Journal of Mathematical Inequalities
Volume 17, Number 2 (2023), 739–759 doi:10.7153/jmi-2023-17-48

# ON SINGULAR INTEGRALS AND MAXIMAL OPERATORS ALONG SURFACES OF REVOLUTION ON PRODUCT DOMAINS

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(*Communicated by L. Liu*)

*Abstract.* We study the mapping properties of singular integral operators along surfaces of revolutions on product domains. For several classes of surfaces, we prove sharp $L^p$ bounds $(1 < p < \infty)$ for these singular integral operators as well as their corresponding maximal operators. By using these $L^p$ bounds and an extrapolation argument we obtain the $L^p$ boundedness of these operators under optimal conditions on the singular kernels. Our results extend and improve several results previously obtained by many authors.

## 1. Introduction

Let $\mathbf{R}^d$ $(d = n \text{ or } d = m)$, $d \geqslant 2$ be the $d$-dimensional Euclidean space and $\mathbf{S}^{d-1}$ be the unit sphere in $\mathbf{R}^d$ equipped with the normalized Lebesgue measure $d\sigma$. Also, we let $\xi'$ denote $\xi/|\xi|$ for $\xi \in \mathbf{R}^n \setminus \{0\}$ and $p'$ denote the exponent conjugate to $p$, that is $1/p + 1/p' = 1$.

Let $h(\cdot,\cdot)$ be a measurable function on $\mathbf{R}^+ \times \mathbf{R}^+$ and let

$$K_{\Omega,h}(x,y) = \frac{\Omega(x',y')}{|x|^n |y|^m} h(|x|,|y|) \tag{1.1}$$

where $\Omega$ is a homogeneous function of degree zero on $\mathbf{R}^n \times \mathbf{R}^m$ and satisfies

$$\int_{\mathbf{S}^{n-1}} \Omega(u,\cdot)\, d\sigma(u) = \int_{\mathbf{S}^{m-1}} \Omega(\cdot,v)\, d\sigma(v) = 0. \tag{1.2}$$

For a measurable real-valued function $h$ on $\mathbf{R}^+ \times \mathbf{R}^+$, we say that $h \in \Delta_\gamma(\mathbf{R}^+ \times \mathbf{R}^+)$, $\gamma > 1$, if

$$\|h\|_{\Delta_\gamma} = \sup_{R_1,R_2>0} \{R_2^{-1} R_1^{-1} \int_{R_2}^{2R_2} \int_{R_1}^{2R_1} |h(t,s)|^\gamma dt ds\}^{\frac{1}{\gamma}} < \infty.$$

Let $\Phi(s,t)$ be a real-valued function on $\mathbf{R}^+ \times \mathbf{R}^+$. For $(x,y) \in \mathbf{R^n} \times \mathbf{R^m}$ and $z \in \mathbf{R}$, let $T_{\Phi,h}$ be the singular integral operator along the surface $\Gamma_\Phi(x,y) = (x,y,\Phi(|x|,|y|))$

$$T_{\Phi,h} f(x,y,z) = \text{p.v.} \int_{\mathbf{R}^n \times \mathbf{R}^m} f(x-u, y-v, z-\Phi(|u|,|v|))\, K_{\Omega,h}(u,v)\, du dv. \tag{1.3}$$

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*Mathematics subject classification* (2020): Primary 42B20; Secondary 42B15, 42B25.

*Keywords and phrases*: Singular integrals, product domains, rough kernels, block spaces, extrapolation.

Also, let $\mathscr{M}_{\Phi,h}$ be the related maximal operator defined initially defined for $f \in C_0^\infty(\mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R})$ by

$$\begin{aligned} &\mathscr{M}_{\Phi h} f(x,y,z) \\ &= \sup_{r_1,r_2>0} \frac{1}{r_1^n r_2^m} \int_{|v|\leqslant r_2} \int_{|u|\leqslant r_1} |f(x-u,y-v,z-\Phi(|u|,|v|))| \left|\Omega\left(u',v'\right)\right| |h(|u|,|v|)|\, du dv. \end{aligned} \tag{1.4}$$

If $\Phi \equiv 0$, we shall let $T_h = T_{0,h}$ and $\mathscr{M}_h = \mathscr{M}_{0,h}$.

The study of the $L^p$ $(1<p<\infty)$ boundedness of $T_h$ and $\mathscr{M}_h$ and their extensions under various conditions on $\Omega$ and $h$ has attracted the attention of many authors (see for example, [6], [9], [17], [18], [20], [21], [22]). In the one parameter case, the study of the $L^p$ boundedness of such kind of operators $T_{\Phi,h}$ and $\mathscr{M}_{\Phi,h}$ was initiated in [25] and continued by many authors. For relevant results one may consult [7], [10], [24], among others.

In [25], the authors proved that the $L^p$ boundedness of singular integrals along certain surfaces of revolution still holds even if the surfaces make an infinite order of contact with their tangent planes at $(0,0)$ (i.e. flat). The result can be described as follows:

THEOREM A. *Let $\phi$ be a $C^2([0,\infty))$, convex and increasing function satisfying $\phi(0)=0$. Let $\Omega \in C^\infty(\mathbf{S}^{n-1})$ and $\mathbf{S}_\phi f$ be given by*

$$\mathbf{S}_\phi f(x,x_{n+1}) = p.v. \int_{\mathbf{R}^n} f(x-y, x_{n+1}-\phi(|y|)) \frac{\Omega(y')}{|y|^n} dy.$$

*Then for $1<p<\infty$, there exists a positive constant $C_p$ such that*

$$\left\|\mathbf{S}_\phi f\right\|_{L^p(\mathbf{R}^{n+1})} \leqslant C_p \|f\|_{L^p(\mathbf{R}^{n+1})}$$

*for all $f \in L^p(\mathbf{R}^{n+1})$.*

This result was improved in several papers (see [7] and [10], among others). An analogue of Theorem A in the product space setting was obtained in [1], which can be described as follows.

THEOREM B. *Let $\phi,\psi$ be $C^2([0,\infty))$, convex and increasing functions satisfying $\phi(0)=\psi(0)=0$. Let $\Omega \in B_q^{(0,1)}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ for some $q>1$, and $h \in \Delta_\gamma(\mathbf{R}^+\times\mathbf{R}^+)$ for some $\gamma>1$ and $\mathbf{S}_{\phi,\psi} f$ be given by*

$$\mathbf{S}_{\phi,\psi} f(\bar{x},\bar{y}) = p.v. \int_{\mathbf{R}^m}\int_{\mathbf{R}^n} f(\bar{x}-\tilde{\Phi}(u), \bar{y}-\tilde{\Psi}(v)) K_{\Omega,h}(u,v) du dv$$

*where $\tilde{\Phi}(x)=(x,\phi(|x|)), \tilde{\Psi}(y)=(y,\psi(|y|))$, $\bar{x}=(x,x_{n+1}) \in \mathbf{R}^n\times\mathbf{R}$ and $\bar{y}=(y,y_{m+1}) \in \mathbf{R}^m\times\mathbf{R}$. Then for $1<p<\infty$, there exists a positive constant $C_p$ such that*

$$\left\|\mathbf{S}_{\phi,\psi} f\right\|_{L^p(\mathbf{R}^{n+1}\times\mathbf{R}^{m+1})} \leqslant C_p \|f\|_{L^p(\mathbf{R}^{n+1}\times\mathbf{R}^{m+1})}$$

*for all* $f \in L^p(\mathbf{R}^{n+1} \times \mathbf{R}^{m+1})$.

The study of the double Hilbert transforms along surfaces has attracted the attention of many authors. See for example [11], [12], [13], [14], [15], [17], [26], [27]. In this paper, we are very much motivated by the work of authors in [11], [15], among others who studied double Hilbert transforms along surfaces of the form $(t,s,\phi(t,s))$.

Our main focus in this paper is to investigate the $L^p$ boundedness of $T_{\Phi,h}$ and $\mathscr{M}_{\Phi,h}$ for several classes of functions $\Phi(s,t)$ and under very weak conditions on $\Omega$ and $h$. We notice that our surfaces are natural extensions of the surfaces of revolutions considered by many authors in the one parameter setting.

Our principal results in this paper are the following:

THEOREM 1.1. *Let* $\Phi \in C^1([0,\infty) \times [0,\infty))$. *Suppose that* $\Omega \in L^q(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ *for some* $1 < q \leqslant 2$ *and* $h \in \Delta_\gamma(\mathbf{R}^+ \times \mathbf{R}^+)$ *for some* $1 < \gamma \leqslant \infty$. *Then*

$$\left\|T_{\Phi,h}(f)\right\|_{L^2(\mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R})} \leqslant C_p (q-1)^{-2} \|\Omega\|_{L^q(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})} \|f\|_{L^2(\mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R})} \tag{1.5}$$

*for every* $f \in L^2(\mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R})$.

THEOREM 1.2. *Suppose that* $\Omega \in L^q(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ *for some* $1 < q \leqslant 2$ *and* $h \in \Delta_\gamma(\mathbf{R}^+ \times \mathbf{R}^+)$ *for some* $1 < \gamma \leqslant \infty$. *Assume that* $\Phi \in C^1([0,\infty) \times [0,\infty))$ *such that for every fixed* $t$ *and* $s$, $\Gamma_t^1(\cdot) = \Phi(t,\cdot), \Gamma_s^2(\cdot) = \Phi(\cdot,s) \in C^2[0,\infty)$ *are convex increasing functions with* $\Gamma_t^1(0) = \Gamma_s^2(0) = 0$. *Then*

*(i) for* $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$, *there exists a positive constant* $C_p$ *such that*

$$\left\|T_{\Phi,h} f\right\|_{L^p(\mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R})} \leqslant C_p (q-1)^{-2} \|\Omega\|_{L^q(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})} \|f\|_{L^p(\mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R})}, \tag{1.6}$$

*(ii) for every* $\gamma' < p \leqslant \infty$, *there exists a positive constant* $C_p$ *such that*

$$\left\|\mathscr{M}_{\Phi,h}(f)\right\|_{L^p(\mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R})} \leqslant C_p (q-1)^{-2} \|\Omega\|_{L^q(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})} \|f\|_{L^p(\mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R})} \tag{1.7}$$

*for all* $f \in L^p(\mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R})$. *The constant* $C_p$ *may depend on* $n, m$, *but is independent of the* $\Omega$ *and* $q$.

