Journal of
Mathematical
Inequalities
Volume 17, Number 2 (2023), 779–797
doi:10.7153/jmi-2023-17-50

IMPROVEMENTS OF THE WEIGHTED HERMITE–HADAMARD INEQUALITY AND APPLICATIONS TO MEAN INEQUALITY

SHIGERU FURUICHI, NICUŞOR MINCULETE AND HAMID REZA MORADI

(*Communicated by T. Burić*)

Abstract. This paper aims to characterize the function appearing in the weighted Hermite-Hadamard inequality. We provide improved inequalities for the weighted means as applications of the obtained results. Modifications of the weighted Hermite-Hadamard inequality are also presented. Our results contain some exciting inequalities and extensions of the known results.

1. Introduction and preliminiaries

For $a,b>0$ and $0\leqslant v\leqslant 1$, the weighted arithmetic-geometric mean inequality asserts that $a\sharp_v b\leqslant a\nabla_v b$, where $a\sharp_v b:=a^{1-v}b^v$ and $a\nabla_v b:=(1-v)a+vb$ are named the weighted geometric mean and the weighted arithmetic mean, respectively. We use the symbols ∇ and $\sharp$ instead of $\nabla_{1/2}$ and $\sharp_{1/2}$. During the past decades, the study of inequalities involving mathematical means has attracted many mathematicians; see, for example, [4, 6, 7, 8, 9, 10].

Recently, in [15, Theorem 2.2], the weighted logarithmic mean was introduced in the following structure:

$$L_v(a,b):=\frac{1}{\log a-\log b}\left\{\frac{1-v}{v}(a-a^{1-v}b^v)+\frac{v}{1-v}(a^{1-v}b^v-b)\right\} \tag{1}$$

for $a,b>0$, $a\neq b$ with $v\in(0,1)$ and $L_v(a,a)=a$. For $v=1/2$, (1) reduces to the logarithmic mean $L_{1/2}(a,b)=L(a,b):=\dfrac{a-b}{\log a-\log b}$. Besides, it has been shown that

$$a\sharp_v b\leqslant L_v(a,b)\leqslant a\nabla_v b. \tag{2}$$

Inequality (2) provides a modification of the famous Young's inequality

$$ab\leqslant\frac{1}{p\log a-q\log b}\left(\frac{q}{p}(a^p-ab)+\frac{p}{q}(ab-b^q)\right)\leqslant\frac{a^p}{p}+\frac{b^q}{q}$$

for $a,b>0$, $a^p\neq b^q$ with $p,q>1$ and $1/p+1/q=1$.

Mathematics subject classification (2020): Primary 26D15; Secondary 26B25, 26E60.

Keywords and phrases: Weighted logarithmic mean, weighted identric mean, convex function, Hermite-Hadamard inequality.

Notice that inequality (2) is an immediate consequence of the following generalization of the Hermite-Hadamard inequality (see [15, Theorem 2.1])

$$\begin{aligned} & f(a\nabla_\nu b) \\ & \leqslant (1-\nu)\int_0^1 f(\nu\lambda(b-a)+a)d\lambda + \nu\int_0^1 f((1-\nu)\lambda(b-a)+\nu b+(1-\nu)a)d\lambda \\ & \leqslant f(a)\nabla_\nu f(b) \end{aligned} \tag{3}$$

for a convex Riemann integrable function $f:[a,b]\to\mathbb{R}$ and $a,b>0$ with $\nu\in[0,1]$. Indeed, by letting $\nu=1/2$ in (3), we recover the Hermite-Hadamard inequality:

$$f(a\nabla b) \leqslant \int_0^1 f(a\nabla_\lambda b)\,d\lambda \leqslant f(a)\nabla f(b). \tag{4}$$

For additional refinements and applications related to Hermite-Hadamard inequality, see [5, 13, 17].

Since $a\nabla_0 b=a$, $a\nabla_1 b=b$, $a\nabla_{1-t}b=b\nabla_t a$, and $(a\nabla_\alpha b)\nabla_\gamma\left(a\nabla_\beta b\right)=a\nabla_{(1-\gamma)\alpha+\gamma\beta}b$ with $\alpha,\beta,\gamma\in[0,1]$, inequality (3) can be written as

$$f(a\nabla_\nu b) \leqslant \mathfrak{C}_{f,\nu}(a,b) \leqslant f(a)\nabla_\nu f(b) \tag{5}$$

where

$$\mathfrak{C}_{f,\nu}(a,b) = \left(\int_0^1 f(a\nabla_{\nu\lambda}b)\,d\lambda\right)\nabla_\nu\left(\int_0^1 f\left(b\nabla_{(1-\nu)\lambda}a\right)d\lambda\right), \tag{6}$$

due to

$$\int_0^1 f\left(b\nabla_{(1-\nu)(1-\lambda)}a\right)d\lambda = \int_0^1 f\left(b\nabla_{(1-\nu)\mu}a\right)d\mu.$$

In [15], the representing function of the weighted logarithmic mean, i.e.,

$$f_\nu(t) := \frac{1}{\log t}\left\{\frac{1-\nu}{\nu}(t^\nu-1)+\frac{\nu}{1-\nu}(t-t^\nu)\right\} = L_\nu(1,t), \quad (1\neq t>0) \tag{7}$$

was studied and characterized by the following inequalities:

$$t^\nu \leqslant f_\nu(t) \leqslant \frac{1}{2}(t^\nu+(1-\nu)+\nu t) \leqslant (1-\nu)+\nu t.$$

The following results have been established in [7]:

THEOREM 1.1. *Let $f:[a,b]\to\mathbb{R}$ be a convex function. Then for any $\nu\in[0,1]$,*

$$f(a\nabla_\nu b) \leqslant \mathfrak{R}^{(1)}_{f,\nu}(a,b) \leqslant \mathfrak{C}_{f,\nu}(a,b) \leqslant \mathfrak{R}^{(2)}_{f,\nu}(a,b) \leqslant f(a)\nabla_\nu f(b),$$

where

$$\mathfrak{R}^{(1)}_{f,v}(a,b) := f(a\nabla_{\frac{v}{2}}b)\nabla_v f(a\nabla_{\frac{1+v}{2}}b),$$

and

$$\mathfrak{R}^{(2)}_{f,v}(a,b) := (f(a)\nabla_v f(b))\,\nabla\,(f(a\nabla_v b)).$$

COROLLARY 1.2. *Let* $a,b>0$ *and* $v\in(0,1)$. *Then*

$$a\sharp_v b \leqslant \left(a\sharp_{\frac{v}{2}}b\right)\nabla_v\left(a\sharp_{\frac{1+v}{2}}b\right) \leqslant L_v(a,b) \leqslant (a\nabla_v b)\,\nabla\,(a\sharp_v b) \leqslant a\nabla_v b.$$

In this paper, we refine inequalities (2). Refinement of the Hermite-Hadamard inequality is also provided. The inequalities demonstrated in the next section can be extended to the positive Hilbert space operators by utilizing the standard functional calculus. We leave this idea for the interested reader.

2. Main results

We begin with the following lemma, which includes two identities for $\mathfrak{C}_{f,v}$.

LEMMA 2.1. *Let* $f:[0,1)\cup(1,\infty)\to\mathbb{R}$ *be a convex function. Then, for any* $v\in(0,1)$,

$$\mathfrak{C}_{f,v}(t,1) = \frac{1}{1-t}\left(\frac{1-v}{v}\int_t^{(1-v)t+v} f(\lambda)d\lambda + \frac{v}{1-v}\int_{(1-v)t+v}^{1} f(\lambda)d\lambda\right), \tag{8}$$

and

$$\mathfrak{C}_{f,v}(1,t) = \frac{1}{t-1}\left(\frac{1-v}{v}\int_1^{(1-v)+vt} f(\lambda)d\lambda + \frac{v}{1-v}\int_{(1-v)+vt}^{t} f(\lambda)d\lambda\right). \tag{9}$$

Proof. Putting $a=t$ and $b=1$ in (6), we deduce the equality (8), with calculations. Relation (9) can be obtained likewise. □

REMARK 2.2.

