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DIMENSION–FREE ESTIMATES FOR HARDY–LITTLEWOOD MAXIMAL FUNCTIONS WITH MIXED HOMOGENEITIES

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Abstract. We mainly study the dimension-free L^p-inequality of the Hardy-Littlewood maximal functions with mixed homogeneities

$$M_*^G f(x,y) = \sup_{t>0} \frac{1}{|G|} \left| \int_G f(x-tu, y-t^2v)\,dudv \right|,$$

where G is a bounded, closed and symmetric convex subset of $\mathbb{R}^{d+1}$. When G is in the isotropic position, we prove that there is a constant C_p independent of d such that

$$\left\| M_*^G f \right\|_{L^p(\mathbb{R}^{d+1})} \leqslant C_p(L(G)) \|f\|_{L^p(\mathbb{R}^{d+1})},$$

for $\frac{3}{2} < p \leqslant \infty$, where $L(G)$ is a constant associated with G.

1. Introduction

The purpose of this paper is to develop a new dimension-free estimate of Hardy-Littlewood maximal functions with mixed homogeneities. We write $\mathbb{R}^{d+1} = \mathbb{R}^d \times \mathbb{R}$ with $(x,y) \in \mathbb{R}^{d+1}$, where $x \in \mathbb{R}^d$ and $y \in \mathbb{R}$. Let G be a convex centrally symmetric body in $\mathbb{R}^{d+1}$, which is also a bounded closed and centrally symmetric convex subset of $\mathbb{R}^{d+1}$ with non-empty interior. For every $t > 0$ and for every $(x,y) \in \mathbb{R}^{d+1}$, we call

$$\mathcal{M}_t^G f(x,y) = \frac{1}{|G|} \int_G f(x-tu, y-tv)\,dudv \tag{1.1}$$

the Hardy-Littlewood averaging function associated with isotropic homogeneity where $(x,y) \in \mathbb{R}^{d+1}$ and $(u,v) \in \mathbb{R}^{d+1}$. For $p \in (1,\infty]$, let $\mathcal{C}_p(d,G) > 0$ be the best constant such that the following maximal inequalities

$$\left\| \sup_{t>0} \left| \mathcal{M}_t^G f \right| \right\|_{L^p(\mathbb{R}^{d+1})} \leqslant \mathcal{C}_p(d,G) \|f\|_{L^p(\mathbb{R}^{d+1})}, \tag{1.2}$$

hold for every $f \in L^p(\mathbb{R}^{d+1})$. It is easy to see that (1.2) holds with $p = \infty$. Using a covering argument for $p = 1$ and a simple interpolation with $p = \infty$, we can obtain that

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$\mathcal{C}_p(d,G) < \infty$ for every $p \in (1,\infty]$ and for every convex symmetric body $G \subset \mathbb{R}^{d+1}$. However, the constant $\mathcal{C}_p(d,G)$ obtained by this method is bounded by an upper bound which depends on the dimension d.

The first dimension-free result for the Hardy-Littlewood maximal operator was obtained by Stein. In [1], he showed that if G is the Euclidean ball B^2, then $\mathcal{C}(d,B^2)$ is bounded independently of the dimension for every $p \in (1,\infty]$, see also [2] for more details. This inspired a lot of generalizations for Hardy-Littlewood maximal operators related to other convex bodies. Bourgain [3] showed that $\mathcal{C}_p(d,G)$ is bounded by an absolute constant, which is independent of the underling convex symmetric body $G \subset \mathbb{R}^{d+1}$ for $p = 2$. Later, Bourgain [4] extended this result for $p \in (\frac{3}{2},\infty]$. At the same time, Carbery [9] obtained the same result independently. Thus, mathematicians guess if $\mathcal{C}_p(d,G)$ can be bounded by a dimension-free constant for all $p \in (1,\infty]$. This result was proved by Müller [12] for the q-balls B^q, $q \in [1,\infty)$ and for cubes B^∞ by Bourgain [5]. In recent years, some interesting results were obtained by Bourgain, Mirek, Stein and Wróbel [6, 8, 7], where the authors proved the dimension-free estimate of discrete Hardy-Littlewood maximal operator defined over ball and cube. More about dimension-free estimates for the Hardy-Littlewood maximal functions can be found in [10, 11, 13].