We notice that our theorem covers several types of natural surfaces. For example, our theorem allows surfaces of the type $\Gamma_\Phi$ with $\Phi(t,s) = s^2 t^2 (e^{-1/s} + e^{-1/t})$, $(s,t > 0)$. This surface has a contact of infinite order at the origin which was studied by Duoandikoetxea in [17]. Also we notice that the interesting special case of $\Gamma_\Phi$ with $\Phi(t,s) = \phi_1(t)\phi_2(s)$, where each $\phi_i \in C^2[0,\infty)$ is a convex increasing function with $\phi_i(0) = 0$. This surface was considered in [15] in studying double Hilbert transforms along surfaces of the form $(t,s,\phi(t)\psi(s))$. A nice example of this surface is $(t,s,e^{-1/s}e^{-1/t})$.

THEOREM 1.3. *Suppose that* $\Omega \in L^q(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ *for some* $q \in (1,2]$ *and* $h \in \Delta_\gamma(\mathbf{R}_+ \times \mathbf{R}^+)$ *for some* $1 < \gamma \leqslant \infty$. *Assume that* $\Phi(t,s) = P(t,s) = \sum_{l=0}^{d_1} \sum_{i=0}^{d_2} a_{i,l} t^{\alpha_i} s^{\beta_l}$ *with* $\alpha_i, \beta_l > 0$ *is a generalized polynomial on* $\mathbf{R}^2$. *Then* *(i) for* $|1/p-1/2|<\min\{1/2,1/\gamma'\}$, *there exists a positive constant* $C_p$ *such that*

$$\left\|T_{\Phi,h}f\right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R})}\leqslant C_p(q-1)^{-2}\|\Omega\|_{L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}\|f\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R})}, \tag{1.8}$$

*(ii) for every* $\gamma'<p\leqslant\infty$, *there exists a constant* $C_p$ *such that*

$$\left\|\mathscr{M}_{\Phi,h}(f)\right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R})}\leqslant C_p(q-1)^{-2}\|\Omega\|_{L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}\|f\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R})} \tag{1.9}$$

*for all* $f\in L^p(\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R})$.

*The constant* $C_p$ *may depend on* $n,m$, *but is independent of the* $\Omega$ *and* $q$ *and the coefficients of P.*

We remark that Theorem 1.3 allows very important special classes of surfaces. If we take $\Phi(t,s)=t^{\alpha}s^{\beta}$ with $\alpha,\beta>0$, then the corresponding surface was considered by many authors in their studying double Hilbert transforms and singular integrals on product domains. See for example, [13], [14], [17], [18], [23]. Also, as a special case of $\Phi$ is $\Phi(t,s)=P(s,t)$ is a polynomial where the study of Double Hilbert transforms along the surface $(t,s,P(t,s))$ has attracted the attention of many authors. See for example [11], [27], among others.

THEOREM 1.4. *Suppose that* $\Omega\in L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ *for some* $q\in(1,2]$ *and* $h\in\Delta_\gamma(\mathbf{R}_+\times\mathbf{R}_+)$ *for some* $1<\gamma\leqslant\infty$. *Assume that* $\Phi(t,s)=\phi(t)P(s)$, *where* $\phi\in C^2[0,\infty)$ *is a convex increasing function with* $\phi(0)=0$ *and P is generalized polynomial on* $\mathbf{R}$. *Then*

*(i) for* $|1/p-1/2|<\min\{1/2,1/\gamma'\}$, *there exists a positive constant* $C_p$ *such that*

$$\left\|T_{\Phi,h}f\right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R})}\leqslant C_p(q-1)^{-2}\|\Omega\|_{L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}\|f\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R})}, \tag{1.10}$$

*(ii) for every* $\gamma'<p\leqslant\infty$, *there exists a constant* $C_p$ *such that*

$$\left\|\mathscr{M}_{\Phi,h}(f)\right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R})}\leqslant C_p(q-1)^{-2}\|\Omega\|_{L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}\|f\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R})} \tag{1.11}$$

*for all* $f\in L^p(\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R})$. *The constant* $C_p$ *may depend on* $n,m$, *but is independent of the* $\Omega,\gamma$ *and* $q$ *and the coefficients of P.*

THEOREM 1.5. *Suppose that* $\Omega\in L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ *for some* $q\in(1,2]$ *and* $h\in\Delta_\gamma(\mathbf{R}_+\times\mathbf{R}_+)$ *for some* $1<\gamma\leqslant\infty$. *Assume that* $\Phi(t,s)=\phi_1(t)+\phi_2(s)$, *where each* $\phi_l$ $(l=1,2)$ *is either a generalized polynomial or is in* $C^2[0,\infty)$, *a convex increasing function with* $\phi_l(0)=0$. *Then*

*(i) for* $|1/p-1/2|<\min\{1/2,1/\gamma'\}$, *there exists a positive constant* $C_p$ *such that*

$$\left\|T_{\Phi,h}f\right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R})}\leqslant C_p(q-1)^{-2}\|\Omega\|_{L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}\|f\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R})}, \tag{1.12}$$

*(ii) for every* $\gamma'<p\leqslant\infty$, *there exists a constant* $C_p$ *such that*

$$\left\|\mathscr{M}_{\Phi,h}(f)\right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R})}\leqslant C_p(q-1)^{-2}\|\Omega\|_{L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}\|f\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R})} \tag{1.13}$$

*for all* $f\in L^p(\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R})$. *The constant* $C_p$ *may depend on* $n,m$, *but is independent of the* $\Omega$ *and* $q$.

By the conclusions in Theorems 1.2, 1.3, 1.4 and 1.5 and applying an extrapolation method as in [8], we get the following results:

THEOREM 1.6. *Let $\Phi$ and $h$ be given as in any of Theorem 1.2, 1.3, 1.4 or 1.5. Assume that $\Omega \in L(\log L)^2\left(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}\right)$ or $\Omega \in B_q^{(0,1)}\left(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}\right)$ for some $q>1$, then*

*(i) for $|1/p-1/2|<\min\{1/2,1/\gamma'\}$, there exists a constant $C_p>0$ such that*

$$\left\|T_{\Phi,h}f\right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R})} \leqslant C_p\|f\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R})}, \tag{1.14}$$

*(ii) for every $\gamma'<p\leqslant\infty$, there exists a constant $C_p$ such that*

$$\left\|\mathscr{M}_{\Phi,h}(f)\right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R})} \leqslant C_p\|f\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R})} \tag{1.15}$$

*for all $f\in L^p(\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R})$.*

We shall also establish the $L^p$ boundedness of the maximal truncated singular integral operator $T^*_{\Phi,h}$ given by

$$(T^*_{\Phi,h}f)(x,y,z) = \sup_{\varepsilon_1,\varepsilon_2>0}\left|\int_{|v|\geqslant\varepsilon_2}\int_{|u|\geqslant\varepsilon_1} f(x-u,y-v,z-\Phi(|u|,|v|))\,K_{\Omega,h}(u,v)\,dudv\right|, \tag{1.16}$$

where $\Phi$ is given as before.

By Theorem 1.6 and following a similar argument as in [6] we have the following result for $T^*_{\Phi,h}$.

THEOREM 1.7. *Suppose that $\Omega\in L(\log L)^2\left(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}\right)$ or $\Omega\in B_q^{0,1}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ for some $q>1$.*

*(i) If $\Phi\in C^1([0,\infty)\times[0,\infty))$ and $h\in\Delta_\gamma(\mathbf{R}^+\times\mathbf{R}^+)$ for some $\gamma>1$,*

$$\left\|T^*_{\Phi,h}(f)\right\|_{L^2(\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R})} \leqslant C\|f\|_{L^2(\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R})} \tag{1.17}$$

*for every $f\in L^2(\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R})$, and*

*(ii) if $h(t,s)=h_1(t)h_2(s)$ with $h_1,\ h_2\in L^\infty(\mathbf{R}^+)$ and $\Phi$ is given as in any of Theorem 1.2, 1.3, 1.4 or 1.5, then*

$$\left\|T^*_{\Phi,h}(f)\right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R})} \leqslant C_p\|f\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R})} \tag{1.18}$$

*holds for all $1<p<\infty$ and $f\in L^p(\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R})$.*

## 2. Some definitions and lemmas

We will begin by recalling some definitions. The class $L(\log L)^\alpha(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ (for $\alpha>0$) denotes the class of all measurable functions $\Omega$ on $\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}$ which satisfy

$$\|\Omega\|_{L(\log L)^\alpha(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})} = \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} |\Omega(x,y)|\log^\alpha(2+|\Omega(x,y)|)d\sigma(x)d\sigma(y)<\infty.$$

Now we define the class of $B_q^{(0,\upsilon-1)}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$. A $q$-block on $\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}$ is an $L^q$ $(1<q\leqslant\infty)$ function $b(x,y)$ that satisfies $b\subset I$ and $\|b\|_{L^q}\leqslant|I|^{-1/q'}$, where $|\cdot|$ denotes the product measure on $\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}$ and $I$ is an interval on $\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}$, i.e.,

$$I=\left\{x'\in\mathbf{S}^{n-1}:\left|x'-x_0'\right|<\alpha\right\}\times\left\{y'\in\mathbf{S}^{m-1}:\left|y'-y_0'\right|<\beta\right\}$$

for some $\alpha,\ \beta>0$, $x_0'\in\mathbf{S}^{n-1}$ and $y_0'\in\mathbf{S}^{m-1}$. The block space $B_q^{(0,\upsilon)}=B_q^{(0,\upsilon)}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ is defined by

$$B_q^{(0,\upsilon)}=\left\{\Omega\in L^1(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}):\ \Omega=\sum_{\mu=1}^{\infty}\lambda_\mu b_\mu,\ M_q^{(0,\upsilon)}\left(\{\lambda_\mu\}\right)<\infty\right\}$$

where each $\lambda_\mu$ is a complex number, each $b_\mu$ is a $q$-block supported on an interval $I_\mu$ on $\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}$, $\upsilon>-1$, and

$$M_q^{(0,\upsilon)}\left(\{\lambda_\mu\}\right)=\sum_{\mu=1}^{\infty}\left|\lambda_\mu\right|\left\{1+\log^{(\upsilon+1)}(\left|I_\mu\right|^{-1})\right\}.$$

Let $\|\Omega\|_{B_q^{(0,\upsilon)}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}=N_q^{(0,\upsilon)}(\Omega)=\inf\{M_q^{(0,\upsilon)}\left(\{\lambda_\mu\}\right):\ \Omega=\sum_{\mu=1}^{\infty}\lambda_\mu b_\mu$ and each $b_\mu$ is a $q$-block function supported on a cap $I_\mu$ on $\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}\}$.