(i) If we take $f(\lambda)=\lambda$ in (9), then we have $\mathfrak{C}_{\lambda,v}(1,t)=(1-v)+vt$ which is the representing function of the weighted arithmetic mean.

(ii) If we take $v=1/2$ in (8) and (9), then we reach

$$\mathfrak{C}_{f,\frac{1}{2}}(t,1) = \frac{1}{1-t}\int_t^1 f(\lambda)d\lambda = \frac{1}{t-1}\int_1^t f(\lambda)d\lambda = \mathfrak{C}_{f,\frac{1}{2}}(1,t), \quad (1\neq t>0).$$

(iii) For $t=0$, in equality (8), we obtain

$$\mathfrak{C}_{f,v}(0,1)=\frac{1-v}{v}\int_0^v f(\lambda)d\lambda+\frac{v}{1-v}\int_v^1 f(\lambda)d\lambda. \tag{10}$$

If we take $f(\lambda)=t^\lambda$ in (10), then we deduce $\mathfrak{C}_{t^\lambda,v}(0,1)=f_v(t)$, where $f_v(t)$ is given as in (7).

On account of Remark 2.2, it is interesting to study the function $\mathfrak{C}_{f,v}(t,1)$. The following result presents an upper and a lower bound for $\mathfrak{C}_{f,v}(t,1)$.

THEOREM 2.3. *Let $f:[0,1)\cup(1,\infty)\to\mathbb{R}_+$ be a convex function. Then for any $v\in(0,1)$,*

$$\min\left\{\frac{1-v}{v},\frac{v}{1-v}\right\}\mathfrak{C}_{f,\frac{1}{2}}(t,1)\leqslant\mathfrak{C}_{f,v}(t,1)\leqslant\max\left\{\frac{1-v}{v},\frac{v}{1-v}\right\}\mathfrak{C}_{f,\frac{1}{2}}(t,1). \tag{11}$$

Proof. Employing Remark 2.2 (ii), we have

$$\begin{aligned}
&\min\left\{\frac{1-v}{v},\frac{v}{1-v}\right\}\mathfrak{C}_{f,\frac{1}{2}}(t,1)\\
&=\min\left\{\frac{1-v}{v},\frac{v}{1-v}\right\}\frac{1}{1-t}\int_t^1 f(\lambda)\,d\lambda\\
&=\min\left\{\frac{1-v}{v},\frac{v}{1-v}\right\}\frac{1}{1-t}\left(\int_t^{(1-v)t+v} f(\lambda)\,d\lambda+\int_{(1-v)t+v}^1 f(\lambda)\,d\lambda\right)\\
&\leqslant\frac{1}{1-t}\left(\frac{1-v}{v}\int_t^{(1-v)t+v} f(\lambda)\,d\lambda+\frac{v}{1-v}\int_{(1-v)t+v}^1 f(\lambda)\,d\lambda\right)\\
&\leqslant\max\left\{\frac{1-v}{v},\frac{v}{1-v}\right\}\frac{1}{1-t}\left(\int_t^{(1-v)t+v} f(\lambda)\,d\lambda+\int_{(1-v)t+v}^1 f(\lambda)\,d\lambda\right)\\
&=\max\left\{\frac{1-v}{v},\frac{v}{1-v}\right\}\frac{1}{1-t}\int_t^1 f(\lambda)\,d\lambda\\
&=\max\left\{\frac{1-v}{v},\frac{v}{1-v}\right\}\mathfrak{C}_{f,v}(t,1).
\end{aligned}$$

Consequently, we prove the inequality of the statement. □

REMARK 2.4. Letting $t=0$ in (11). Then for any $\nu \in (0,1)$,

$$\min\left\{\frac{1-\nu}{\nu},\frac{\nu}{1-\nu}\right\}\int_0^1 f(\lambda)d\lambda \leqslant \mathfrak{C}_{f,\nu}(0,1) \leqslant \max\left\{\frac{1-\nu}{\nu},\frac{\nu}{1-\nu}\right\}\int_0^1 f(\lambda)d\lambda. \tag{12}$$

If we take $f(\lambda)=t^{\lambda}$ in (12), we infer

$$\min\left\{\frac{1-\nu}{\nu},\frac{\nu}{1-\nu}\right\}\underbrace{\frac{t-1}{\log t}}_{L(t,1)} \leqslant f_\nu(t) \leqslant \max\left\{\frac{1-\nu}{\nu},\frac{\nu}{1-\nu}\right\}\underbrace{\frac{t-1}{\log t}}_{L(t,1)}.$$

The above inequalities have been demonstrated in [6, Theorem 2.2]. More precisely, Theorem 2.3 provides an extension of [6, Theorem 2.2].

In the following lemma, the difference between the weighted arithmetic mean and the weighted geometric mean has been represented by the representing function of the weighted logarithmic mean $L_\nu(t,1)$.

LEMMA 2.5. *Let* $t\in[0,1)\cup(1,\infty)$ *and* $\nu\in(0,1)$. *Then*

$$L(t,1)-L(t^{\nu},1)=\frac{(1-\nu)+\nu t-t^{\nu}}{\nu\log t}.$$

Proof. It is easy to see that

$$\nu\log t\,\{L(t,1)-L(t^{\nu},1)\}=(\nu\log t)L(t,1)-(\nu\log t)L(t^{\nu},1)=\nu t-\nu-t^{\nu}+1,$$

which proves the equality of the statement. □

In the sequel, we need the following refinements and reverses of Young inequality.

(i) Kittaneh-Manasrah's inequality [11, 12]: For any $t>0$,

$$r(\sqrt{t}-1)^2 \leqslant (1-\nu)+\nu t-t^{\nu} \leqslant R(\sqrt{t}-1)^2, \tag{13}$$

where $r=\min\{\nu,1-\nu\}$, $R=\max\{\nu,1-\nu\}$, and $\nu\in[0,1]$.

(ii) Cartwright-Field's inequality [2]: For any $t>0$ and $0\leqslant\nu\leqslant1$,

$$\frac{1}{2}\nu(1-\nu)\frac{(t-1)^2}{\max\{t,1\}} \leqslant (1-\nu)+\nu t-t^{\nu} \leqslant \frac{1}{2}\nu(1-\nu)\frac{(t-1)^2}{\min\{t,1\}}. \tag{14}$$

(iii) Alzer-Fonseca-Kovačec's inequality [1]: For any $t>0$ and $0<\nu,\lambda<1$,

$$\frac{1}{2}\nu(1-\nu)\min\{t,1\}\log^2 t \leqslant (1-\nu)+\nu t-t^{\nu} \leqslant \frac{1}{2}\nu(1-\nu)\max\{t,1\}\log^2 t, \tag{15}$$

and

$$\min\left\{\frac{\nu}{\lambda},\frac{1-\nu}{1-\lambda}\right\}\left(\lambda t+(1-\lambda)-t^{\lambda}\right)\leqslant(1-\nu)+\nu t-t^{\nu}\leqslant\max\left\{\frac{\nu}{\lambda},\frac{1-\nu}{1-\lambda}\right\}\left(\lambda t+(1-\lambda)-t^{\lambda}\right). \tag{16}$$

