$.

(2) The assumption $4|\alpha|+|\beta|<4$ is *best possible* in the sense that: If $4|\alpha|+|\beta|=4$, then there exists $f:\mathbb{C}\to\mathbb{C}$ such that f satisfies (1.5) for all $x,y,z\in\mathbb{C}$ but f is not additive. Let α,β be such that $4|\alpha|+|\beta|=4$. We define $f:\mathbb{C}\to\mathbb{C}$ by $f(x):=1$ for all $x\in\mathbb{C}$. Then f satisfies (1.5) for all $x,y,z\in\mathbb{C}$ and f is not additive.

To discuss Theorem SJPL2, we recall the stability result proposed by Forti [6] which will be used later in the following two subsections.

THEOREM F. [6] *Let X be a vector space and $(Y,\|\cdot\|)$ be a complex Banach space. Suppose that $f:X\to Y, g:Y\to Y, h:X\to X, \delta:X\to[0,\infty)$, and $\phi:[0,\infty)\to[0,\infty)$ satisfy the following two inequalities*

$$\|(g\circ f\circ h)(x)-f(x)\|\leqslant\delta(x)\quad \textit{for all } x,y\in X;$$
$$\|g(u)-g(v)\|\leqslant\phi(\|u-v\|)\quad \textit{for all } u,v\in Y.$$

If ϕ is non-decreasing subadditive, g is continuous, and $\Phi(x):=\sum_{n=0}^{\infty}\phi^n(\delta(h^n(x)))<\infty$ for all $x\in X$, then the mapping $F:X\to Y$, determined by

$$F(x):=\lim_{n\to\infty}(g^n\circ f\circ h^n)(x)\quad \textit{for all } x\in X,$$

is well-defined and it is the unique mapping such that $g\circ F\circ h=F$ and

$$\|F(x)-f(x)\|\leqslant\Phi(x)\quad \textit{for all } x\in X.$$

3.1. Some remarks on Theorem SPJL2(a)

Theorem SJPL2(a) with $\alpha=\beta:=0$ is not true as shown in the following example.

EXAMPLE 3.6. Let $f:\mathbb{C}\to\mathbb{C}$ and $\varphi:\mathbb{C}^3\to\mathbb{C}$ be defined as in Example 2.1. Obviously that f and φ satisfy (1.5) for all $x,y,z\in\mathbb{C}$. Moreover, it is not hard to see that φ satisfies Theorem SJPL2(a) since

$$\widetilde{\varphi}(x,y,z):=\sum_{n=0}^{\infty}\frac{\varphi(2^nx,2^ny,2^nz)}{4^n}\leqslant\sum_{n=0}^{\infty}\frac{\varphi(2^nx,2^ny,2^nz)}{2^n}<\infty$$

for all $x,y,z\in\mathbb{C}$. Moreover, we note here that

$$\varphi(x,0,x)=\left|(\sqrt{2}-4)\sqrt{|x|}+4\right|=\varphi(-x,0,x)\quad \text{for all } x\in\mathbb{C}. \tag{3.2}$$

Now, we suppose that there exists a Drygas mapping $\mathscr{D}:\mathbb{C}\to\mathbb{C}$ satisfying

$$|\mathscr{D}(x)-f(x)-f(-x)|\leqslant\frac{\widetilde{\varphi}(x,0,x)+\widetilde{\varphi}(-x,0,x)}{4}=\frac{\widetilde{\varphi}(x,0,x)}{2}\quad \text{for all } x\in\mathbb{C}. \tag{3.3}$$

We show that

$$\left|2\sqrt{|x|}-2\right| = |f(x)+f(-x)| \leqslant \frac{\widetilde{\varphi}(x,0,x)}{2} \quad \text{for all } x \in \mathbb{C}. \tag{3.4}$$