REMARK. For any $q>1$ and $0<\upsilon\leqslant1$, the following inclusions hold and are proper:

$$L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})\subset L(\log L)^{\alpha}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})\subset L^1(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})\text{ for }\alpha>0,$$

$$\bigcup_{r>1}L^r(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})\subset B_q^{(0,\upsilon)}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})\text{ for any }-1<\upsilon\text{ and }q>1,$$

$$L(\log L)^{\beta}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})\subset L(\log L)^{\alpha}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})\text{ if }0<\alpha<\beta.$$

The question with regard to the relationship between $B_q^{(0,\upsilon-1)}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ and $L(\log^+L)^{\upsilon}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ (for $\upsilon>0$) remains open.

We shall need the following two lemmas from [6] which are extensions of the corresponding results of Duoandikoetxea in [17].

LEMMA 2.1. *Let $\{\mu_{k,j}\}$ be a sequence of Borel measures on $\mathbf{R}^n\times\mathbf{R}^m$. Suppose that for some $q>1$ and $B>0$,*

$$\|\mu^*(f)\|_{L^q(\mathbf{R}^n\times\mathbf{R}^m)}\leqslant B\|f\|_{L^q(\mathbf{R}^n\times\mathbf{R}^m)}$$

*holds for every $f$ in $L^q(\mathbf{R}^n\times\mathbf{R}^m)$. Then the following vector-valued inequality*

$$\begin{aligned}&\left\|\left(\sum_{k,j\in\mathbf{Z}}\left|\mu_{k,j}*g_{k,j}\right|^2\right)^{1/2}\right\|_{L^{p_0}(\mathbf{R}^n\times\mathbf{R}^m)}\\&\leqslant\left(B\sup_{k,j\in\mathbf{Z}}\left\|\mu_{k,j}\right\|\right)^{1/2}\left\|\left(\sum_{k,j\in\mathbf{Z}}\left|g_{k,j}\right|^2\right)^{1/2}\right\|_{L^{p_0}(\mathbf{R}^n\times\mathbf{R}^m)}\end{aligned}$$

*holds for* $|1/p_0 - 1/2| = 1/(2q)$ *and for arbitrary functions* $\{g_{k,j}\}$ *on* $\mathbf{R}^n \times \mathbf{R}^m$.

LEMMA 2.2. *Let* $L : \mathbf{R}^n \longrightarrow \mathbf{R}^{j_1}$ *and* $Q : \mathbf{R}^m \longrightarrow \mathbf{R}^{j_2}$ *be linear transformations. Let* $\{\mho_{k,j}\colon k,\ j \in \mathbf{Z}\}$ *be a sequence of Borel measures on* $\mathbf{R}^n \times \mathbf{R}^m$. *Suppose that for some* $a \geqslant 2,\ b \geqslant 2,\ \alpha,\ \beta,\ C > 0,\ B > 1$ *and* $p_o \in (2,\infty)$ *the following hold for* $k$, $j \in \mathbf{Z}$, $(\xi,\eta) \in \mathbf{R}^n \times \mathbf{R}^m$ *and arbitrary functions* $\{g_{k,j}\}$ *on* $\mathbf{R}^n \times \mathbf{R}^m$*:*

*(i)* $\left|\hat{\mho}_{k,j}(\xi,\eta)\right| \leqslant CB^2 \left(a^{kB}|L(\xi)|\right)^{\pm\frac{\alpha}{B}} \left(b^{jB}|Q(\eta)|\right)^{\pm\frac{\beta}{B}}$,

*(ii)* $\left\|\left(\sum_{k,j\in\mathbf{Z}} \left|\mho_{k,j} * g_{k,j}\right|^2\right)^{1/2}\right\|_{L^{p_0}(\mathbf{R}^n\times\mathbf{R}^m)} \leqslant CB^2 \left\|\left(\sum_{k,j\in\mathbf{Z}} \left|g_{k,j}\right|^2\right)^{1/2}\right\|_{L^{p_0}(\mathbf{R}^n\times\mathbf{R}^m)}$.

*Then for* $p_0' < p < p_0$ *there exists a positive constant* $C_p$ *such that*

$$\left\|\sum_{k,j\in\mathbf{Z}} \mho_{k,j} * f\right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)} \leqslant C_p\, B^2\, \|f\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)}$$

*and*

$$\left\|\left(\sum_{k,j\in\mathbf{Z}} \left|\mho_{k,j} * f\right|^2\right)^{1/2}\right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)} \leqslant C_p\, B^2\, \|f\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)}$$

*hold for all* $f$ *in* $L^p(\mathbf{R}^n \times \mathbf{R}^m)$. *The constant* $C_p$ *is independent of* $B$ *and the linear transformations* $L$ *and* $Q$.

Let $\theta \geqslant 2$. For a suitable function $\Omega(\cdot,\cdot)$ on $\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$ we define the measures $\{\lambda_{k,j,\theta,\Phi} : k, j \in \mathbf{Z}\}$ and the corresponding maximal operator $\lambda^*_{\Phi,\theta}$ on $\mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}$ by

$$\int_{\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R}} f\, d\lambda_{k,j,\theta,\Phi} = \int_{D_{k,j,\theta}} f\left(u,v,\Phi(|u|,|v|)\right) K_{\Omega,h}(u,v)\,du\,dv, \tag{2.1}$$

and

$$\lambda^*_{\Phi,\theta} f(x,y) = \sup_{k,j\in\mathbf{Z}} \left|\left|\lambda_{k,j,\theta,\Phi}\right| * f(x,y)\right| \tag{2.2}$$

where $D_{k,j,\theta} = \left\{(u,v) \in \mathbf{R}^n \times \mathbf{R}^m : \theta^k \leqslant |u| < \theta^{k+1}, \theta^j \leqslant |v| < \theta^{j+1}\right\}$ and $\Phi(t,s)$ is an arbitrary function on $\mathbf{R} \times \mathbf{R}$. Let $t^{\pm\alpha} = \inf\left(t^{\alpha},\ t^{-\alpha}\right)$.

LEMMA 2.3. *Assume that* $\Phi \in C^1\left([0,\infty) \times C^1[0,\infty)\right)$ *and let* $h \in \Delta_\gamma\left(\mathbf{R}^+ \times \mathbf{R}^+\right)$ *for some* $\gamma$, $1 < \gamma \leqslant 2$. *Let* $\Omega \in L^q(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ *for some* $1 < q \leqslant 2$ *and satisfy (1.2). Then there exist a positive constant* $C$, $0 < \alpha < 1/q'$ *such that for all* $k, j \in \mathbf{Z}$, $(\xi,\eta,\mu) \in \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}$ *we have*

$$\left|\hat{\lambda}_{k,j,\theta,\Phi}(\xi,\eta,\mu)\right| \leqslant C(\log\theta)^2 \|\Omega\|_q \left|\theta^k\, \xi\right|^{\pm\frac{\alpha}{q'}} \left|\theta^j\, \eta\right|^{\pm\frac{\alpha}{q'}}. \tag{2.3}$$

*The constant* $C$ *is independent of* $k, j,\ \theta$ *and* $\Phi(\cdot,\cdot)$.

*Proof.* By using Hölder's inequality we get

$$\begin{aligned}
&\left|\hat{\lambda}_{k,j,\theta,\Phi}(\xi,\eta,\mu)\right| \\
&\leqslant \left(\int_{\theta^j}^{\theta^{j+1}}\int_{\theta^k}^{\theta^{k+1}} |h(t,s)|^{\gamma}\frac{dtds}{ts}\right)^{1/\gamma} \\
&\quad \times \int_{\mathbf{S}^{m-1}}\left(\int_{\theta^j}^{\theta^{j+1}}\int_{\theta^k}^{\theta^{k+1}}\left|\int_{\mathbf{S}^{n-1}} e^{-i(t\xi\cdot x+\mu\Phi(t,s))}\Omega(x,y)d\sigma(x)\right|^{\gamma'}\frac{dtds}{ts}\right)^{1/\gamma'} d\sigma(v).
\end{aligned}$$

Since

$$\begin{aligned}
&\int_{\theta^j}^{\theta^{j+1}}\int_{\theta^k}^{\theta^{k+1}} |h(t,s)|^{\gamma}\frac{dtds}{ts} \\
&\leqslant \sum_{s=0}^{(\log\theta)/(\log 2)}\sum_{l=0}^{(\log\theta)/(\log 2)}\int_{\theta^j 2^s}^{\theta^j 2^{s+1}}\int_{\theta^k 2^l}^{\theta^k 2^{l+1}} |h(t,s)|^{\gamma}\frac{dt}{t}\frac{ds}{s} \\
&\leqslant C(\log\theta)^2\,\|h\|_{\Delta_\gamma}^{\gamma}, \qquad (2.4)
\end{aligned}$$

and $\gamma' \geqslant 2,$ we obtain

$$\begin{aligned}
\left|\hat{\lambda}_{k,j,\theta,\Phi}(\xi,\eta,\mu)\right| &\leqslant C\log\theta)^{(1+1/\gamma)}\int_{\mathbf{S}^{m-1}}\|\Omega(\cdot,v)\|_{L^1(\mathbf{S}^{n-1})}^{(1-\frac{2}{\gamma'})} \\
&\quad \times\left(\int_{\theta^k}^{\theta^{k+1}}\left|\int_{\mathbf{S}^{n-1}} e^{-i(t\xi\cdot x+\mu\Phi(t,s))}\Omega(u,v)d\sigma(u)\right|^2\frac{dt}{t}\right)^{\frac{1}{\gamma'}} d\sigma(v).
\end{aligned}$$