Let P be a polynomial from $\mathbb{R}^{d+1}$ to $\mathbb{R}^{d+1}$ and fix a family of (possible non-isotropic) dilations

$$(x,y) \mapsto t\cdot(x,y) = (t^{\lambda_1}x_1,\ldots,t^{\lambda_d}x_d,t^{\lambda_{d+1}}x_{d+1}),$$

with λ_1, ..., $\lambda_{d+1} > 0$. Then the maximal operator M_P on $\mathbb{R}^{d+1}$ can be defined as

$$M_P f(x,y) = \sup_{t>0} \frac{1}{|B^2|} \left| \int_{B^2} f((x,y) - P(t\cdot(u,v)))\,dudv \right|.$$

In [14], Stein pointed that M_P is bounded on $L^p(\mathbb{R}^{d+1})$. Thus we want to study the dimension-free estimate of M_P. In this paper, we mainly pay our attention to the special case $\lambda_1 = \cdots \lambda_d = 1$, $\lambda_{d+1} = 2$ and $P(x,y) = (x,y)$.

DEFINITION 1.1. Let G be central symmetric convex set and f a locally integrable function defined on $\mathbb{R}^{d+1}$. Then

$$M_t^G f(x,y) = \frac{1}{|G|}\int_G f(x-tu, y-t^2v)\,dudv \tag{1.3}$$

is called the Hardy-Littlewood averaging function with mixed homogeneities. Correspondingly, we called

$$M_*^G f(x,y) = \sup_{t>0} \left| M_t^G f(x,y) \right|.$$

the Hardy-Littlewood maximal function associated with mixed homogeneities.

Obviously, M_*^G is bounded on $L^p(\mathbb{R}^{d+1})$ for $p > 1$. A convex symmetric body $G \subset \mathbb{R}^{d+1}$ is called in the isotropic position, if it has Lebesgue measure $|G| = 1$, and there is a constant $L = L(G) > 0$ which depends on G such that

$$\int_G \langle x, \xi \rangle^2 dx = L(G)^2 |\xi|^2$$

for any $\xi \in \mathbb{R}^{d+1}$. The constant $L(G)$ is called the isotropic constant of G. Our dimension-free estimate about M_*^G is as following.

THEOREM 1.2. *Suppose G is in the isotropic position. For* $1 < p \leqslant \infty$, *there is a constant* $C_p(L(G))$ *such that*

$$\left\| \sup_{n \in \mathbb{Z}} |M_{2^n}^G f| \right\|_{L^p(\mathbb{R}^{d+1})} \leqslant C_p(L(G)) \, \|f\|_{L^p(\mathbb{R}^{d+1})} . \tag{1.4}$$

THEOREM 1.3. *Suppose G is in the isotropic position. For* $\frac{3}{2} < p \leqslant \infty$, *there is a constant* $C_p(L(G))$ *such that*

$$\left\| M_*^G f \right\|_{L^p(\mathbb{R}^{d+1})} \leqslant C_p(G) \|f\|_{L^p(\mathbb{R}^{d+1})}.$$

Since, when G is the q-ball, $C_p(L(G))$ is not dependent on d, we have the following two corollaries.

COROLLARY 1.4. *For* $1 < p \leqslant \infty$ *and* $1 \leqslant q \leqslant \infty$, *there is a constant* C_p *independent on dimension* d *such that*

$$\left\| \sup_{n \in \mathbb{Z}} |M_{2^n}^{B^q} f| \right\|_{L^p(\mathbb{R}^{d+1})} \leqslant C_p \, \|f\|_{L^p(\mathbb{R}^{d+1})} . \tag{1.5}$$

COROLLARY 1.5. *For* $\frac{3}{2} < p \leqslant \infty$ *and* $1 \leqslant q \leqslant \infty$, *there is a constant* C_p *independent on dimension* d *such that*

$$\left\| M_*^{B^q} f \right\|_{L^p(\mathbb{R}^{d+1})} \leqslant C_p \|f\|_{L^p(\mathbb{R}^{d+1})}.$$

We finish this section by fixing some further notations and terminologies.

- Throughout the whole paper $C_p > 0$ denotes a constant, which does not depend on the dimension, but it may vary from occurrence to occurrence.
- We write that $A \lesssim B$ to say that there is an absolute constant $C > 0$ such that $A \leqslant CB$.
- The Euclidean space $\mathbb{R}^{d+1}$ is endowed with the standard inner product

$$\langle (x,y), (\xi, \eta) \rangle = \sum_{k=1}^{d} x_k \xi_k + y\eta$$

for every $(x,y) = (x_1, \dots, x_d, y)$ and $(\xi, \eta) = (\xi_1, \dots, \xi_d, \eta)$.