To prove this, we show that $\mathscr{D} \equiv \mathbf{0}$. Since $\mathscr{D}$ is Drygas, there exist a quadratic mapping $\mathscr{Q} : \mathbb{C} \to \mathbb{C}$ and an additive mapping $\mathscr{A} : \mathbb{C} \to \mathbb{C}$ such that $\mathscr{D}(x) = \mathscr{Q}(x) + \mathscr{A}(x)$ for all $x \in \mathbb{C}$ (see [3, Corollary 3]). Now, let $x \in \mathbb{C}$ be given. For each $k \in \mathbb{N}$, we see from (3.3) that

$$\left|\frac{\mathscr{D}(2^k x)}{4^k} - \frac{f(2^k x)}{4^k} - \frac{f(-2^k x)}{4^k}\right| \leqslant \frac{\widetilde{\varphi}(2^k x,0,2^k x)}{2\cdot 4^k} = \frac{1}{2}\sum_{n=k}^{\infty} \frac{\varphi(2^n x,0,2^n x)}{4^n}. \tag{3.5}$$

We easily see that

$$\lim_{k\to\infty} \frac{f(2^k x)}{4^k} = \lim_{k\to\infty} \frac{2^{k/2}\sqrt{|x|}-1}{4^k} = 0 \quad \text{and} \quad \lim_{k\to\infty} \frac{f(-2^k x)}{4^k} = 0;$$
$$\lim_{k\to\infty} \sum_{n=k}^{\infty} \frac{\varphi(2^n x,0,2^n x)}{4^n} = 0.$$

Hence, $\lim_{k\to\infty} \frac{\mathscr{D}(2^k x)}{4^k} = 0$. It follows that

$$0 = \lim_{k\to\infty} \frac{\mathscr{D}(2^k x)}{4^k} = \lim_{k\to\infty}\left(\frac{\mathscr{Q}(2^k x)}{4^k} + \frac{\mathscr{A}(2^k x)}{4^k}\right) = \lim_{k\to\infty}\left(\mathscr{Q}(x) + \frac{\mathscr{A}(x)}{2^k}\right) = \mathscr{Q}(x).$$

Hence, $\mathscr{D}(x) = \mathscr{A}(x)$. It follows from (3.3) and the evenness of f that

$$|\mathscr{A}(x) - 2f(x)| \leqslant \frac{\widetilde{\varphi}(x,0,x)}{2}. \tag{3.6}$$

Next, we show that $\mathscr{A}(x) = 0$. It follows from the direct computation that

$$\lim_{k\to\infty} \frac{f(2^k x)}{2^k} = \lim_{k\to\infty} \frac{2^{k/2}\sqrt{|x|}-1}{2^k} = 0;$$
$$\lim_{k\to\infty} \frac{\widetilde{\varphi}(2^k x,0,2^k x)}{2^k} = \lim_{k\to\infty} \sum_{n=0}^{\infty} \frac{\varphi(2^{n+k}x,0,2^{n+k}x)}{2^k\cdot 4^n} \leqslant \lim_{k\to\infty} \sum_{n=k}^{\infty} \frac{\varphi(2^n x,0,2^n x)}{2^n} = 0.$$

It follows from (3.6) that

$$|\mathscr{A}(x)| = \lim_{k\to\infty} \left|\frac{\mathscr{A}(2^k x)}{2^k} - \frac{f(2^k x)}{2^k}\right| \leqslant \lim_{k\to\infty} \frac{\widetilde{\varphi}(2^k x,0,2^k x)}{2\cdot 2^k} = 0.$$

So, we have that $\mathscr{A}(x) = 0$. Hence, we prove (3.4).

By letting $x_0 := (\sqrt{2}+4)^2$, we see from (3.2) that

$$\varphi(2^n x_0,0,2^n x_0) = \left|2^{n/2}(\sqrt{2}-4)(\sqrt{2}+4)+4\right| = 14\cdot 2^{n/2} - 4 \quad \text{for all } n \in \mathbb{N}_0.$$

It follows that

$$\widetilde{\varphi}(x_0,0,x_0) := \sum_{n=0}^{\infty} \frac{\varphi(2^n x_0, 0, 2^n x_0)}{4^n} = \sum_{n=0}^{\infty} \frac{14}{2^{3n/2}} - \sum_{n=0}^{\infty} \frac{4}{4^n} = \frac{32+12\sqrt{2}}{3}.$$

One can see that

$$\left|2(\sqrt{2}+4)-2\right| = 6+2\sqrt{2} = \frac{18+6\sqrt{2}}{3} > \frac{16+6\sqrt{2}}{3} = \frac{\widetilde{\varphi}(x_0,0,x_0)}{2},$$

which contradicts to (3.6). Hence, the conclusion of Theorem SJPL2(a) does not hold.