We notice that

$$\left|H_{k,j,y}(t,s)\right|^2 = \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}}\Omega(x,y)\,\overline{\Omega(u,y)}e^{i\theta^k t(x-u)\cdot\xi}\, d\sigma(x)\,d\sigma(u)$$

and

$$\begin{aligned}
\left|\int_1^{\theta} e^{i\theta^k t\xi\cdot(x-u)}\,\frac{dt}{t}\right| &\leqslant C\min\left\{\log\theta, \left|\theta^k\xi\cdot(x-u)\right|^{-1}\right\} \\
&\leqslant C\,(\log\theta)\left|\theta^k\xi\right|^{-\alpha}\left|\xi'\cdot(x-u)\right|^{-\alpha},
\end{aligned}$$

where $\xi' = \xi/|\xi|$, and $0<\alpha<1$. By choosing $\alpha$ with $\alpha q' < 1$ we get

$$\begin{aligned}
&\left|\hat{\lambda}_{k,j,\theta,\Phi}(\xi,\eta,\mu)\right| \\
&\leqslant C(\log\theta)^2\,\|h\|_{\Delta_\gamma}\left|\theta^k\xi\right|^{-\frac{\alpha}{\gamma'}}\int_{\mathbf{S}^{m-1}}\|\Omega(\cdot,y)\|_{L^1(\mathbf{S}^{n-1})}^{(1-2/\gamma')} \\
&\quad\times\left(\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}}\Omega(x,y)\,\overline{\Omega(u,y)}\left|\xi'\cdot(x-u)\right|^{-\alpha}d\sigma(x)\,d\sigma(u)\right)^{\frac{1}{\gamma'}}.
\end{aligned}$$

Since

$$\left(\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}} |x_1-u_1|^{-\alpha q'}\,d\sigma(x)\,d\sigma(u)\right)^{\frac{1}{\gamma' q'}} < \infty,$$

by Hölder's inequality we get

$$\begin{aligned}&\left|\hat{\lambda}_{k,j,\theta,\Phi}(\xi,\eta,\mu)\right|\\ &\leqslant C(\log\theta)^2\,\|h\|_{\Delta_\gamma}\left|\theta^k\xi\right|^{-\frac{\alpha}{\gamma'}}\int_{\mathbf{S}^{m-1}}\|\Omega(\cdot,y)\|_{L^1(\mathbf{S}^{n-1})}^{(1-2/\gamma')}\|\Omega(\cdot,y)\|_{L^q(\mathbf{S}^{n-1})}^{2/\gamma'}\,d\sigma(y),\end{aligned}$$

which easily implies

$$\left|\hat{\lambda}_{k,j,\theta,\Phi}(\xi,\eta,\mu)\right| \leqslant C(\log\theta)^2\,\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\left|\theta^k\xi\right|^{-\alpha/\gamma'}.$$

By combining the last estimate with the trivial estimate

$$\left|\hat{\lambda}_{k,j,\theta,\Phi}(\xi,\eta,\mu)\right| \leqslant C(\log\theta)^2\,\|\Omega\|_{L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}, \tag{2.5}$$

we get

$$\left|\hat{\lambda}_{k,j,\theta,\Phi}(\xi,\eta,\mu)\right| \leqslant C(\log\theta)^2\,\|\Omega\|_{L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}\left|\theta^k\xi\right|^{-\frac{\alpha}{q'\gamma'}}. \tag{2.6}$$

Similarly, we have

$$\left|\hat{\lambda}_{k,j,\theta,\Phi}(\xi,\eta,\mu)\right| \leqslant C(\log\theta)^2\,\|\Omega\|_{L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}\left|\theta^j\eta\right|^{-\frac{\alpha}{q'\gamma'}}. \tag{2.7}$$

Now, by (1.2) we get that

$$\begin{aligned}&\left|\hat{\lambda}_{k,j,\theta,\Phi}(\xi,\eta,\mu)\right|\\ &\leqslant \int_{\mathbf{S}^{n-1}}\left(\int_1^\theta\int_1^\theta\left|\int_{\mathbf{S}^{m-1}}\Omega(x,y)e^{-i\theta^j s\eta\cdot y}d\sigma(y)\right|\right.\\ &\quad\left.\times\left|h(\theta^k t,\theta^j s)\right|\left|e^{-i\{\theta^k t\xi\cdot x+\mu\Phi(\theta^k t,\theta^j s)\}}-e^{-i\mu\Phi(\theta^k t,\theta^j s)}\right|\frac{dt}{t}\frac{ds}{s}\right)d\sigma(x).\end{aligned}$$

By the last inequality and Hölder's inequality we get

$$\begin{aligned}&\left|\hat{\lambda}_{k,j,\theta,\Phi}(\xi,\eta,\mu)\right|\\ &\leqslant C\left|\theta^k\xi\right|\left(\int_1^\theta\int_1^\theta\left|h(\theta^k t,\theta^j s)\right|^{\gamma'}\frac{dt}{t}\frac{ds}{s}\right)^{1/\gamma'}\\ &\quad\times\int_{\mathbf{S}^{n-1}}\left(\int_1^\theta\int_1^\theta\left|\int_{\mathbf{S}^{m-1}}\Omega(x,y)e^{-i\theta^j s\eta\cdot y}d\sigma(y)\right|^{\gamma'}\frac{dt}{t}\frac{ds}{s}\right)^{1/\gamma'}d\sigma(x)\end{aligned} \tag{2.8}$$

and hence by (2.4) we obtain

$$\left|\hat{\lambda}_{k,j,\theta,\Phi}(\xi,\eta,\mu)\right| \leqslant C(\log\theta)^2 \|\Omega\|_{L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})} \left|\theta^k\xi\right|. \tag{2.9}$$

By (2.5) and (2.9) we get

$$\left|\hat{\lambda}_{k,j,\theta,\Phi}(\xi,\eta,\mu)\right| \leqslant C(\log\theta)^2 \|\Omega\|_{L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})} \left|\theta^k\xi\right|^{\frac{\alpha}{2q'\gamma'}}. \tag{2.10}$$

Similarly we have

$$\left|\hat{\lambda}_{k,j,\theta,\Phi}(\xi,\eta,\mu)\right| \leqslant C(\log\theta)^2 \|\Omega\|_{L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})} \left|\theta^j\eta\right|^{\frac{\alpha}{2q'\gamma'}}. \tag{2.11}$$

By combining (2.5)–(2.7) and (2.10)–(2.11) we get

$$\left|\hat{\lambda}_{k,j,\theta,\Phi}(\xi,\eta,\mu)\right| \leqslant C(\log\theta)^2 \|\Omega\|_{L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})} \left|\theta^k\xi\right|^{\pm\frac{\alpha}{2q'\gamma'}} \left|\theta^j\eta\right|^{\pm\frac{\alpha}{2q'\gamma'}}. \tag{2.12}$$

The lemma is proved. □

LEMMA 2.4. *Let* $h \in \Delta_\gamma\left(\mathbf{R}^+\times\mathbf{R}^+\right)$ *for some* $1<\gamma\leqslant\infty$, $\Omega\in L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ *for some* $1<q\leqslant 2$ *and* $\theta=2^{q'}$. *Assume that* $\Phi\in C^1([0,\infty)\times[0,\infty))$ *such that for every fixed* $t$ *and* $s$, $\Gamma_t^1(\cdot)=\Phi(t,\cdot)$, $\Gamma_s^2(\cdot)=\Phi(\cdot,s)\in C^2[0,\infty)$ *are convex increasing functions with* $\Gamma_t^1(0)=\Gamma_s^2(0)=0$. *Then for* $\gamma'<p\leqslant\infty$ *and* $f\in L^p(\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R})$ *there exists a positive constant* $C_p$ *which is independent of* $\Omega$ *and* $h$ *such that*

$$\left\|\lambda^*_{\Phi,\theta}(f)\right\|_p \leqslant C_p(q-1)^{-2}\|\Omega\|_q\|f\|_p. \tag{2.13}$$

*Proof.* Without loss of generality, we may assume that $\Omega\geqslant 0$. We shall first prove the lemma for the special case $\Phi(t,s)=\phi(t)\psi(s)$, where $\phi,\psi\in C^2([0,\infty))$, and $\phi$ and $\psi$ are convex increasing functions with $\phi(0)=\psi(0)=0$. By Hölder's inequality and (2.4), there exists a positive constant $C$ such that

$$\lambda^*_{\Phi,\theta}(f) \leqslant C(\log\theta)^{2/\gamma}(\sigma^*_{\theta,\Phi}(|f|^{\gamma'}))^{1/\gamma'} \tag{2.14}$$

where

$$\int_{\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R}} f\,d\sigma_{k,j,\theta,\Phi} = \int_{D_{k,j,\theta}} f(u,v,\Phi(|u|,|v|))\frac{\Omega(u\prime,v\prime)}{|u|^n|v|^m}dudv$$

and

$$\sigma^*_{\Phi,\theta}(f) = \sup_{k,j\in\mathbf{Z}}\left|\left|\sigma_{k,j,\theta,\Phi}\right|*f\right|. \tag{2.15}$$