In particular, if $\lambda=1-\nu$, in (16), then

$$\begin{aligned}&\min\left\{\frac{1-\nu}{\nu},\frac{\nu}{1-\nu}\right\}\left((1-\nu)t+\nu-t^{1-\nu}\right)\\&\leqslant(1-\nu)+\nu t-t^{\nu}\\&\leqslant\max\left\{\frac{\nu}{1-\nu},\frac{1-\nu}{\nu}\right\}\left((1-\nu)t+\nu-t^{1-\nu}\right).\end{aligned} \tag{17}$$

Employing the above inequalities together with Lemma 2.5, we get the following result:

PROPOSITION 2.6. *Let* $\nu\in(0,1)$. *If* $t>1$, *then*

$$\frac{r}{\nu}\frac{(\sqrt{t}-1)^2}{\log t}\leqslant L(t,1)-L(t^{\nu},1)\leqslant\frac{R}{\nu}\frac{(\sqrt{t}-1)^2}{\log t},$$

$$\frac{1-\nu}{2}\frac{(t-1)^2}{\max\{t,1\}\log t}\leqslant L(t,1)-L(t^{\nu},1)\leqslant\frac{1-\nu}{2}\frac{(t-1)^2}{\min\{t,1\}\log t},$$

$$\frac{1-\nu}{2}\min\{t,1\}\log t\leqslant L(t,1)-L(t^{\nu},1)\leqslant\frac{1-\nu}{2}\max\{t,1\}\log t,$$

and

$$\begin{aligned}&\frac{1-\nu}{\nu}\min\left\{\frac{\nu}{1-\nu},\frac{1-\nu}{\nu}\right\}\left(L(t,1)-L(t^{1-\nu},1)\right)\\&\leqslant L(t,1)-L(t^{\nu},1)\\&\leqslant\frac{1-\nu}{\nu}\max\left\{\frac{\nu}{1-\nu},\frac{1-\nu}{\nu}\right\}\left(L(t,1)-L(t^{1-\nu},1)\right).\end{aligned}$$

The reversed inequalities hold when $0<t<1$.

Proof. Using Lemma 2.5, we find

$$(1-\nu)+\nu t-t^{\nu}=\nu\log t\left(L(t,1)-L(t^{\nu},1)\right).$$

Replacing the expression $(1-\nu)+\nu t-t^{\nu}$ in inequalities (13), (14), (15), and (17), we deduce the inequalities from the statement. □

We can obtain the alternative expression of the difference between the weighted arithmetic mean and the weighted geometric mean by the weighted logarithmic mean and the logarithmic mean. Related to this, we state the following lemma.

LEMMA 2.7. *Let* $t \in [0,1) \cup (1,\infty)$ *and* $\nu \in (0,1)$. *Then*

$$L_\nu(1,t) - L(1,t) = \frac{(2\nu-1)}{\nu(1-\nu)\log t}\left\{(1-\nu)+\nu t - t^\nu\right\}.$$

Proof. Making the difference between the weighted logarithmic mean and the logarithmic mean of t and 1, we have:

$$\begin{aligned} L_\nu(1,t) - L(1,t) &= \frac{1}{\log t}\left(\frac{1-\nu}{\nu}(t^\nu - 1) + \frac{\nu}{1-\nu}(t-t^\nu) - t + 1\right) \\ &= \frac{1}{\log t}\left\{\left(\frac{1-\nu}{\nu} - \frac{\nu}{1-\nu}\right)t^\nu + \left(\frac{\nu}{1-\nu} - 1\right)t + 1 - \frac{1-\nu}{\nu}\right\} \\ &= \frac{1-2\nu}{\log t}\left\{\frac{1}{\nu(1-\nu)}t^\nu - \frac{1}{1-\nu}t - \frac{1}{\nu}\right\} \\ &= \frac{1-2\nu}{\nu(1-\nu)\log t}\left\{t^\nu - \nu t - (1-\nu)\right\} \end{aligned}$$

for all $t > 0$, $t \neq 1$ and $\nu \in (0,1)$. $\square$

REMARK 2.8. Using Lemma 2.7, we can obtain similar results like Proposition 2.6 with the help of inequalities (13), (14), (15), and (17). However, we leave them for interested readers.

Inequality (2) can be improved by using Theorem 1.1. Indeed, we have:

THEOREM 2.9. *Let* $a,b > 0$, $a \neq b$. *Then for any* $\nu \in (0,1)$,

$$\begin{aligned} a\sharp_\nu b &\leqslant \left(a\sharp_{\frac{3\nu}{4}} b\right)\nabla_\nu\left(a\sharp_{\frac{1+3\nu}{4}} b\right) \\ &\leqslant \left(\sqrt{a}\sharp_\nu\sqrt{b}\right)L_\nu\left(\sqrt{a},\sqrt{b}\right) \\ &\leqslant (a\sharp_\nu b)\nabla\left(\left(a\sharp_{\frac{\nu}{2}} b\right)\nabla_\nu\left(a\sharp_{\frac{1+\nu}{2}} b\right)\right) \\ &\leqslant \left(a\sharp_{\frac{\nu}{2}} b\right)\nabla_\nu\left(a\sharp_{\frac{1+\nu}{2}} b\right) \\ &\leqslant L_\nu(a,b) \\ &\leqslant (a\sharp_\nu b)\nabla(a\nabla_\nu b) \\ &\leqslant a\nabla_\nu b. \end{aligned}$$

Proof. We set $a = 0$, $b = 1$, and $f(\lambda) = t^\lambda$, $(t > 0)$ in Theorem 1.1. Then we have

$$t^\nu \leqslant \mathfrak{R}^{(1)}_{t^\lambda,\nu}(0,1) \leqslant \mathfrak{C}_{t^\lambda,\nu}(0,1) \leqslant \mathfrak{R}^{(2)}_{t^\lambda,\nu}(t^\lambda,1) \leqslant \nu t + (1-\nu),$$

where

$$\mathfrak{R}^{(1)}_{t^\lambda,\nu}(0,1) = t^{\frac{\nu}{2}}\nabla_\nu t^{\frac{1+\nu}{2}} = (1-\nu)t^{\frac{\nu}{2}} + \nu t^{\frac{1+\nu}{2}},$$

and

$$\mathfrak{R}^{(2)}_{t^{\lambda},\nu}(0,1) = (\nu t + (1-\nu))\nabla t^{\nu} = \frac{1}{2}[t^{\nu} + (1-\nu) + \nu t].$$

That is, we obtain

$$t^{\nu} \leqslant (1-\nu)t^{\frac{\nu}{2}} + \nu t^{\frac{1+\nu}{2}} \leqslant f_{\nu}(t) \leqslant \frac{1}{2}\left(t^{\nu} + (1-\nu) + \nu t\right) \leqslant \nu t + (1-\nu) \tag{18}$$

for all $t \in [0,1) \cup (1,\infty)$ and $\nu \in (0,1)$.