The following example illustrates that the *uniqueness part* of Theorem SJPL2(a) is *not true* although we assume additionally that $\varphi(0,0,0)=0$.

EXAMPLE 3.7. Let $f:\mathbb{C}\to\mathbb{C}$ and $\varphi:\mathbb{C}^3\to[0,\infty)$ be defined by

$$f(x) := x \quad \text{for all } x\in\mathbb{C};$$
$$\varphi(x,y,z) := |x|+|y|+|z| \quad \text{for all } x,y,z\in\mathbb{C}.$$

For any $x,y,z\in\mathbb{C}$, we can easily see that

$$\begin{aligned}
&|f(x+y+z)+f(x+y-z)-2f(x)-2f(y)-f(z)-f(-z)| \\
&\leqslant \frac{1}{2}|f(x+y-z)+f(x-y+z)-2f(x)-f(y)-f(-y)-f(z)-f(-z)| \\
&\quad + \frac{1}{4}|f(x+y+z)-f(x+z)-f(y)|+\varphi(x,y,z).
\end{aligned}$$

It follows that f and φ satisfy (1.5) for all $x,y,z\in\mathbb{C}$ with $\alpha := \frac{1}{2}$ and $\beta := \frac{1}{4}$. We also see that

$$\widetilde{\varphi}(x,y,z) := \sum_{k=0}^{\infty} \frac{1}{4^k}\varphi(2^k x, 2^k y, 2^k z) = 2\varphi(x,y,z).$$

Hence, φ satisfies the assumption (a) of Theorem SJPL2. Obviously that the Drygas mappings $\mathscr{D}_1,\mathscr{D}_2:\mathbb{C}\to\mathbb{C}$, defined by $\mathscr{D}_1(x) := x$ and $\mathscr{D}_2(x) := 2x$ for all $x\in\mathbb{C}$, satisfy

$$|\mathscr{D}_i(x)-f(x)-f(-x)| \leqslant 4|x| = \frac{(4|x|)+(4|x|)}{4(1-\frac{1}{2})} = \frac{\widetilde{\varphi}(x,0,x)+\widetilde{\varphi}(-x,0,x)}{4(1-|\alpha|)}$$

for all $x\in\mathbb{C}$.

According to Theorem SJPL2(a) and Examples 3.6, 3.7, we present the following result.

PROPOSITION 3.8. *Suppose that X is a vector space, Y is a complex Banach space, and $\alpha,\beta\in\mathbb{C}$ satisfy $4|\alpha|+|\beta|<4$. Suppose that $f:X\to Y$ and $\varphi:X^3\to[0,\infty)$ satisfy (1.5) for all $x,y,z\in X$. If φ satisfies $\varphi(0,0,0)=0$ and the assumption (a) of*

Theorem SJPL2, then the zero mapping $\mathbf{0}: X \to Y$ (that is, $\mathbf{0}(x) := 0$ for all $x \in X$) is the unique quadratic mapping such that

$$\|f(x)+f(-x)\| = \|\mathbf{0}(x)-f(x)-f(-x)\| \leqslant \frac{\widetilde{\varphi}(x,0,x)+\widetilde{\varphi}(-x,0,x)}{4(1-|\alpha|)}$$

for all $x \in X$.

Proof. For each $x \in X$, we first define a sequence $(Q_n(x))_{n=0}^{\infty}$ by

$$Q_n(x) := \frac{1}{4^n}(f(2^n x)+f(-2^n x)) \quad \text{for all } n \in \mathbb{N}_0.$$

Sun *et al.* in [21] proved that such a sequence converges and hence we can define $\mathscr{Q}: X \to Y$ by

$$\mathscr{Q}(x) := \lim_{n\to\infty} Q_n(x) = \lim_{n\to\infty} \frac{1}{4^n}(f(2^n x)+f(-2^n x)).$$