To prove (2.13), by (2.14) it suffices to prove that

$$\left\|\sigma^*_{\Phi,\theta}(f)\right\|_p \leqslant C_p(\log\theta)^2\|\Omega\|_q\|f\|_p \quad \text{for } 1<p\leqslant\infty. \tag{2.16}$$

By the arguments in the proof of Lemma 2.3 we obtain the following:

$$\left|\hat{\sigma}_{k,j,\theta,\Phi}(\xi,\eta,\mu)\right| \leqslant C(\log\theta)^2 \|\Omega\|_q \left|\theta^k\, \xi\right|^{-\frac{\alpha}{2q'\gamma'}} \left|\theta^j\, \eta\right|^{-\frac{\alpha}{2q'\gamma'}}; \tag{2.17}$$

$$\begin{aligned}&\left|\hat{\sigma}_{k,j,\theta,\Phi}(\xi,\eta,\mu)-\hat{\sigma}_{k,j,\theta,\Phi}(0,\eta,\mu)\right|\\ \leqslant\ & C(\log\theta)^2 \|\Omega\|_q \left|\theta^k\, \xi\right|^{\frac{\alpha}{2q'\gamma'}} \left|\theta^j\, \eta\right|^{-\frac{\alpha}{2q'\gamma'}};\end{aligned} \tag{2.18}$$

$$\begin{aligned}&\left|\hat{\sigma}_{k,j,\theta,\Phi}(\xi,\eta,\mu)-\hat{\sigma}_{k,j,\theta,\Phi}(\xi,0,\mu)\right|\\ \leqslant\ & C(\log\theta)^2 \|\Omega\|_q \left|\theta^k\, \xi\right|^{-\frac{\alpha}{2q'\gamma'}} \left|\theta^j\, \eta\right|^{\frac{\alpha}{2q'\gamma'}};\end{aligned} \tag{2.19}$$

$$\begin{aligned}&\left|\hat{\sigma}_{k,j,\theta,\Phi}(\xi,\eta,\mu)-\hat{\sigma}_{k,j,\theta,\Phi}(0,\eta,\mu)-\hat{\sigma}_{k,j,\theta,\Phi}(\xi,0,\mu)+\hat{\sigma}_{k,j,\theta,\Phi}(0,0,\mu)\right|\\ \leqslant\ & C(\log\theta)^2 \|\Omega\|_q \left|\theta^k\, \xi\right|^{\frac{\alpha}{2q'\gamma'}} \left|\theta^j\, \eta\right|^{\frac{\alpha}{2q'\gamma'}},\end{aligned} \tag{2.20}$$

where $\xi \in \mathbf{R}^n$, $\eta \in \mathbf{R}^m$ and $\mu \in \mathbf{R}$.

Let $\Psi^1 \in \mathscr{S}(\mathbf{R}^n)$ and $\Psi^2 \in \mathscr{S}(\mathbf{R}^m)$ be two Schwartz functions such that $\widehat{\Psi^l}(\xi_l) = 1$ for $|\xi_l| \leqslant \frac{1}{2}$ and $(\widehat{\Psi^l})(\xi_l) = 0$ for $|\xi_l| \geqslant 1, l = 1,2$. Let $\widehat{\Psi^1_k}(\xi) = \widehat{\Psi^1}\left(\theta^k\xi\right)$ and $\widehat{\Psi^2_j}(\eta) = \widehat{\Psi^2}\left(\theta^j\eta\right)$. Define the sequence of measures $\left\{\nu_{k,j}\right\}$ by

$$\begin{aligned}\hat{\nu}_{k,j}(\xi,\eta,\mu) = \hat{\sigma}_{k,j,\theta,\Phi}(\xi,\eta,\mu) &- \widehat{\Psi^1_k}(\xi)\,\hat{\sigma}_{k,j,\theta,\Phi}(0,\eta,\mu) - \widehat{\Psi^2_j}(\eta)\,\hat{\sigma}_{k,j,\theta,\Phi}(\xi,0,\mu)\\ &+\widehat{\Psi^1_k}(\xi)\,\widehat{\Psi^2_j}(\eta)\,\hat{\sigma}_{k,j,\theta,\Phi}(0,0,\mu).\end{aligned} \tag{2.21}$$

By a standard argument we get

$$\left|\hat{\nu}_{k,j}(\xi,\eta,\mu)\right| \leqslant C(\log\theta)^2 \|\Omega\|_q \left|\theta^k\, \xi\right|^{\pm\frac{\alpha}{4q'\gamma'}} \left|\theta^j\, \eta\right|^{\pm\frac{\alpha}{4q'\gamma'}}. \tag{2.22}$$

Set

$$\begin{aligned}
g(f)(x,y,z) &= \left(\sum_{k,j\in\mathbf{Z}} \left|\nu_{k,j} * f\,(x,y,z)\right|^2\right)^{\frac{1}{2}}, \nu^*(f) = \sup_{k,j\in\mathbf{Z}} \left|\left|\nu_{k,j}\right| * f\right|,\\
\sigma^{(1)}_{\Phi,\theta} f(x,y,z) &= \sup_{k,j\in\mathbf{Z}} \int_{\theta^j\leqslant|v|<\theta^{j+1}} \left(\int_{\theta^k}^{\theta^{k+1}} |f(x,y-v,z-\phi(t)\psi(|v|)|\,\Omega_2(v)\right)\frac{dt}{t}dv,\\
\sigma^{(2)}_{\Phi,\theta} f(x,y,z) &= \sup_{k,j\in\mathbf{Z}} \int_{\theta^k\leqslant|u|<\theta^{k+1}} \int_{\theta^j}^{\theta^{j+1}} |f(x-u,y,z-\phi(|u|)\psi(s))|\,\Omega_1(u)\frac{ds}{s}du,\\
\sigma^{(3)}_{\Phi,\theta} f(x,y,z) &= \|\Omega\|_q \sup_{k,j\in\mathbf{Z}} \int_{\theta^k}^{\theta^{k+1}} \int_{\theta^j}^{\theta^{j+1}} |f(x,y,z-\phi(t)\psi(s))|\,\frac{dtds}{ts},
\end{aligned}$$

where

$$\Omega_1(u) = \int_{\mathbf{S}^{m-1}} |\Omega(u,v)|\,d\sigma(v) \text{ and } \Omega_2(v) = \int_{\mathbf{S}^{n-1}} |\Omega(u,v)|\,d\sigma(u).$$

It is clear that $\Omega_1 \in L^q(\mathbf{S}^{n-1})$ and $\Omega_2 \in L^q(\mathbf{S}^{m-1})$. Now, by (2.21) we have

$$\begin{aligned} \nu^*(f)(x,y,z) \leqslant\ & g(f)(x,y,z) + C\left((\mathscr{M}_{\mathbf{R}^n} \otimes id_{\mathbf{R}^m} \otimes id_{\mathbf{R}^1}) \circ \sigma^{(1)}_{\Phi,\theta}\right)(f)(x,y,z) \\ & + C\left(id_{\mathbf{R}^m} \otimes \mathscr{M}_{\mathbf{R}^m} \otimes id_{\mathbf{R}^1}) \circ \sigma^{(2)}_{\Phi,\theta}\right)(f)(x,y,z) \\ & + C\left(\mathscr{M}_{\mathbf{R}^n} \otimes \mathscr{M}_{\mathbf{R}^m} \otimes id_{\mathbf{R}^1}) \circ \sigma^{(3)}_{\Phi,\theta}\right)(f)(x,y,z), \end{aligned} \tag{2.23}$$

where $\mathscr{M}_{\mathbf{R}^s}$ denotes the Hardy-Littlewood maximal function on $\mathbf{R}^s$.

We need now to study the $L^p$ boundedness of the maximal operators $\sigma^{(l)}_{\Phi,\theta}(f)$, $l = 1,2$. First, by definition of $\sigma^{(1)}_{\Phi,\theta}(f)$ we have

$$\begin{aligned} & \sigma^{(1)}_{\Phi,\theta}(f)(x,y,z) \\ & \leqslant \sup_{k,j\in\mathbf{Z}} \left( \int_{\theta^k}^{\theta^{k+1}} \left( \int_{\theta^j \leqslant |v| < \theta^{j+1}} f(x, y-v, z-\phi(t)\psi(|v|)) \frac{\Omega_2(v)}{|v|^m} dv \right) \frac{dt}{t} \right) \\ & \leqslant \sup_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^{k+1}} \mathscr{M}_{\phi(t),\Omega_2} f(x,\cdot,\cdot)(y,z) \frac{dt}{t}, \end{aligned} \tag{2.24}$$

where

$$\mathscr{M}_{\alpha,\Omega_2} g(y,z) = \sup_{j\in\mathbf{Z}} \left| \int_{\theta^j \leqslant |v| < \theta^{j+1}} g(y-v, z-\alpha\psi(|v|)) \frac{\Omega_2(v)}{|v|^m} dv \right|.$$

By employing the same argument as in the proof of Proposition 14 in [7] we get for $1 < p \leqslant \infty$, there exists, positive constant $C_p$ independent of $\alpha$ such that

$$\left\| \mathscr{M}_{\alpha,\Omega_2}(g) \right\|_{L^p(\mathbf{R}^{m+1})} \leqslant C_p (\log\theta) \|\Omega\|_q \|g\|_{L^p(\mathbf{R}^{m+1})}. \tag{2.25}$$

By (2.24)–(2.25), for every $1 < p \leqslant \infty$ we have

$$\left\| \sigma^{(1)}_{\Phi,\theta}(f) \right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R})} \leqslant C_p (\log\theta)^2 \|\Omega\|_q \|f\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R})}. \tag{2.26}$$

Similarly, for every $1 < p \leqslant \infty$ we have

$$\left\| \sigma^{(2)}_{\Phi,\theta}(f) \right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R})} \leqslant C_p (\log\theta)^2 \|\Omega\|_q \|f\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R})}. \tag{2.27}$$