- Let $(X,\mathcal{B}(X),\mu)$ be a σ-finite measure space. Let $p\in[1,\infty]$ and suppose that $(T_t : t\in Z)$ is a family of linear operators such that T_t maps $L^p(X)$ to itself for every $t\in Z\subset(0,\infty)$. Then the corresponding maximal function will be denoted by
$$T_{*,Z}f := \sup_{t\in Z}|T_t f|$$
for every $f\in L^p(X)$.
- $\mathbb{N}_0=\mathbb{N}\cup\{0\}$, $\mathbb{N}$ is the set of positive integers.

2. Preliminaries and lemmas

In this section we give some important useful lemmas.

2.1. Fourier transform estimate

The method of dimension-free estimates in this paper is mainly based on the properties of the Fourier transform. From [6], we know that there is a linear positive transformation U of $\mathbb{R}^{d+1}$ such that $\overline{G}=U(G)$ is in the isotropic position. However, $M_t^G f = M_t^{U(G)}(f\circ U^{-1})\circ U$ is not true. It implies that one can not get
$$\left\|M_*^G\right\|_{L^p\to L^p}=\left\|M_*^{\overline{G}}\right\|_{L^p\to L^p}.$$
By [3] we know that $1\lesssim L=L(G)$. Let $m(\xi,\eta)$ denote the Fourier transform of $\frac{1}{|G|}\chi_G=\chi_G$. It follows that
$$\widehat{M_t f}(\xi,\eta)=m(t\xi,t^2\eta)\widehat{f}(\xi,\eta). \tag{2.1}$$
The following estimate can be found in [3].

LEMMA 2.1. *Let G be a symmetric convex body $G\subset\mathbb{R}^{d+1}$ which is in the isotropic position. Let $L=L(G)$ be the isotropic constant of G. Then for every $\xi\in\mathbb{R}^{d+1}\setminus\{0\}$ we have*
$$|m(\xi,\eta)|\lesssim (L\max\{|\xi|,|\eta|\})^{-1} \tag{2.2}$$
$$|m(\xi,\eta)-1|\lesssim L\max\{|\xi|,|\eta|\} \tag{2.3}$$
and
$$|\langle\nabla m(\xi,\eta),(\xi,\eta)\rangle|\lesssim C. \tag{2.4}$$

Using Lemma 2.1, we obtain the following important estimates which will be used in the almost orthogonality principle.

LEMMA 2.2. *When $|\xi|^2>|\eta|$, for $j\in\mathbb{Z}$ we have*
$$\left|m\left(2^n\xi,2^{2n}\eta\right)-e^{-4^nL^2\left(|\xi|^2+|\eta|\right)}\right|\left|e^{-4^{j+n+1}L^2\left(|\xi|^2+|\eta|\right)}-e^{-4^{j+n}L^2\left(|\xi|^2+|\eta|\right)}\right|$$
$$\lesssim 2^{-\frac{|j|}{2}}\min\left\{(2^nL|\xi|)^{-\frac{1}{2}},(2^nL|\xi|)^{\frac{1}{2}}\right\}.$$

Proof. It follows from inequality (2.2) that

$$\begin{aligned}\left|m\left(2^n\xi,2^{2n}\eta\right)-e^{-4^nL^2\left(|\xi|^2+|\eta|\right)}\right| &\lesssim \left(L\max\{2^n|\xi|,2^{2n}|\eta|\}\right)^{-1}+e^{-4^n(L|\xi|)^2}\\ &\lesssim \frac{1}{L2^n|\xi|}.\end{aligned}$$

Note that $|\xi|^2>|\eta|$. It can be deduced that $\max\{2^n|\xi|,2^{2n}|\eta|\}\leqslant\max\{2^n|\xi|,(2^n|\xi|)^2\}$. Recalling that $1\lesssim L$, we have

$$\begin{aligned}&\left|m\left(2^n\xi,2^{2n}\eta\right)-e^{-4^nL^2\left(|\xi|^2+|\eta|\right)}\right|\\ &\lesssim \left|m\left(2^n\xi,2^{2n}\eta\right)-1\right|+\left|e^{-4^nL^2\left(|\xi|^2+|\eta|\right)}-1\right|\\ &\lesssim L\max\{2^n|\xi|,2^{2n}|\eta|\}+(2^nL|\xi|)^2\\ &\lesssim \max\{2^nL|\xi|,(2^nL|\xi|)^2\}.\end{aligned}$$