Moreover, they also proved that $\mathscr{Q}$ is a Drygas mapping that satisfies

$$\|\mathscr{Q}(x)-f(x)-f(-x)\| \leqslant \frac{\widetilde{\varphi}(x,0,x)+\widetilde{\varphi}(-x,0,x)}{4(1-|\alpha|)}.$$

Next, we show that $\mathscr{Q} \equiv \mathbf{0}$. To prove this, let $x,y \in X$ and $n \in \mathbb{N}$ be given. It follows from Proposition 3.1(2) that there exists a real number K which fulfills the following:

$$\begin{aligned}
&\|Q_n(x+y)+Q_n(x-y)-2Q_n(x)\| \\
&\leqslant \frac{1}{4^n}\|f(2^n x+2^n y)+f(2^n x-2^n y)-2f(2^n x)\| \\
&\quad + \frac{1}{4^n}\|f(-2^n x-2^n y)+f(-2^n x+2^n y)-2f(-2^n x)\| \\
&\leqslant \frac{K}{4^n}\left(\varphi(2^n x,0,2^n y)+\varphi(2^n y,-2^n y,-2^n y)+\varphi(-2^n x,0,-2^n y)+\varphi(-2^n y,2^n y,2^n y)\right).
\end{aligned}$$

It follows from the condition (a) of Theorem SJPL2 that

$$\lim_{n\to\infty} \frac{\varphi(\pm 2^n x,0,\pm 2^n y)}{4^n} = 0 = \lim_{n\to\infty} \frac{\varphi(\pm 2^n y,\mp 2^n y,\mp 2^n y)}{4^n}.$$

One gets that $\mathscr{Q}(x+y)+\mathscr{Q}(x-y) = 2\mathscr{Q}(x)$. This means that $\mathscr{Q}$ is additive (see the proof of Corollary 3.4). Since $\mathscr{Q}$ is even and additive, we can conclude that $\mathscr{Q} \equiv \mathbf{0}$.

Finally, we prove the uniqueness part. Suppose that there exists a quadratic mapping $\mathscr{Q}': X \to Y$ such that

$$\|\mathscr{Q}'(x)-f(x)-f(-x)\| \leqslant \frac{\widetilde{\varphi}(x,0,x)+\widetilde{\varphi}(-x,0,x)}{4(1-|\alpha|)} \quad \text{for all } x \in X. \tag{3.7}$$

Let $x \in X$ be given. By using (3.7), we see that

$$\begin{aligned}\|\mathscr{Q}'(x) - \mathbf{0}(x)\| &= \lim_{k\to\infty} \left\| \mathscr{Q}'(x) - Q_k(x) \right\| \\ &= \lim_{k\to\infty} \left\| \frac{\mathscr{Q}'(2^k x)}{4^k} - \frac{f(2^k x)}{4^k} - \frac{f(-2^k x)}{4^k} \right\| \\ &\leqslant \frac{1}{4(1-|\alpha|)} \lim_{k\to\infty} \frac{\widetilde{\varphi}(2^k x, 0, 2^k x) + \widetilde{\varphi}(-2^k x, 0, 2^k x)}{4^k} \\ &= \frac{1}{4(1-|\alpha|)} \lim_{k\to\infty} \left(\sum_{n=k}^{\infty} \frac{\varphi(2^n x, 0, 2^n x)}{4^n} + \sum_{n=k}^{\infty} \frac{\varphi(-2^n x, 0, 2^n x)}{4^n} \right) = 0.\end{aligned}$$

Hence, $\mathscr{Q}'(x) = \mathbf{0}(x) = 0$ and the proof is complete. $\square$

REMARK 3.9. Proposition 3.1(1) tells us that the even part of f is also approximately zero as Proposition 3.8. In fact, the mapping f satisfies

$$\|f(x) + f(-x)\| \leqslant \frac{1}{2(1-|\alpha|)} \min\{\varphi(x,-x,-x), \varphi(-x,x,x)\} \quad \text{for all } x \in X.$$

The inequality above provides another stability of the even part of functions satisfying (1.5). Comparing the result of Proposition 3.8 to Proposition 3.1, we see that the completeness of Y and the assumption (a) of Theorem SJPL2 are not necessary.

By using Proposition 3.1(2) and Theorem F, we improve Theorem SJPL2(a) as follows.