Also, by a change of variable we have

$$\begin{aligned} & \int_{\theta^j}^{\theta^{j+1}} \int_{\theta^k}^{\theta^{k+1}} |f(x,y,z-\phi(t)\psi(s))| \frac{dtds}{ts} \\ & = \int_{\theta^j}^{\theta^{j+1}} \int_{\phi(\theta^k)}^{\phi(\theta^{k+1})} |f(x,y,z-u\psi(s))| \frac{du}{\phi^{-1}(u)\phi'(\phi^{-1}(u))} ds \\ & \leqslant C(\log\theta) \left( \int_{\theta^j}^{\theta^{j+1}} \mathscr{M}_{\psi(s),\mathbf{R}^1} f(x,y,\cdot)(z) ds \right), \end{aligned}$$

where $\mathscr{M}_{\alpha,\mathbf{R}^1}$ is the directional Hardy-Littlewood maximal function on $\mathbf{R}$ in the direction of $\alpha$. Since $\mathscr{M}_{s,\mathbf{R}^1}$ is bounded on $L^p$ with bound independent of $s$, for every $1 < p \leqslant \infty$ we easily get

$$\left\|\sigma^{(3)}_{\Phi,\theta}(f)\right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R})} \leqslant C_p(\log\theta)^2 \|\Omega\|_q \|f\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R})}. \tag{2.28}$$

Now, by (2.22) and Plancherel's theorem we have

$$\|g(f)\|_{L^2} \leqslant C(\log\theta)^2 \|\Omega\|_q \|f\|_{L^2} \tag{2.29}$$

and hence by (2.23), (2.26)–(2.28) we get

$$\|\nu^*(f)\|_{L^2} \leqslant C(\log\theta)^2 \|\Omega\|_q \|f\|_{L^2} \tag{2.30}$$

for some positive constant $C$ independent of $\theta$. By applying Lemma 2.1 (with $q=2$) along with the trivial estimate $\left\|\nu_{k,j}\right\| \leqslant C\|\Omega\|_q(\log\theta)^2$ we get

$$\left\|\Big(\sum_{k,j\in\mathbf{Z}}\left|\nu_{k,j}*g_{k,j}\right|^2\Big)^{\frac12}\right\|_{p_0} \leqslant C_{p_0}(\log\theta)^2\|\Omega\|_q \left\|\Big(\sum_{k,j\in\mathbf{Z}}\left|g_{k,j}\right|^2\Big)^{\frac12}\right\|_{p_0} \tag{2.31}$$

if $1/4 = |1/p_0 - 1/2|$. Now, by (2.22), (2.31) and Lemma 2.2 we obtain

$$\|g(f)\|_{L^p} \leqslant C_p(\log\theta)^2 \|\Omega\|_q \|f\|_{L^p} \tag{2.32}$$

for all $p$ satisfying $4/3 < p < 4$ which, when combined with (2.23), (2.26)–(2.28) and the $L^p$ boundedness of the Hardy-Littlewood maximal function, implies

$$\|\nu^*(f)\|_{L^p} \leqslant C(\log\theta)^2 \|\Omega\|_q \|f\|_{L^p} \tag{2.33}$$

for all $p$ satisfying $4/3 < p < 4$. Now by (2.23), (2.33) and applying Lemma 2.1 and Lemma 2.2 we get

$$\|g(f)\|_{L^p} \leqslant C_p(\log\theta)^2 \|\Omega\|_q \|f\|_{L^p} \tag{2.34}$$

for every $p$ satisfying $8/7 < p < 8$. By successive applications of Lemma 2.1 and Lemma 2.2 along with (2.23) and (2.26)–(2.28) we get

$$\|g(f)\|_{L^p} \leqslant C_p(\log\theta)^2 \|\Omega\|_q \|f\|_{L^p} \tag{2.35}$$

and hence

$$\|\nu^*(f)\|_{L^p} \leqslant C_p(\log\theta)^2 \|\Omega\|_q \|f\|_{L^p} \tag{2.36}$$

for all $p \in (1,\infty)$. By (2.21) and (2.23) we have

$$\begin{aligned}\sigma^*_{\Phi,\theta}(f)(x,y,z) \leqslant\; &\nu^*(f)(x,y,z) + 2C[(\mathscr{M}_{\mathbf{R}^n}\otimes id_{\mathbf{R}^m}\otimes id_{\mathbf{R}^1})\circ\sigma^{(1)}_{\Phi,\theta}](f)(x,y,z)\\ &+2C[(id_{\mathbf{R}^m}\otimes\mathscr{M}_{\mathbf{R}^m}\otimes id_{\mathbf{R}^1})\circ\sigma^{(2)}_{\Phi,\theta}](f)(x,y,z)\\ &+2C[(\mathscr{M}_{\mathbf{R}^n}\otimes\mathscr{M}_{\mathbf{R}^m}\otimes id_{\mathbf{R}^1})\circ\sigma^{(3)}_{\Phi,\theta}](f)(x,y,z)\end{aligned} \tag{2.37}$$

which when combined with (2.26)–(2.28), (2.36) and the $L^p$ boundedness of the Hardy-Littlewood maximal function we get

$$\left\|\sigma^*_{\Phi,\theta}(f)\right\|_{L^p} \leqslant C_p(\log\theta)^2 \|\Omega\|_q \|f\|_{L^p} \text{ for } p\in(1,\infty). \tag{2.38}$$

Since the inequality

$$\left\|\sigma^*_{\Phi,\theta}(f)\right\|_{L^\infty} \leqslant C(\log\theta)^2 \|\Omega\|_q \|f\|_{L^\infty}$$

holds trivially, the proof of (2.13) is complete for the case $\Phi(t,s)=\phi(t)\psi(s)$, where $\phi,\psi \in C^2([0,\infty))$, and $\phi$ and $\psi$ are convex increasing functions.

Now we need to prove the lemma for the general case of $\Phi$ as stated above. To this end, we first need to prove the following: For $f\geqslant 0$, let

$$\lambda^*_\Phi(f)(z) = \sup_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^{k+1}} |f(z-\Phi(t,s))| \frac{dt}{t} = \sup_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^{k+1}} \left|f(z-\Gamma^2_s(t))\right| \frac{dt}{t}.$$

Our purpose now is to prove that for every $1<p<\infty$, there exists a positive constant $C_p$ independent of $\Phi$ such that

$$\|\lambda^*_\Phi(f)\|_{L^p(\mathbf{R})} \leqslant C_p(\log\theta)\|f\|_{L^p(\mathbf{R})}. \tag{2.39}$$

By a change of variable we have

$$\lambda^*_\Phi(f)(z) = \sup_{k\in\mathbf{Z}} \left( \int_{\Gamma^2_s(\theta^k)}^{\Gamma^2_s(\theta^{k+1})} f(z-u) \frac{du}{(\Gamma^2_s)^{-1}(u)\,(\Gamma^2_s)'\left((\Gamma^2_s)^{-1}(u)\right)} \right).$$

Since the function $\frac{1}{(\Gamma^2_s)^{-1}(u)(\Gamma^2_s)'\left((\Gamma^2_s)^{-1}(u)\right)}$ is non-negative, decreasing and its integral over $[\Gamma^2_s(\theta^k),\ \Gamma^2_s(\theta^{k+1})]$ is equal to $\log(\theta)$ we have

$$\lambda^*_\Phi(f)(z) \leqslant C\log(\theta)\mathscr{M}_{\mathbf{R}^1} f(z),$$

where $\mathscr{M}_{\mathbf{R}^1} f(z)$ is the Hardy-Littlewood maximal function on $\mathbf{R}^1$. By the $L^p$ boundedness of $\mathscr{M}_{\mathbf{R}^1} f(z)$ and the last inequality we get (2.39).

Now we notice that the proof of the lemma for the general case $\Phi(t,s)$ will be the same as its proof in the special case $\Phi(t,s)=\phi(t)\psi(s)$ until we reach (2.24). Now we verify (2.24).

First, by definition of $\sigma^{(1)}_{\Phi,\theta}(f)$ we have

$$\begin{aligned}\sigma^{(1)}_{\Phi,\theta}(f)(x,y,z) &\leqslant \sup_{k,j\in\mathbf{Z}} \left( \int_{\theta^k}^{\theta^{k+1}} \left( \int_{\theta^j\leqslant|v|<\theta^{j+1}} f\left(x,y-v,z-\Gamma^1_t(|v|)\right) \frac{\Omega_2(v)}{|v|^m} dv \right) \frac{dt}{t} \right) \\ &\leqslant \sup_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^{k+1}} \mathscr{M}_{t,\Omega_2} f(x,\cdot,\cdot)(y,z)\frac{dt}{t},\end{aligned}$$

where

$$\mathscr{M}_{t,\Omega_2}g(y,z) = \sup_{j\in\mathbf{Z}}\left|\int_{\theta^j\leqslant|v|<\theta^{j+1}} g(y-v,z-\Gamma_t^1(|v|))\frac{\Omega_2(v)}{|v|^m}dv\right|.$$

By (2.39) and employing the same argument as in the proof of Proposition 14 in [7] we get for $1<p\leqslant\infty$, there exists a positive constant $C_p$ independent of $\Gamma_t^1$ such that

$$\left\|\mathscr{M}_{t,\Omega_2}(g)\right\|_{L^p(\mathbf{R}^{m+1})} \leqslant C_p(\log\theta)\|\Omega\|_q\|g\|_{L^p(\mathbf{R}^{m+1})}$$

which in turn implies

$$\left\|\sigma_{\Phi,\theta}^{(1)}(f)\right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R})} \leqslant C_p(\log\theta)^2\|\Omega\|_q\|f\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R})}.$$

Similarly we can prove (2.27).