If we replace t by $t^{\frac{1}{2}}$ in inequality (18), then we deduce the following sequence of inequalities:

$$t^{\frac{\nu}{2}} \leqslant (1-\nu)t^{\frac{\nu}{4}} + \nu t^{\frac{1+\nu}{4}} \leqslant f_{\nu}(t^{\frac{1}{2}}) \leqslant \frac{1}{2}\left(t^{\frac{\nu}{2}} + (1-\nu) + \nu t^{\frac{1}{2}}\right) \leqslant \nu t^{\frac{1}{2}} + (1-\nu).$$

Multiplying by $t^{\frac{\nu}{2}}$ the above sequence of inequalities, we have

$$t^{\nu} \leqslant (1-\nu)t^{\frac{3\nu}{4}} + \nu t^{\frac{1+3\nu}{4}} \leqslant t^{\frac{\nu}{2}} f_{\nu}(t^{\frac{1}{2}}) \leqslant \frac{1}{2}\left(t^{\nu} + (1-\nu)t^{\frac{\nu}{2}} + \nu t^{\frac{\nu+1}{2}}\right) \leqslant \nu t^{\frac{\nu+1}{2}} + (1-\nu)t^{\frac{\nu}{2}},$$

for all $t \in (0,1) \cup (1,\infty)$ and $\nu \in (0,1)$. From the first and the second inequalities in (18), we find

$$\frac{1}{2}\left(t^{\nu} + (1-\nu)t^{\frac{\nu}{2}} + \nu t^{\frac{1+\nu}{2}}\right) \leqslant (1-\nu)t^{\frac{\nu}{2}} + \nu t^{\frac{1+\nu}{2}} \leqslant f_{\nu}(t).$$

Thus we have the inequalities

$$\begin{aligned} t^{\nu} &\leqslant (1-\nu)t^{\frac{3\nu}{4}} + \nu t^{\frac{1+3\nu}{4}} \\ &\leqslant t^{\frac{\nu}{2}} f_{\nu}\left(t^{\frac{1}{2}}\right) \\ &\leqslant \frac{1}{2}\left(t^{\nu} + (1-\nu)t^{\frac{\nu}{2}} + \nu t^{\frac{1+\nu}{2}}\right) \\ &\leqslant (1-\nu)t^{\frac{\nu}{2}} + \nu t^{\frac{1+\nu}{2}} \\ &\leqslant f_{\nu}(t) \\ &\leqslant \frac{1}{2}\left(t^{\nu} + (1-\nu) + \nu t\right) \\ &\leqslant (1-\nu) + \nu t. \end{aligned} \tag{19}$$

Putting $t = \frac{b}{a} \neq 1$ in inequalities (19) and multiplying by a to both sides, we deduce the sequence of inequalities. $\square$

The following corollary gives an interpolation between the weighted geometric mean and the weighted logarithmic mean by the self-improving inequality technique.

COROLLARY 2.10. *Let $m \in \mathbb{N}$ and $0 < \nu < 1$. Then for any $t > 0$,*

$$\begin{aligned} t^{\nu} \leqslant \cdots \leqslant t^{\left(1-\frac{1}{2^m}\right)\nu} f_{\nu}\left(t^{\frac{1}{2^m}}\right) \leqslant t^{\left(1-\frac{1}{2^{m-1}}\right)\nu} f_{\nu}\left(t^{\frac{1}{2^{m-1}}}\right) \leqslant \cdots \\ \leqslant t^{\left(1-\frac{1}{4}\right)\nu} f_{\nu}\left(t^{\frac{1}{4}}\right) \leqslant t^{\frac{\nu}{2}} f_{\nu}\left(t^{\frac{1}{2}}\right) \leqslant f_{\nu}(t), \end{aligned} \tag{20}$$

where the function $f_{\nu}(t)$ is defined as in (7).

Proof. It is sufficient to prove the third inequality in (20) for any $m \in \mathbb{N}$. In the process of the proof of Theorem 2.9, we found the inequality $t^{\frac{\nu}{2}} f_\nu\left(t^{\frac{1}{2}}\right) \leqslant f_\nu(t)$ for $t > 0$ and $0 < \nu < 1$. In this inequality, we put $t = s^{\frac{1}{2^{m-1}}}$. Then we have $s^{\frac{\nu}{2^m}} f_\nu\left(s^{\frac{1}{2^m}}\right) \leqslant f_\nu\left(s^{\frac{1}{2^{m-1}}}\right)$. Multiplying $s^{\frac{2^{m-1}-1}{2^{m-1}}\nu}$ to both sides of this inequality, we get the third inequality in (20). Taking $m \to \infty$, we have $t^{\left(1-\frac{1}{2^m}\right)\nu} f_\nu\left(t^{\frac{1}{2^m}}\right) \to t^\nu$, since $f_\nu(1) = \lim_{t \to 1} f_\nu(t) = 1$. $\square$

Before expressing the next result, we recall an interesting inequality for convex functions [3]: If f is a convex function on the interval $J \subseteq \mathbb{R}$, then for any $x, y \in J$,

$$2r\left(f(x)\nabla f(y) - f(x\nabla y)\right) \leqslant f(x)\nabla_t f(y) - f(x\nabla_t y) \tag{21}$$

holds, where $r = \min\{t, 1-t\}$ and $0 \leqslant t \leqslant 1$. In the same paper, it has been shown that

$$f(x)\nabla_t f(y) - f(x\nabla_t y) \leqslant 2R\left(f(x)\nabla f(y) - f(x\nabla y)\right) \tag{22}$$

where $R = \max\{t, 1-t\}$.

The following theorem provides an improvement and a reverse for the first inequality in (5), with the help of (21) and (22).

THEOREM 2.11. *Let $f : [a,b] \to \mathbb{R}$ be a convex function. Then for any $0 \leqslant \nu \leqslant 1$,*

$$2r\int_0^1 \left(\left(f(a\nabla_{\nu\lambda} b)\nabla f\left(b\nabla_{(1-\nu)\lambda} a\right)\right) - f\left(a\nabla_{\frac{1+\lambda(2\nu-1)}{2}} b\right)\right) d\lambda \leqslant \mathfrak{C}_{f,\nu}(a,b) - f(a\nabla_\nu b),$$

and

$$\mathfrak{C}_{f,\nu}(a,b) - f(a\nabla_\nu b) \leqslant 2R\int_0^1 \left(\left(f(a\nabla_{\nu\lambda} b)\nabla f\left(b\nabla_{(1-\nu)\lambda} a\right)\right) - f\left(a\nabla_{\frac{1+\lambda(2\nu-1)}{2}} b\right)\right) d\lambda,$$

where $r = \min\{\nu, 1-\nu\}$ and $R = \max\{\nu, 1-\nu\}$.

Proof. By substituting $x = a\nabla_{\nu\lambda} b$ and $y = b\nabla_{(1-\nu)\lambda} a$, in (21), we obtain

$$\begin{aligned} & f\left((a\nabla_{\nu\lambda} b)\nabla_\nu \left(b\nabla_{(1-\nu)\lambda} a\right)\right) \\ & \leqslant f(a\nabla_{\nu\lambda} b)\nabla_\nu f\left(b\nabla_{(1-\nu)\lambda} a\right) - 2r\Big(f(a\nabla_{\nu\lambda} b)\nabla f\left(b\nabla_{(1-\nu)\lambda} a\right) \\ & \quad - f\left((a\nabla_{\nu\lambda} b)\nabla \left(b\nabla_{(1-\nu)\lambda} a\right)\right)\Big). \end{aligned}$$

Since $(a\nabla_{\nu\lambda} b)\nabla_\nu \left(b\nabla_{(1-\nu)\lambda} a\right) = (a\nabla_{\nu\lambda} b)\nabla_\nu \left(a\nabla_{1-(1-\nu)\lambda} b\right) = a\nabla_{(1-\nu)\nu\lambda+\nu(1-(1-\nu)\lambda)} b = a\nabla_\nu b$, we have

$$f(a\nabla_\nu b) = f\left((a\nabla_{\nu\lambda} b)\nabla_\nu \left(b\nabla_{(1-\nu)\lambda} a\right)\right).$$

Consequently, we prove

$$f(a\nabla_v b) \leqslant f(a\nabla_{v\lambda} b)\nabla_v f\left(b\nabla_{(1-v)\lambda} a\right) \\ -2r\left(f(a\nabla_{v\lambda} b)\nabla f\left(b\nabla_{(1-v)\lambda} a\right) - f\left(a\nabla_{\frac{1+(2v-1)\lambda}{2}} b\right)\right).$$