Using these two estimates above, one has

$$\begin{aligned}\left|m(2^n\xi,4^n\eta)-e^{-4^nL^2\left(|\xi|^2+|\eta|\right)}\right| &\lesssim \min\left\{\frac{1}{2^nL|\xi|},\max\left\{2^nL|\xi|,(2^nL|\xi|)^2\right\}\right\}\\ &\lesssim \min\left\{\frac{1}{2^nL|\xi|},2^nL|\xi|\right\}.\end{aligned}$$

Thus, it enough to estimate

$$\begin{aligned}&\min\left\{(2^nL|\xi|)^{-1},2^nL|\xi|\right\}\left|e^{-4^{j+n+1}L^2\left(|\xi|^2+|\eta|\right)}-e^{-4^{j+n}L^2\left(|\xi|^2+|\eta|\right)}\right|\\ &\lesssim 2^{-\frac{|j|}{2}}\min\{(2^nL|\xi|)^{-\frac12},(2^nL|\xi|)^{\frac12}\}.\end{aligned}$$

If $j\geqslant 0$, we have

$$\begin{aligned}&\min\left\{(2^nL|\xi|)^{-1},2^nL|\xi|\right\}\left|e^{-4^{j+n+1}L^2\left(|\xi|^2+|\eta|\right)}-e^{-4^{j+n}L^2\left(|\xi|^2+|\eta|\right)}\right|\\ &\lesssim \min\left\{(2^nL|\xi|)^{-\frac12},(2^nL|\xi|)^{\frac12}\right\}\\ &\quad\times(2^nL|\xi|)^{\frac12}\left|e^{-4^{j+n+1}L^2\left(|\xi|^2+|\eta|\right)}-e^{-4^{j+n}L^2\left(|\xi|^2+|\eta|\right)}\right|\\ &\lesssim \min\left\{(2^nL|\xi|)^{-\frac12},(2^nL|\xi|)^{\frac12}\right\}(2^nL|\xi|)^{\frac12}e^{-4^{j+n}L^2\left(|\xi|^2+|\eta|\right)}\\ &\lesssim 2^{-\frac{j}{2}}\min\left\{(2^nL|\xi|)^{-\frac12},(2^nL|\xi|)^{\frac12}\right\}.\end{aligned}$$

If $j<0$, we have

$$\begin{aligned}&\min\left\{(2^nL|\xi|)^{-1},2^nL|\xi|\right\}\left|e^{-4^{j+n+1}L^2\left(|\xi|^2+|\eta|\right)}-e^{-4^{j+n}L^2\left(|\xi|^2+|\eta|\right)}\right|\\ &\lesssim \min\left\{(2^nL|\xi|)^{-\frac12},(2^nL|\xi|)^{\frac12}\right\}\\ &\quad\times(2^nL|\xi|)^{-\frac12}e^{-4^{j+n}L^2\left(|\xi|^2+|\eta|\right)}\left|e^{-3\cdot4^{j+n}L^2\left(|\xi|^2+|\eta|\right)}-1\right|\end{aligned}$$

$$
\begin{aligned}
&\lesssim \min\left\{(2^nL|\xi|)^{-\frac{1}{2}},(2^nL|\xi|)^{\frac{1}{2}}\right\}(2^nL|\xi|)^{-\frac{1}{2}}\,e^{-4^{j+n}(L|\xi|)^2}4^{j+n}L^2\left(|\xi|^2+|\eta|\right)\\
&\lesssim \min\left\{(2^nL|\xi|)^{-\frac{1}{2}},(2^nL|\xi|)^{\frac{1}{2}}\right\}(2^nL|\xi|)^{-\frac{1}{2}}\,e^{-4^{j+n}(L|\xi|)^2}4^{j+n}(L|\xi|)^2\\
&\lesssim 2^{\frac{j}{2}}\min\left\{(2^nL|\xi|)^{-\frac{1}{2}},(2^nL|\xi|)^{\frac{1}{2}}\right\}.
\end{aligned}
$$

The proof is completed. □

Note that $2^n \leqslant 2^n+2^{n-l}s \leqslant 2\cdot 2^n$ holds for $0\leqslant l\leqslant n$ and $0\leqslant s\leqslant 2^l$. Using the same method as in the proof of Lemma 2.2, for every $0<\varepsilon<1$ and $|\eta|\leqslant|\xi|^2$, we have the following estimate.