Now, it is left to prove (2.28). We notice that

$$\begin{aligned}&\int_{\theta^j}^{\theta^{j+1}}\int_{\theta^k}^{\theta^{k+1}}|f(x,y,z-\Phi(t,s))|\frac{dtds}{ts}\\ &\leqslant \int_{\theta^j}^{\theta^{j+1}}\left(\sup_{k\in\mathbf{Z}}\int_{\theta^k}^{\theta^{k+1}}\left|f(z-\Gamma_s^2(t))\right|\frac{dt}{t}\right)\frac{ds}{s}.\end{aligned}$$

By the last inequality and (2.39) we get (2.28). Now the rest of the proof will be exactly the same as in the special case $\Phi(t,s)=\phi(t)\psi(s)$. These details will be omitted. The proof of the lemma is complete. □

LEMMA 2.5. *Let $h\in\Delta_\gamma\left(\mathbf{R}^+\times\mathbf{R}^+\right)$ for some $1<\gamma\leqslant\infty$, $\Omega\in L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ for some $1<q\leqslant 2$ and $\theta=2^{q'}$. Assume*

$$\Phi(t,s)=P(t,s)=\sum_{q=0}^{d_2}\sum_{l=0}^{d_1}a_{l,q}t^{\alpha_l}s^{\beta_q}$$

*with $\alpha_l,\beta_q>0$ is a generalized polynomial on $\mathbf{R}^2$. Then for $\gamma'<p\leqslant\infty$ and $f\in L^p(\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R})$ there exists a positive constant $C_p$ which is independent of $\Omega$, $h$ and the coefficients of $P$ such that*

$$\left\|\sigma_{P,\theta}^*(f)\right\|_p \leqslant C_p(q-1)^{-2}\|\Omega\|_q\|f\|_p. \tag{2.40}$$

*Proof.* The proof follows exactly the same lines of the proof of Lemma 2.4 except that we need to prove (2.26)–(2.28) when $\Phi(t,s)=P(t,s)$ is a generalized polynomial on $\mathbf{R}^2$. Now $P$ can be written as $P(t,s)=Q_s(t)=\sum_{l=0}^{d_1}b_l(s)t^{\alpha_l}$ and $P(t,s)=$

$R_t(s) = \sum_{q=0}^{d_2} c_q(t)s^{\beta_q}$, where $b_l(s) = \sum_{q=0}^{d_2} a_{l,q}s^{\beta_q}$ and $c_q(t) = \sum_{l=0}^{d_1} a_{l,q}t^{\alpha_l}$. We start by proving (2.26). To this end, by definition of $\sigma_{P,\theta}^{(1)}(f)$ we have

$$\begin{aligned}\sigma_{P,\theta}^{(1)}(f)(x,y,z) &\leqslant \sup_{k,j\in\mathbf{Z}}\left(\int_{\theta^k}^{\theta^{k+1}}\left(\int_{\theta^j\leqslant|v|<\theta^{j+1}} f(x,y-v,z-R_t(|v|))\frac{\Omega_2(v)}{|v|^m}dv\right)\frac{dt}{t}\right)\\ &\leqslant \sup_{k\in\mathbf{Z}}\int_{\theta^k}^{\theta^{k+1}} \mathscr{F}_{R_t,\Omega_2}f(x,\cdot,\cdot)(y,z)\frac{dt}{t},\end{aligned} \tag{2.41}$$

where

$$\mathscr{F}_{R_t,\Omega_2}g(y,z) = \sup_{j\in\mathbf{Z}}\left|\int_{\theta^j\leqslant|v|<\theta^{j+1}} g(y-v,z-R_t(|v|))\frac{\Omega_2(v)}{|v|^m}dv\right|.$$

Now,

$$\begin{aligned}&\left\|\mathscr{F}_{R_t,\Omega_2}(g)\right\|_{L^p(\mathbf{R}^{m+1})}\\ &\leqslant \int_{\mathbf{S}^{m-1}}|\Omega_2(v)|\left(\int_{\mathbf{R}^{m+1}}\left(\sup_{j\in\mathbf{Z}}\int_{\theta^j}^{\theta^{j+1}} g(y-sv,z-R_t(s))\frac{dt}{t}\right)^p dydz\right)^{1/p}\\ &\leqslant \sum_{l=1}^{[\log\theta]+1}\int_{\mathbf{S}^{m-1}}|\Omega_2(v)|\left(\int_{\mathbf{R}^{m+1}}\left(\sup_{j\in\mathbf{Z}}\int_{\theta^j2^{l-1}}^{\theta^j2^l} g(y-sv,z-R_t(s))\frac{dt}{t}\right)^p dydz\right)^{1/p}.\end{aligned}$$

Since $R_t(s)$ is a generalized polynomial in $s$ with coefficients depending on $t$, by a result established in [28] we get

$$\left(\int_{\mathbf{R}^{m+1}}\left(\sup_{j\in\mathbf{Z}}\int_{\theta^j2^{l-1}}^{\theta^j2^l} g(y-sv,z-R_t(s))\frac{dt}{t}\right)^p dydz\right)^{1/p} \leqslant C_p\|g\|_{L^p(\mathbf{R}^{m+1})}$$

where $C_p$ is a positive constant independent of $t$. By the last two inequalities we easily get that for every $1<p\leqslant\infty$, there exists a positive constant $C_p$ independent of $t$ such that

$$\left\|\mathscr{F}_{R_t,\Omega_2}(g)\right\|_{L^p(\mathbf{R}^{m+1})} \leqslant C_p(\log\theta)\|\Omega\|_q\|g\|_{L^p(\mathbf{R}^{m+1})}. \tag{2.42}$$

It is clear that the proof of (2.27) will be the same. We omit the details. Finally we prove (2.28). We notice that

$$\begin{aligned}&\int_{\theta^j}^{\theta^{j+1}}\int_{\theta^k}^{\theta^{k+1}}|f(x,y,z-P(t,s))|\frac{dtds}{ts}\\ &= \sum_{l=1}^{[\log\theta]+1}\int_{\theta^j}^{\theta^{j+1}}\int_{\theta^j2^{l-1}}^{\theta^j2^l}|f(x,y,z-R_t(s))|\frac{dtds}{ts}\\ &\leqslant C(\log\theta)\int_{\theta^j}^{\theta^{j+1}} M^*_{R_t,\mathbf{R}^1}f(x,y,z)\frac{dt}{t},\end{aligned}$$

where

$$M^{*}_{R_t,\mathbf{R}^1}f(x)=\sup_{r>0}\frac{1}{r}\int_{|s|<r}|f(x-R_t(s))|\,ds.$$

As above, by the last inequality and the $L^p$ boundedness of $M^{*}_{R_t,\mathbf{R}^1}f$ proved in [28] we get (2.28). The lemma is proved. □

LEMMA 2.6. *Let $h\in\Delta_\gamma\left(\mathbf{R}^+\times\mathbf{R}^+\right)$ for some $1<\gamma\leqslant\infty$, $\Omega\in L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ for some $1<q\leqslant 2$ and $\theta=2^{q'}$. Assume $\Phi(t,s)=\phi(t)P(s)$, where $\phi\in C^2([0,\infty))$, and $\phi$ is a convex increasing function and $P$ is a generalized polynomial given by $P(s)=\sum_{l=0}^{d}a_l s^{\alpha_l}$. Then for $\gamma'<p\leqslant\infty$ and $f\in L^p\left(\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R}\right)$ there exists a positive constant $C_p$ which is independent of $\Omega$ and $h$ such that*

$$\left\|\sigma^{*}_{\Phi,\theta}(f)\right\|_p\leqslant C_p(q-1)^{-2}\|\Omega\|_q\|f\|_p. \tag{2.43}$$

*Proof.* Again as in the proof of Lemma 2.5, we follow the same lines of the proof of Lemma 2.4 and hence we only need to prove (2.26)–(2.28) for $\Phi(t,s)=\phi(t)P(s)$. We prove first (2.26). We notice that

$$\begin{aligned}
&\sigma^{(1)}_{\Phi,\theta}(f)(x,y,z)\\
&\leqslant \sup_{k,j\in\mathbf{Z}}\left(\int_{\theta^k}^{\theta^{k+1}}\left(\int_{\theta^j\leqslant|v|<\theta^{j+1}}f(x,y-v,z-\phi(t)P(|v|))\frac{\Omega_2(v)}{|v|^m}dv\right)\frac{dt}{t}\right)\\
&\leqslant \sup_{k\in\mathbf{Z}}\int_{\theta^k}^{\theta^{k+1}}\mathscr{I}_{H_t,\Omega_2}f(x,\cdot,\cdot)(y,z)\frac{dt}{t},
\end{aligned}\tag{2.44}$$

where

$$\mathscr{I}_{H_t,\Omega_2}g(y,z)=\sup_{j\in\mathbf{Z}}\left|\int_{\theta^j\leqslant|v|<\theta^{j+1}}g(y-v,z-H_t(|v|))\frac{\Omega_2(v)}{|v|^m}dv\right|$$

and $H_t(s)=\phi(t)P(s)$. We notice that if $g\geqslant 0$ we have

$$\begin{aligned}
&\int_{\theta^j\leqslant|v|<\theta^{j+1}}g(y-v,z-H_t(|v|))\frac{\Omega_2(v)}{|v|^m}dv\\
&=\sum_{l=1}^{[\log\theta]+1}\int_{\mathbf{S}^{m-1}}|\Omega_2(v)|\int_{\theta^j2^{l-1}}^{\theta^j2^l}g(y-sv,z-H_t(s))\frac{ds}{s}d\sigma(v).
\end{aligned}\tag{2.45}$$

Since $H_t(s)$ is a generalized polynomial in $s$ with coefficients depending on $t$, by (2.45) and the same argument as in the proof (2.42) we get

$$\left\|\mathscr{I}_{H_t,\Omega_2}(g)\right\|_{L^p(\mathbf{R}^{m+1})}\leqslant C_p(\log\theta)\|\Omega\|_q\|g\|_{L^p(\mathbf{R}^{m+1})}\quad\text{for } 1<p\leqslant\infty. \tag{2.46}$$

Also, as above we have

$$\sigma^{(2)}_{\Phi,\theta}(f)(x,y,z)\leqslant\sup_{j\in\mathbf{Z}}\int_{\theta^j}^{\theta^{j+1}}L_{G_s,\Omega_1}f(\cdot,y,\cdot)(x,z)\frac{dt}{t}, \tag{2.47}$$

where

$$L_{G_s,\Omega_1} g(y,z) = \sup_{k\in\mathbf{Z}} \left| \int_{\theta^k \leqslant |v| < \theta^{k+1}} g(x-u, z - G_s(|u|)) \frac{\Omega_1(u)}{|u|^n} dv \right|,$$

and $G_s(t) = \phi(t)P(s)$.