By taking integral over $\lambda \in [0,1]$, we reach to

$$f(a\nabla_v b) \leqslant \int_0^1 \left(f(a\nabla_{v\lambda} b)\nabla_v f\left(b\nabla_{(1-v)\lambda} a\right)\right) d\lambda \\ -2r\int_0^1 \left(f(a\nabla_{v\lambda} b)\nabla f\left(b\nabla_{(1-v)\lambda} a\right) - f\left(a\nabla_{\frac{1+(2v-1)\lambda}{2}} b\right)\right) d\lambda,$$

which is the first inequality. The second inequality follows likewise by employing inequality (22) instead of inequality (21). □

REMARK 2.12. Note that

$$f(a\nabla_{v\lambda} b)\nabla f\left(b\nabla_{(1-v)\lambda} a\right) - f\left(a\nabla_{\frac{1+\lambda(2v-1)}{2}} b\right) \geqslant 0, \quad (0 \leqslant v, \lambda \leqslant 1,\ a,b > 0)$$

by the convexity of f.

The following corollary gives a refinement and a reverse for the inequality $a^{1-v}b^v \leqslant L_v(a,b)$.

COROLLARY 2.13. *Let $a,b > 0$ and $0 < v < 1$ with $v \neq 1/2$. Then*

$$\frac{r}{\log b - \log a}\left(\frac{a\sharp_v b - a}{v} + \frac{b - a\sharp_v b}{1-v} - \frac{4(a\sharp b - a\sharp_v b)}{1-2v}\right) \leqslant L_v(a,b) - a\sharp_v b, \tag{23}$$

and

$$L_v(a,b) - a\sharp_v b \leqslant \frac{R}{\log b - \log a}\left(\frac{a\sharp_v b - a}{v} + \frac{b - a\sharp_v b}{1-v} - \frac{4(a\sharp b - a\sharp_v b)}{1-2v}\right), \tag{24}$$

where $r = \min\{v, 1-v\}$ and $R = \max\{v, 1-v\}$. In the limit of $v \to 1/2$, both sides in inequalities (23) and (24) coincide.

Proof. Letting $f(t) = e^t$ in Theorem 2.11. A simple calculation reveals that

$$\int_0^1 e^{a\nabla_{v\lambda} b} d\lambda = \frac{e^{a\nabla_v b} - e^a}{v(b-a)}, \quad \int_0^1 e^{b\nabla_{(1-v)\lambda} a} d\lambda = \frac{e^b - e^{a\nabla_v b}}{(1-v)(b-a)},$$

and

$$\int_0^1 e^{a\nabla_{\frac{1+\lambda(2v-1)}{2}} b} d\lambda = \frac{2\left(e^{a\nabla b} - e^{a\nabla_v b}\right)}{(1-2v)(b-a)}.$$

Hence we have

$$r\left(\frac{e^{a\nabla_\nu b}-e^a}{\nu(b-a)}+\frac{e^b-e^{a\nabla_\nu b}}{(1-\nu)(b-a)}-\frac{4\left(e^{a\nabla b}-e^{a\nabla_\nu b}\right)}{(1-2\nu)(b-a)}\right)$$
$$\leqslant\left(\frac{e^{a\nabla_\nu b}-e^a}{\nu(b-a)}\right)\nabla_\nu\left(\frac{e^b-e^{a\nabla_\nu b}}{(1-\nu)(b-a)}\right)-e^{a\nabla_\nu b},$$

and

$$\left(\frac{e^{a\nabla_\nu b}-e^a}{\nu(b-a)}\right)\nabla_\nu\left(\frac{e^b-e^{a\nabla_\nu b}}{(1-\nu)(b-a)}\right)-e^{a\nabla_\nu b}$$
$$\leqslant R\left(\frac{e^{a\nabla_\nu b}-e^a}{\nu(b-a)}+\frac{e^b-e^{a\nabla_\nu b}}{(1-\nu)(b-a)}-\frac{4\left(e^{a\nabla b}-e^{a\nabla_\nu b}\right)}{(1-2\nu)(b-a)}\right).$$

We obtain desired inequalities by replacing e^a and e^b by a and b in the above two inequalities.

Finally, we quickly find that

$$\lim_{\nu\to 1/2}\frac{4\left(a\sharp b-a\sharp_\nu b\right)}{1-2\nu}=\lim_{\nu\to 1/2}\frac{4(a\sharp_\nu b)\left(\log a-\log b\right)}{-2}=2(a\sharp b)\left(\log b-\log a\right),$$

which implies the last statement by simple calculations. □

COROLLARY 2.14. *Let $f:[a,b]\to\mathbb{R}$ be a convex function. Then for any $0\leqslant\nu\leqslant 1$,*

$$f(a\nabla_\nu b)\leqslant f(a)\nabla_\nu f(b)-2r\int_0^1\left(\left(f(a\nabla_{\nu\lambda}b)\nabla f\left(b\nabla_{(1-\nu)\lambda}a\right)\right)-f\left(a\nabla_{\frac{1+\lambda(2\nu-1)}{2}}b\right)\right)d\lambda,$$

where $r=\min\{\nu,1-\nu\}$.

Proof. By the second inequality in (5), we understand that

$$\mathfrak{C}_{f,\nu}(a,b)\leqslant f(a)\nabla_\nu f(b).$$

Combining this with Theorem 2.11 finishes the proof. □

The following result improves the second inequality in (4).

COROLLARY 2.15. *Let $f:[a,b]\to\mathbb{R}$ be a convex function. Then*

$$\int_0^1 f(a\nabla_\nu b)\,d\nu$$
$$\leqslant f(a)\nabla f(b)$$
$$-\int_0^1\left((1-|2\nu-1|)\int_0^1\left(\left(f(a\nabla_{\nu\lambda}b)\nabla f\left(b\nabla_{(1-\nu)\lambda}a\right)\right)-f\left(a\nabla_{\frac{1+\lambda(2\nu-1)}{2}}b\right)\right)d\lambda\right)d\nu.$$

Proof. Since $2\min\{\nu,1-\nu\}=1-|2\nu-1|$, we infer from Corollary 2.14 that

$$\begin{aligned}
&f(a\nabla_\nu b)\\
&\leqslant f(a)\nabla_\nu f(b)\\
&\quad-(1-|2\nu-1|)\int_0^1\left(\left(f(a\nabla_{\nu\lambda}b)\nabla f\left(b\nabla_{(1-\nu)\lambda}a\right)\right)-f\left(a\nabla_{\frac{1+\lambda(2\nu-1)}{2}}b\right)\right)d\lambda.
\end{aligned}$$

We obtain the desired result if we take integral over $\nu\in[0,1]$. $\square$

REMARK 2.16. The case $\nu=1/2$ in Corollary 2.14, recovers the second inequality of (4). Indeed,

$$\begin{aligned}
f(a\nabla b)&\leqslant f(a)\nabla f(b)-\left(\int_0^1\left(f\left(a\nabla_{\frac{\lambda}{2}}b\right)\nabla f\left(b\nabla_{\frac{\lambda}{2}}a\right)-f(a\nabla b)\right)d\lambda\right)\\
&=f(a)\nabla f(b)-\int_0^1\left(f\left(a\nabla_{\frac{\lambda}{2}}b\right)\nabla f\left(b\nabla_{\frac{\lambda}{2}}a\right)\right)d\lambda+f(a\nabla b)\\
&=f(a)\nabla f(b)-\int_0^1\left(f\left(a\nabla_{\frac{\lambda}{2}}b\right)\nabla f\left(a\nabla_{1-\frac{\lambda}{2}}b\right)\right)d\lambda+f(a\nabla b).
\end{aligned}$$

Equalities

$$\int_0^1 f\left(a\nabla_{\frac{\lambda}{2}}b\right)d\lambda=2\int_0^{1/2}f(a\nabla_x b)\,dx$$

and

$$\int_0^1 f\left(a\nabla_{1-\frac{\lambda}{2}}b\right)d\lambda=2\int_{1/2}^{1}f(a\nabla_x b)\,dx$$

imply

$$\int_0^1\left(f\left(a\nabla_{\frac{\lambda}{2}}b\right)\nabla f\left(a\nabla_{1-\frac{\lambda}{2}}b\right)\right)d\lambda=\int_0^1 f(a\nabla_\lambda b)\,d\lambda.$$

Thus we have

$$\int_0^1 f(a\nabla_\lambda b)\,d\lambda\leqslant f(a)\nabla f(b).$$

The following result gives a refinement of the second inequality in (5).