THEOREM 3.10. *Suppose that X is a vector space, Y is a complex Banach space, and $\alpha, \beta \in \mathbb{C}$ satisfy $4|\alpha| + |\beta| < 4$. Suppose that $f : X \to Y$ and $\varphi : X^3 \to [0,\infty)$ satisfy (1.5) for all $x, y, z \in X$. If $\varphi(0,0,0) = 0$ and the following two conditions are satisfied:*

(1) $\Phi_1(x) := \sum_{n=0}^{\infty} \frac{1}{2^n} \varphi(2^n x, 0, 2^n x) < \infty$ *and* $\Phi_2(x) := \sum_{n=0}^{\infty} \frac{1}{2^k} \varphi(2^n x, -2^n x, -2^n x) < \infty$ *for all $x \in X$;*

(2) $\lim_{n\to\infty} \frac{1}{2^n} \varphi(2^n x, 2^n y, 2^n z) = 0$ *for all $x, y, z \in X$,*

then there exists a unique additive mapping $\mathscr{A} : X \to Y$ such that

$$\|\mathscr{A}(x) - f(x)\| \leqslant \frac{\Phi_1(x)}{2(1-|\alpha|)} + \frac{1+|\alpha|}{4(1-|\alpha|)^2} \Phi_2(x) \quad \textit{for all } x \in X. \tag{3.8}$$

Proof. Proposition 3.1(2) shows that f satisfies

$$\|f(x+y) + f(x-y) - 2f(x)\| \leqslant \frac{\varphi(x,0,y)}{1-|\alpha|} + \frac{1+|\alpha|}{2(1-|\alpha|)^2} \varphi(y,-y,-y) \tag{3.9}$$

for all $x,y\in X$. We define $\delta : X\to[0,\infty)$ by

$$\delta(x) := \frac{\varphi(x,0,x)}{1-|\alpha|}+\frac{1+|\alpha|}{2(1-|\alpha|)^2}\varphi(x,-x,-x)\quad\text{for all } x\in X.$$

By letting $x=y$ in (3.9), we have that

$$\|f(2x)-2f(x)\|\leqslant\delta(x)\quad\text{for all } x\in X,\tag{3.10}$$

or equivalently,

$$\left\|\frac{1}{2}f(2x)-f(x)\right\|\leqslant\frac{1}{2}\delta(x)\quad\text{for all } x\in X.$$

We define $g:Y\to Y$, $h:X\to X$, and $\phi:[0,\infty)\to[0,\infty)$ by $g(u):=u/2$ for all $u\in Y$, $h(x):=2x$ for all $x\in X$, and $\phi(t):=t/2$ for all $t\in[0,\infty)$, respectively. It follows that

$$\|(g\circ f\circ h)(x)-f(x)\|=\left\|\frac{1}{2}f(2x)-f(x)\right\|\leqslant\frac{1}{2}\delta(x)\quad\text{for all } x\in X;$$

$$\|g(u)-g(v)\|=\frac{1}{2}\|u-v\|=\phi(\|u-v\|)\quad\text{for all } u,v\in Y.$$

Obviously, ϕ is non-decreasing subadditive and g is continuous. Note that

$$\sum_{n=0}^{\infty}\phi^n\left(\delta(h^n(x))\right)=\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{2^n}\delta(2^nx)=\frac{\Phi_1(x)}{2(1-|\alpha|)}+\frac{1+|\alpha|}{4(1-|\alpha|)^2}\Phi_2(x)<\infty$$

for all $x\in X$. Theorem F asserts that the mapping $\mathscr{A}:=\lim_{n\to\infty}g^n\circ f\circ h^n$ exists and is the unique mapping such that $\mathscr{A}(2x)=2\mathscr{A}(x)$ for all $x\in X$ and (3.8) holds. Note that $\mathscr{A}(x)=\lim_{n\to\infty}\frac{1}{2^n}f(2^nx)$ for all $x\in X$. Next, we prove that $\mathscr{A}$ is additive. To see this, let $x,y\in X$ be given. We see from (3.9) and Condition (2) that

$$\begin{aligned}&\|\mathscr{A}(x+y)+\mathscr{A}(x-y)-2\mathscr{A}(x)\|\\&=\lim_{n\to\infty}\frac{1}{2^n}\|f(2^nx+2^ny)+f(2^nx-2^ny)-2f(2^nx)\|\\&\leqslant\lim_{n\to\infty}\frac{1}{2^n}\left(\frac{\varphi(2^nx,0,2^ny)}{1-|\alpha|}+\frac{1+|\alpha|}{2(1-|\alpha|)^2}\varphi(2^ny,-2^ny,-2^ny)\right)=0.\end{aligned}$$