By following the same argument employed in the proof of (2.26) in Lemma 2.4 we obtain (2.27). Finally we prove (2.28). We notice that

$$\begin{aligned}
&\int_{\theta^j}^{\theta^{j+1}} \int_{\theta^k}^{\theta^{k+1}} |f(x,y,z-\phi(t)P(s))| \frac{dtds}{ts} \\
&= \int_{\theta^j}^{\theta^{j+1}} \int_{\phi(\theta^k)}^{\phi(\theta^{k+1})} |f(x,y,z-uP(s))| \frac{du}{\phi^{-1}(u)\phi'(\phi^{-1}(u))} \frac{ds}{s} \\
&\leqslant C(\log\theta) \left( \int_{\theta^j}^{\theta^{j+1}} \mathscr{M}_{P(s),\mathbf{R}^1} f(x,y,z) \frac{ds}{s} \right),
\end{aligned}$$

where $\mathscr{M}_{P(s),\mathbf{R}^1}$ is the directional Hardy-Littlewood maximal function on $\mathbf{R}$ in the direction of $s$. Since $\mathscr{M}_{P(s),\mathbf{R}^1}$ is bounded on $L^p$ with bound independent of $P(s)$ we easily get

$$\left\| \sigma^{(3)}_{\Phi,\theta}(f) \right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R})} \leqslant C_p (\log\theta)^2 \|\Omega\|_q \|f\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R})} \tag{2.48}$$

for $1 < p \leqslant \infty$. The lemma is proved. □

LEMMA 2.7. *Let $h \in \Delta_\gamma(\mathbf{R}^+ \times \mathbf{R}^+)$ for some $1 < \gamma \leqslant \infty$, $\Omega \in L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ for some $1 < q \leqslant 2$ and $\theta = 2^{q'}$. Assume $\Phi(t,s) = \phi_1(t) + \phi_2(s)$, where each $\phi_l$ $(l = 1,2)$ is either a generalized polynomial or is in $C^2[0,\infty)$, a convex increasing function with $\phi_l(0) = 0$. Then for $\gamma' < p \leqslant \infty$ and $f \in L^p(\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R})$ there exists a positive constant $C_p$ which is independent of $\Omega$ such that*

$$\left\| \lambda^*_{\Phi,\theta}(f) \right\|_p \leqslant C_p (q-1)^{-2} \|\Omega\|_q \|f\|_p. \tag{2.49}$$

*Proof.* We shall consider $\Phi(t,s) = \phi_1(t) + \phi_2(s)$, where $\phi_1$ is in $C^2[0,\infty)$, a convex increasing function with $\phi_1(0) = 0$ and $\phi_2$ is a generalized polynomial given by $\phi_2(s) = \sum_{l=0}^{d} a_l s^{\alpha_l}$. The other cases can be handled in a similar way. As in the previous lemmas, the proof follows the same lines of the proof of Lemma 2.4 and hence we only need to prove (2.26)–(2.28) for the case $\Phi(t,s) = \phi_1(t) + \phi_2(s)$. We start proving (2.26). We notice that

$$\begin{aligned}
&\sigma^{(1)}_{\Phi,\theta}(f)(x,y,z) \\
&\leqslant \sup_{k,j\in\mathbf{Z}} \left( \int_{\theta^k}^{\theta^{k+1}} \left( \int_{\theta^j \leqslant |v| < \theta^{j+1}} f(x, y-v, z-\phi_1(t) - \phi_2(|v|)) \frac{\Omega_2(v)}{|v|^m} dv \right) \frac{dt}{t} \right) \\
&\leqslant \sup_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^{k+1}} \mathscr{I}_{H_t,\Omega_2} f(x,\cdot,\cdot)(y,z) \frac{dt}{t},
\end{aligned} \tag{2.50}$$

where

$$\mathscr{J}_{H_t,\Omega_2} g(y,z) = \sup_{j\in\mathbf{Z}} \left| \int_{\theta^j \leqslant |v| < \theta^{j+1}} g(y-v, z-H_t(|v|)) \frac{\Omega_2(v)}{|v|^m} dv \right|$$

and $H_t(s) = \phi_1(t) + \phi_2(s)$. By the argument as in (2.45), noticing that $H_t(s)$ is a generalized polynomial in $s$ with a constant term depending on $t$ and using a result established in [28], we get

$$\left\| \mathscr{J}_{H_t,\Omega_2}(g) \right\|_{L^p(\mathbf{R}^{m+1})} \leqslant C_p (\log\theta) \|\Omega\|_q \|g\|_{L^p(\mathbf{R}^{m+1})} \quad \text{for } 1 < p \leqslant \infty,$$

which in turn leads to (2.26). As for proving (2.27), by the argument in (2.47) we have

$$\sigma^{(2)}_{\Phi,\theta}(f)(x,y,z) \leqslant \sup_{j\in\mathbf{Z}} \int_{\theta^j}^{\theta^{j+1}} L_{G_s,\Omega_1} f(\cdot,y,\cdot)(x,z) \frac{dt}{t},$$

where

$$L_{G_s,\Omega_1} g(y,z) = \sup_{k\in\mathbf{Z}} \left| \int_{\theta^k \leqslant |v| < \theta^{k+1}} g(x-u, z-G_s(|u|)) \frac{\Omega_1(u)}{|u|^n} dv \right|,$$

and $G_s(t) = \phi_1(t) + \phi_2(s) = \tilde{\phi}(t)$.

Now we notice $\tilde{\phi}(t)$ is a $C^2([0,\infty))$, convex and increasing function satisfying $\tilde{\phi}(0) = 0$. By following the same argument employed in the proof of (2.26) in Lemma 2.4, we obtain (2.27). Finally we prove (2.28). We notice that

$$\begin{aligned}
&\int_{\theta^j}^{\theta^{j+1}} \int_{\theta^k}^{\theta^{k+1}} |f(x,y,z-\phi_1(t)-\phi_2(s))| \frac{dt\,ds}{ts} \\
&= \int_{\theta^j}^{\theta^{j+1}} \int_{\phi_1(\theta^k)}^{\phi_1(\theta^{k+1})} |f(x,y,z-u-\phi_2(s))| \frac{du}{\phi_1^{-1}(u)\phi_1'(\phi_1^{-1}(u))} \frac{ds}{s} \\
&\leqslant C(\log\theta)^2 \mathscr{M}_{\mathbf{R}^1} f(x,y,.)(z),
\end{aligned}$$

where $\mathscr{M}_{\mathbf{R}^1}$ denotes the Hardy-Littlewood maximal function on $\mathbf{R}^1$. Since $\mathscr{M}_{\mathbf{R}^1}$ is bounded on $L^p$, we easily get

$$\left\| \sigma^{(3)}_{\Phi,\theta}(f) \right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R})} \leqslant C_p (\log\theta)^2 \|\Omega\|_q \|f\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R})} \tag{2.51}$$

for $1 < p \leqslant \infty$. The lemma is proved. □

## 3. Proofs of main theorems

Since $\Delta_\gamma(\mathbf{R}^+ \times \mathbf{R}^+) \subseteq \Delta_2(\mathbf{R}^+ \times \mathbf{R}^+)$ when $\gamma \geqslant 2$, we may assume that $1 < \gamma \leqslant 2$ and $|1/p - 1/2| < 1/\gamma'$. First, we notice that

$$Tf(x,y,z) = \sum_{k,j\in\mathbf{Z}} \lambda_{k,j,\theta,\Phi} * f(x,y,z).$$

Now, by invoking Lemmas 2.4–2.6 and following arguments similar to the proof of Theorem 7.5 (in the one-parameter setting) in ([19], p. 824) we have

$$\left\| \Big( \sum_{k,j\in\mathbf{Z}} \left|\lambda_{k,j,\theta,\Phi} * g_{k,j}\right|^2 \Big)^{\frac{1}{2}} \right\|_p \leqslant C_p (\log\theta)^2 \|\Omega\|_q \left\| \Big( \sum_{k,j\in\mathbf{Z}} \left|g_{k,j}\right|^2 \Big)^{\frac{1}{2}} \right\|_p \tag{3.1}$$

for $p$ satisfying $|1/p-1/2|<1/\gamma'$ and for arbitrary functions $\{g_{k,j}\}$ on $\mathbf{R}^n\times\ \mathbf{R}^m\times\mathbf{R}$.

Now by Lemmas 2.4–2.6, (3.1), Lemma 2.3 and invoking Lemma 2.2 we get

$$\|Tf\|_p = \left\| \sum_{k,j\in\mathbf{Z}} \lambda_{k,j,\theta,\Phi} * f \right\|_p \leqslant C_p (\log\theta)^2 \|\Omega\|_q \|f\|_p \tag{3.2}$$

for $p$ satisfying $|1/p-1/2| < 1/\gamma'$, which in turn ends the proof of each one of the inequalities (1.5), (1.6), (1.8), (1.10) and (1.12). Now, by Lemmas 2.4–2.6 and a standard argument we get (1.7), (1.9), (1.11) and (1.13). This completes the proofs of Theorems 1.1–1.5. Now the proof of Theorem 1.6 can be obtained by the estimates (1.5)–(1.13) and employing an extrapolation method similar to the one employed in [4]. We omit the details.

Finally we can prove Theorem 1.7 by the above estimates and following the same arguments as in [6]. Again we omit the details. This completes the proofs of our theorems. □

*Acknowledgement.* The authors express their sincere gratitude to the anonymous referee for his/her helpful remarks and valuable suggestions.

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(Received June 15, 2022)

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