THEOREM 2.17. *Let $f:[a,b]\to\mathbb{R}$ be a convex function. Then for any $0\leqslant\nu\leqslant 1$,*

$$\begin{aligned}
2\widetilde{r}(\nu)\left(f(a)\nabla f(b)-f(a\nabla b)\right)&\leqslant f(a)\nabla_\nu f(b)-\mathfrak{C}_{f,\nu}(a,b)\\
&\leqslant 2\widetilde{R}(\nu)\left(f(a)\nabla f(b)-f(a\nabla b)\right)
\end{aligned}$$

where

$$\widetilde{r}(v) := \int_0^1 (r_1 \nabla_v r_2)\, d\lambda, \quad \widetilde{R}(v) := \int_0^1 (R_1 \nabla_v R_2)\, d\lambda,$$

$r_1 = \min\{v\lambda, 1 - v\lambda\}$, $r_2 = \min\{(1-v)\lambda, 1-(1-v)\lambda\}$, $R_1 = \max\{v\lambda, 1 - v\lambda\}$, *and* $R_2 = \max\{(1-v)\lambda, 1-(1-v)\lambda\}$.

Proof. If we take $\alpha = f(a)$ and $\beta = f(b)$, in the equality $(\alpha \nabla_{v\lambda} \beta) \nabla_v \left(\beta \nabla_{(1-v)\lambda} \alpha\right) = \alpha \nabla_v \beta$, we deduce $(f(a) \nabla_{v\lambda} f(b)) \nabla_v \left(f(b) \nabla_{(1-v)\lambda} f(a)\right) = f(a) \nabla_v f(b)$, which implies the following:

$$\begin{aligned} & f(a)\nabla_v f(b) - f(a\nabla_{v\lambda} b)\nabla_v f\left(b\nabla_{(1-v)\lambda} a\right) \\ & = (f(a)\nabla_{v\lambda} f(b))\nabla_v \left(f(b)\nabla_{(1-v)\lambda} f(a)\right) - f(a\nabla_{v\lambda} b)\nabla_v f\left(b\nabla_{(1-v)\lambda} a\right) \\ & = (f(a)\nabla_{v\lambda} f(b) - f(a\nabla_{v\lambda} b))\nabla_v \left(f(b)\nabla_{(1-v)\lambda} f(a) - f\left(b\nabla_{(1-v)\lambda} a\right)\right). \end{aligned} \tag{25}$$

If we replace t by $v\lambda$ in (21) and (22), then we deduce

$$\begin{aligned} 2r_1 (f(a)\nabla f(b) - f(a\nabla b)) & \leqslant f(a)\nabla_{v\lambda} f(b) - f(a\nabla_{v\lambda} b) \\ & \leqslant 2R_1 (f(a)\nabla f(b) - f(a\nabla b)) \end{aligned} \tag{26}$$

where $r_1 = \min\{v\lambda, 1 - v\lambda\}$ and $R_1 = \max\{v\lambda, 1 - v\lambda\}$. In the same manner, if we replace t by $(1-v)\lambda$ in (21) and (22), then we obtain

$$2r_2 (f(a)\nabla f(b) - f(a\nabla b)) \leqslant f(b)\nabla_{(1-v)\lambda} f(a) - f\left(b\nabla_{(1-v)\lambda} a\right) \tag{27}$$

$$\leqslant 2R_2 (f(a)\nabla f(b) - f(a\nabla b)) \tag{28}$$

where $r_2 = \min\{(1-v)\lambda, 1-(1-v)\lambda\}$ and $R_2 = \max\{(1-v)\lambda, 1-(1-v)\lambda\}$. Using equality (25) and inequalities (26) and (27), we find the following inequality

$$\begin{aligned} & 2(r_1\nabla_v r_2)(f(a)\nabla f(b) - f(a\nabla b)) \\ & \leqslant (f(a)\nabla_{v\lambda} f(b) - f(a\nabla_{v\lambda} b))\nabla_v \left(f(b)\nabla_{(1-v)\lambda} f(a) - f\left(b\nabla_{(1-v)\lambda} a\right)\right) \\ & \leqslant 2(R_1\nabla_v R_2)(f(a)\nabla f(b) - f(a\nabla b)). \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} & 2(r_1\nabla_v r_2)(f(a)\nabla f(b) - f(a\nabla b)) \\ & \leqslant f(a)\nabla_v f(b) - f(a\nabla_{v\lambda} b)\nabla_v f\left(b\nabla_{(1-v)\lambda} a\right) \\ & \leqslant 2(R_1\nabla_v R_2)(f(a)\nabla f(b) - f(a\nabla b)). \end{aligned}$$

By taking integral over $\lambda \in [0,1]$, we deduce the inequalities of the statement. □

Finding a maximum value of $\widetilde{r}(v)$ and a minimum value of $\widetilde{R}(v)$ for $0 < v < 1$, we state the following corollary.

COROLLARY 2.18. *Let $f:[a,b]\to\mathbb{R}$ be a convex function. Then for any $0<\nu<1$,*

$$\frac{1}{2}\left(f(a)\nabla f(b)-f(a\nabla b)\right)\leqslant f(a)\nabla_{\nu}f(b)-\mathfrak{C}_{f,\nu}(a,b)\leqslant\frac{3}{2}\left(f(a)\nabla f(b)-f(a\nabla b)\right).$$

Proof. To calculate the constants $\widetilde{r}(\nu)$ and $\widetilde{R}(\nu)$ appeared in Theorem 2.17, notice that

$$\int_0^1 r_1 d\lambda=-\frac{(2\nu-1)|2\nu-1|-4\nu+1}{8\nu}\quad\text{and}\quad\int_0^1 r_2 d\lambda=-\frac{(2\nu-1)|2\nu-1|-4\nu+3}{8(\nu-1)},$$

$$\int_0^1 R_1 d\lambda=\frac{(2\nu-1)|2\nu-1|+4\nu+1}{8\nu}\quad\text{and}\quad\int_0^1 R_2 d\lambda=\frac{(2\nu-1)|2\nu-1|+4\nu-5}{8(\nu-1)}.$$