Hence, $\mathscr{A}$ is additive as desired. Moreover, the uniqueness is obvious. □

3.2. Some remarks on Theorem SJPL2(b)

Following the same proof of Proposition 3.8, we obtain the following proposition.

PROPOSITION 3.11. *Suppose that X is a vector space, Y is a complex Banach space, and $\alpha,\beta\in\mathbb{C}$ satisfy $4|\alpha|+|\beta|<4$. Suppose that $f:X\to Y$ and $\varphi:X^3\to[0,\infty)$ satisfy (1.5) for all $x,y,z\in X$. If φ satisfies the assumption (b) of Theorem SJPL2, then the zero mapping $\mathbf{0}:X\to Y$ is the unique quadratic mapping such that*

$$\|f(x)+f(-x)\|=\|\mathbf{0}(x)-f(x)-f(-x)\|\leqslant\frac{\widetilde{\varphi}(x,0,x)+\widetilde{\varphi}(-x,0,x)}{4(1-|\alpha|)}$$

for all $x\in X$.

To improve Theorem SJPL2(b), we present the following stability result which is a consequence of Theorem F. Since the proof follows similarly to that of Theorem 3.10, we omit the proof.

THEOREM 3.12. *Suppose that X is a vector space, Y is a complex Banach space, and $\alpha, \beta \in \mathbb{C}$ satisfy $4|\alpha| + |\beta| < 4$. Suppose that $f : X \to Y$ and $\varphi : X^3 \to [0, \infty)$ satisfy (1.5) for all $x, y, z \in X$. If the following two conditions are satisfied:*

(1) $\Phi_1(x) := \sum_{n=1}^{\infty} 2^k \varphi\left(\frac{x}{2^n}, 0, \frac{x}{2^n}\right) < \infty$ *and* $\Phi_2(x) := \sum_{n=1}^{\infty} 2^k \varphi\left(\frac{x}{2^n}, \frac{-x}{2^n}, \frac{-x}{2^n}\right) < \infty$ *for all* $x \in X$*;*

(2) $\lim_{n\to\infty} 2^k \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) = 0$ *for all* $x, y, z \in X$*,*

then there exists a unique additive mapping $\mathscr{A} : X \to Y$ such that

$$\|\mathscr{A}(x) - f(x)\| \leqslant \frac{\Phi_1(x)}{2(1-|\alpha|)} + \frac{1+|\alpha|}{4(1-|\alpha|)^2}\Phi_2(x) \quad \text{for all } x \in X.$$

4. Final remark

According to Theorems 2.3, 3.10, and 3.12, the following two inequalities:

$$\begin{aligned} &\|f(x+y+z) + f(x+y-z) - 2f(x) - 2f(y) - f(z) - f(-z)\| \\ &\leqslant \|\alpha(f(x+y+z) - f(x) - f(y) - f(z))\| \\ &\quad + \|\beta(f(x+y-z) - f(x) - f(y) - f(-z))\| + \varphi(x,y,z) \end{aligned}$$

and

$$\begin{aligned} &\|f(x+y+z) + f(x+y-z) - 2f(x) - 2f(y) - f(z) - f(-z)\| \\ &\leqslant \|\alpha(f(x+y-z) + f(x-y+z) - 2f(x) - f(y) - f(-y) - f(z) - f(-z))\| \\ &\quad + \|\beta(f(x+y+z) - f(x+z) - f(y))\| + \varphi(x,y,z) \end{aligned}$$

are stable with respect to *additive mappings*. In particular, the name "*3-variable double ρ-functional inequalities of Drygas*" of the preceding two inequalities is not appropriate.

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