Accordingly,

$$\widetilde{r}(\nu)=\frac{(2\nu-1)^2|2\nu-1|+6\nu(1-\nu)-1}{8\nu(1-\nu)},$$

and for $0<\nu<1$

$$\widetilde{R}(\nu)=\frac{1+2\nu(1-\nu)-(2\nu-1)^2|2\nu-1|}{8\nu(1-\nu)}.$$

One can easily check that

$$\frac{d\widetilde{r}(\nu)}{d\nu}=-\frac{(2\nu-1)((2\nu(\nu-1)-1)|2\nu-1|+1)}{8\nu^2(1-\nu)^2}$$

and for $0<\nu<1$

$$\frac{\widetilde{r}(\nu)}{d\nu}=0\Longleftrightarrow\nu=\frac{1}{2}.$$

A direct computation shows that

$$\begin{cases}\widetilde{r}'(\nu)>0; & \text{if } 0<\nu<\frac{1}{2},\\ \widetilde{r}'(\nu)<0; & \text{if } \frac{1}{2}<\nu<1.\end{cases}$$

Notice that $\max\limits_{0<\nu<1}\widetilde{r}(\nu)=1/4$, for $\nu=1/2$. Besides,

$$\frac{\widetilde{R}(\nu)}{d\nu}=\frac{(2\nu-1)((2\nu(\nu-1)-1)|2\nu-1|+1)}{8\nu^2(1-\nu)^2}$$

and

$$\frac{\widetilde{R}(\nu)}{d\nu}=0\Longleftrightarrow\nu=\frac{1}{2}.$$

Direct calculations show that

$$\begin{cases} \widetilde{R}'(v) < 0; & \text{if } 0 < v < \dfrac{1}{2}, \\ \widetilde{R}'(v) > 0; & \text{if } \dfrac{1}{2} < v < 1. \end{cases}$$

Notice that $\min\limits_{0<v<1} \widetilde{R}(v) = 3/4$, for $v = 1/2$. We thus have inequalities in this corollary. □

REMARK 2.19. If we take $v = 1/2$ in Corollary 2.18, then we have

$$\frac{1}{2}(f(a)\nabla f(b) - f(a\nabla b)) \leqslant f(a)\nabla f(b) - \int_0^1 f(a\nabla_\lambda b)d\lambda \leqslant \frac{3}{2}(f(a)\nabla f(b) - f(a\nabla b)),$$

which is equivalent to

$$\frac{3}{2}f(a\nabla b) - \frac{1}{2}f(a)\nabla f(b) \leqslant \int_0^1 f(a\nabla_\lambda b)d\lambda \leqslant \frac{1}{2}f(a\nabla b) + \frac{1}{2}f(a)\nabla f(b).$$

The second inequality above represents Bullen's inequality (see, e.g., [14], [16]). Naturally, the second inequality above improves the second inequality in (4), since Theorem 2.17 improves the second inequality of (5) in a more general form.

To state the following corollary, we review Kantorovich constant $K(a,b) := \dfrac{(a+b)^2}{4ab}$ and the weighted identric mean introduced in [15]:

$$I_v(a,b) := \frac{1}{e}(a\nabla_v b)^{\frac{(1-2v)(a\nabla_v b)}{v(1-v)(b-a)}} \left(\frac{b^{\frac{vb}{1-v}}}{a^{\frac{(1-v)a}{v}}} \right)^{\frac{1}{b-a}}.$$

COROLLARY 2.20. *Let* $a,b > 0$ *and* $0 < v < 1$. *Then*

$$\widetilde{r}(v)\left(\sqrt{a} - \sqrt{b}\right)^2 \leqslant a\nabla_v b - L_v(a,b) \leqslant \widetilde{R}(v)\left(\sqrt{a} - \sqrt{b}\right)^2 \tag{29}$$

and

$$K(a,b)^{\widetilde{r}(v)} a\sharp_v b \leqslant I_v(a,b) \leqslant K(a,b)^{\widetilde{R}(v)} a\sharp_v b, \tag{30}$$

where $\tilde{r}(v)$, $\tilde{R}(v)$, r_1, r_2, R_1, *and* R_2 *are defined as in Theorem* 2.17.

Proof. The results follow from in Theorem 2.17, by take $f(t) = e^t$ and $f(t) = -\log t$, respectively. We note that $\mathfrak{C}_{e^x,v}(\log a, \log b) = L_v(a,b)$ and $\mathfrak{C}_{-\log x,v}(a,b) = \log \dfrac{1}{I_v(a,b)}$. The latter is due to

$$\int_0^1 f\left(b\nabla_{(1-v)(1-\lambda)}a\right)d\lambda = \int_0^1 f\left(b\nabla_{(1-v)\mu}a\right)d\mu. \quad \square$$

Note that $\tilde{r}(v) \geqslant 0$ and $K(a,b) \geqslant 1$. The first inequality and the second inequality in (29) respectively give a refinement and reverse of the inequality $L_v(a,b) \leqslant a\nabla_v b$. Also the first inequality and the second inequality in (30) respectively a refinement and a reverse of the inequality $a\sharp_v b \leqslant I_v(a,b)$.

We reach the following result by combining Theorems 2.11 and 2.17.

COROLLARY 2.21. *Let $f:[a,b]\to\mathbb{R}$ be a convex function. Then for any $0\leqslant v\leqslant 1$,*

$$
\begin{aligned}
&2\left(r\int_0^1\left(\left(f\left(a\nabla_{v\lambda}b\right)\nabla f\left(b\nabla_{(1-v)\lambda}a\right)\right)-f\left(a\nabla_{\frac{1+\lambda(2v-1)}{2}}b\right)\right)d\lambda\right.\\
&\left.\quad+\widetilde{r}(v)\left(f(a)\nabla f(b)-f(a\nabla b)\right)\right)\\
&\leqslant f(a)\nabla_v f(b)-f(a\nabla_v b)\\
&\leqslant 2\left(R\int_0^1\left(\left(f\left(a\nabla_{v\lambda}b\right)\nabla f\left(b\nabla_{(1-v)\lambda}a\right)\right)-f\left(a\nabla_{\frac{1+\lambda(2v-1)}{2}}b\right)\right)d\lambda\right.\\
&\left.\quad+\widetilde{R}(v)\left(f(a)\nabla f(b)-f(a\nabla b)\right)\right)
\end{aligned}
$$

where $r=\min\{v,1-v\}$, $R=\max\{v,1-v\}$, and $\tilde{r}(v)$, $\tilde{R}(v)$ are defined as in Theorem 2.17.

REMARK 2.22. Letting $f(t)=-\log t$ in Corollary 2.21. Simple calculations reveal that

$$\int_0^1 -\log\left(a\nabla_{v\lambda}b\right)d\lambda = 1+\log\left(\frac{a^a}{\left(a\nabla_v b\right)^{a\nabla_v b}}\right)^{\frac{1}{v(b-a)}},$$

$$\int_0^1 -\log\left(b\nabla_{(1-v)\lambda}a\right)d\lambda = 1+\log\left(\frac{\left(a\nabla_v b\right)^{a\nabla_v b}}{b^b}\right)^{\frac{1}{(1-v)(b-a)}},$$

and

$$\int_0^1 -\log\left(a\nabla_{\frac{1+(2v-1)\lambda}{2}}b\right)d\lambda = 1+\log\left(\frac{\left(a\nabla b\right)^{a\nabla b}}{\left(a\nabla_v b\right)^{a\nabla_v b}}\right)^{\frac{2}{(2v-1)(b-a)}}.$$

Thus, we have

$$\left(\int_0^1 -\log\left(a\nabla_{\nu\lambda}b\right)d\lambda\right)\nabla\left(\int_0^1 -\log\left(b\nabla_{(1-\nu)\lambda}a\right)d\lambda\right)$$
$$-\int_0^1 -\log\left(a\nabla_{\frac{1+(2\nu-1)\lambda}{2}}b\right)d\lambda$$
$$=\log\left(\frac{a^{\frac{a}{\nu}}}{b^{\frac{b}{1-\nu}}}\right)^{\frac{1}{2(b-a)}}\left(\frac{(a\nabla_\nu b)^{\frac{(a\nabla_\nu b)}{\nu(1-\nu)(2\nu-1)}}}{(a\nabla b)^{\frac{4(a\nabla b)}{(2\nu-1)}}}\right)^{\frac{1}{2(b-a)}}\geqslant 0.$$

The last inequality is due to Remark 2.12. Therefore, we obtain in terms of Kantorovich constant

$$\alpha_\nu(a,b)^r K(a,b)^{\widetilde{r}(\nu)}\leqslant\frac{a\nabla_\nu b}{a\sharp_\nu b}\leqslant\alpha_\nu(a,b)^R K(a,b)^{\widetilde{R}(\nu)} \tag{31}$$

where $\tilde{r}(\nu)$, $\tilde{R}(\nu)$ are defined as in Theorem 2.17, $r=\min\{\nu,1-\nu\}$, $R=\max\{\nu,1-\nu\}$, and

$$\alpha_\nu(a,b):=\left(\frac{a^{\frac{a}{\nu}}}{b^{\frac{b}{1-\nu}}}\frac{(a\nabla_\nu b)^{\frac{(a\nabla_\nu b)}{\nu(1-\nu)(2\nu-1)}}}{(a\nabla b)^{\frac{4(a\nabla b)}{(2\nu-1)}}}\right)^{\frac{1}{b-a}}\geqslant 1, \tag{32}$$

where $\nu\in(0,1/2)\cup(1/2,1)$. We easily find that $\lim_{b\to a}\alpha_\nu(a,b)=1$ and

$$\lim_{\nu\to\frac{1}{2}}\alpha_\nu(a,b)=\left(e\,(a\nabla b)\left(\frac{a^a}{b^b}\right)^{\frac{1}{b-a}}\right)^2=\left(\frac{a\nabla b}{I_{1/2}(a,b)}\right)^2\geqslant 1. \tag{33}$$

Thus, the first and the second inequalities of (31), respectively, give a refinement and a reverse of the weighted arithmetic-geometric mean inequality.

In addition, (32) together with (33) give an upper bound for the weighted identric mean:

$$I_\nu(a,b)\leqslant\frac{1}{e}\left(\frac{a^a}{b^b}\frac{(a\nabla b)^{\frac{4(a\nabla b)}{1-2\nu}}}{(a\nabla_\nu b)^{\frac{4(a\nabla_\nu b)}{1-2\nu}}}\right)^{\frac{1}{b-a}},\quad\left(0<\nu<1,\quad\nu\neq\frac{1}{2}\right) \tag{34}$$

since

$$1\leqslant\alpha_\nu(a,b)=\frac{1}{I_\nu(a,b)}\frac{1}{e}\left(\frac{a^a}{b^b}\frac{(a\nabla b)^{\frac{4(a\nabla b)}{1-2\nu}}}{(a\nabla_\nu b)^{\frac{4(a\nabla_\nu b)}{1-2\nu}}}\right)^{\frac{1}{b-a}}.$$

Taking the limit of $\nu\to 1/2$ in (34), we have the bound of $I_{1/2}(a,b)$ in the following way.

$$I_{1/2}(a,b)\leqslant e(a\nabla b)^2\left(\frac{a^a}{b^b}\right)^{\frac{1}{b-a}}=\frac{(a\nabla b)^2}{I_{1/2}(a,b)}$$

which implies $I_{1/2}(a,b)\leqslant a\nabla b$. This fits the inequality in (33), understandably.

Declarations

Availability of data and materials. Not applicable.

Competing interests. The authors declare that they have no competing interests.

Funding. This research is supported by a grant (JSPS KAKENHI, Grant Number: 21K03341) awarded to the author, S. Furuichi.

Authors' contributions. Authors declare that they have contributed equally to this paper. All authors have read and approved this version.

Acknowledgement. The authors would like to thank the referees for their careful and insightful comments to improve our manuscript.

REFERENCES

[1] H. ALZER, C. M. DA FONSECA AND A. KOVAČEC, *Young-type inequalities and their matrix analogues*, Linear Multilinear Algebra., **63** (3) (2015), 622–635.

[2] D. I. CARTWRIGHT AND M. J. FIELD, *A refinement of the arithmetic mean-geometric mean inequality*, Proc. Am. Math. Soc., **71** (1978), 36–38.

[3] S. S. DRAGOMIR, *Bounds for the normalised Jensen functional*, Bull. Austral. Math. Soc., **3** (2006), 471–478.

[4] S. S. DRAGOMIR AND R. P. AGARWAL, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Appl. Math. Lett., **11** (5) (1998), 91–95.

[5] A. EL FARISSI, Z. LATREUCH AND B. BALAIDI, *Hadamard type inequalities for near convex functions*, Gazeta Matematica Seria A., **28** (157) (2010), 9–14.

[6] S. FURUICHI, K. YANAGI AND H. R. MORADI, *Mathematical inequalities on some weighted means*, J. Math. Inequal., to appear, **17** (2) (2023), 447–457.

[7] S. FURUICHI AND N. MINCULETE, *Refined inequalities on the weighted inequalities*, J. Math. Inequal., **14** (4) (2020), 1347–1357.

[8] S. FURUICHI AND N. MINCULETE, *Bounds for the differences between arithmetic and geometric means and their applications to inequalities*, Symmetry., **13** (12) (2021), https://doi.org/10.3390/sym13122398.

[9] S. FURUICHI AND H. R. MORADI, *Advances in Mathematical Inequalities*, De Gruyter, 2020.

[10] S. FURUICHI, H. R. MORADI AND M. SABABHEH, *New inequalities for interpolational operator means*, J. Math. Inequal., **15** (1) (2021), 107–116.

[11] F. KITTANEH AND Y. MANASRAH, *Improved Young and Heinz inequalities for matrices*, J. Math. Anal. Appl., **361** (2010), 262–269.

[12] F. KITTANEH AND Y. MANASRAH, *Reverse Young and Heinz inequalities for matrices*, Linear Multilinear Algebra., **59** (2011), 1031–1037.

[13] H. R. MORADI, M. SABABHEH AND S. FURUICHI, *On the operator Hermite-Hadamard inequality*, Complex Anal. Oper. Theory., **15**, 122 (2021), https://doi.org/10.1007/s11785-021-01172-w.

[14] C. P. NICULESCU AND L.-E. PERSSON, *Convex Functions and Their Applications: A Contemporary Approach*, Springer-Verlag, New York, 2006.

[15] R. PAL, M. SINGH, M. S. MOSLEHIAN AND J. S. AUJLA, *A new class of operator monotone functions via operator means*, Linear Multilinear Algebra., **64** (12) (2016), 2463–2473.

[16] J. PEČARIĆ, F. PROSCHAN AND Y. L. TONG, *Convex Functions, Partial Orderings and Statistical Applications*, Academic Press, Inc., 1992.

[17] M. SABABHEH, H. R. MORADI AND S. FURUICHI, *Integrals refining convex inequalities*, Bull. Malays. Math. Sci. Soc., **43** (2020), 2817–2833.

(Received December 4, 2022)

Shigeru Furuichi
Department of Information Science
College of Humanities and Sciences, Nihon University
Setagaya-ku, Tokyo, Japan
e-mail: furuichi@chs.nihon-u.ac.jp

Nicuşor Minculete
Faculty of Mathematics and Computer Science
Transilvania University
Iuliu Maniu street 50, 500091 Braşov, Romania
e-mail: minculeten@yahoo.com

Hamid Reza Moradi
Department of Mathematics
Mashhad Branch, Islamic Azad University
Mashhad, Iran
e-mail: hrmoradi.68@gmail.com