LEMMA 2.3. *Suppose* $|\xi|^2\geqslant|\eta|$, $0\leqslant l\leqslant n$ *and* $0\leqslant s\leqslant 2^l-1$. *Then we have*

$$
\begin{aligned}
&\Bigg|m\left(\left(2^n+2^{n-l}(s+1)\right)\xi,\left(2^n+2^{n-l}(s+1)\right)^2\eta\right)\\
&\quad-m\left(\left(2^n+2^{n-l}s\right)\xi,\left(2^n+2^{n-l}s\right)^2\eta\right)\Bigg|^{\varepsilon}\left|e^{-4^{j+n+1}(|\xi|^2+|\eta|)}-e^{4^{j+n}(|\xi|^2+|\eta|)}\right|\\
&\leqslant 2^{-\varepsilon\frac{|j|}{2}}\min\left\{(2^nL|\xi|)^{-\frac{1}{2}\varepsilon},(2^nL|\xi|)^{\frac{1}{2}\varepsilon}\right\}.
\end{aligned}
$$

LEMMA 2.4. *When* $|\xi|^2<|\eta|$, *we have*

$$
\begin{aligned}
&\left|m\left(2^n\xi,2^{2n}\eta\right)-e^{-4^nL^2\left(|\xi|^2+|\eta|\right)}\right|\\
&\times\left|e^{-4^{j+n+1}L^2\left(|\xi|^2+|\eta|\right)}-e^{-4^{j+n}L^2\left(|\xi|^2+|\eta|\right)}\right|\\
&\lesssim L\min\{2^j,2^{-\frac{j}{2}}\}\min\left\{\left(2^nL|\eta|^{\frac{1}{2}}\right)^{-1},\left(2^nL|\eta|^{\frac{1}{2}}\right)^{\frac{1}{2}}\right\}.
\end{aligned}
$$

Proof. Since $|\xi|^2\leqslant|\eta|$, we have

$$
\begin{aligned}
&\left|m\left(2^n\xi,2^{2n}\eta\right)-e^{-4^nL^2\left(|\xi|^2+|\eta|\right)}\right|\\
&\lesssim\left|m\left(2^n\xi,2^{2n}\eta\right)-1\right|+\left|e^{-4^nL^2\left(|\xi|^2+|\eta|\right)}-1\right|\\
&\lesssim\max\left\{2^nL|\eta|^{\frac{1}{2}},2^{2n}L^2|\eta|\right\}.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
\left|m(2^n\xi,2^{2n}\eta)-e^{-4^nL^2\left(|\xi|^2+|\eta|\right)}\right|&\lesssim\left|m(2^n\xi,2^{2n}\eta)\right|+e^{-4^nL^2\left(|\xi|^2+|\eta|\right)}\\
&\lesssim\max\left\{2^nL|\xi|,2^{2n}L|\eta|\right\}^{-1}+\left(2^{2n}L^2|\eta|\right)^{-1}\\
&\lesssim\left(2^{2n}L|\eta|\right)^{-1}+\left(2^{2n}L^2|\eta|\right)^{-1}\\
&\lesssim L\cdot\left(2^{2n}L^2|\eta|^2\right)^{-1}.
\end{aligned}
$$

Thus, we have

$$\left|m(2^n\xi,2^{2n}\eta)-e^{-4^nL^2\left(|\xi|^2+|\eta|\right)}\right|\lesssim L\min\left\{\left(2^nL|\eta|^{\frac12}\right)^{-2},\left(2^nL|\eta|^{\frac12}\right)\right\}.$$

So, it is enough to estimate

$$\begin{aligned}&\min\left\{\left(2^nL|\eta|^{\frac12}\right)^{-2},2^nL|\eta|^{\frac12}\right\}\\&\quad\times\left|e^{-4^{j+n+1}L^2\left(|\xi|^2+|\eta|\right)}-e^{-4^{j+n}L^2\left(|\xi|^2+|\eta|\right)}\right|\\&\lesssim\min\{2^j,2^{-\frac{j}{2}}\}\min\left\{\left(2^nL|\eta|^{\frac12}\right)^{-1},\left(2^nL|\eta|^{\frac12}\right)^{\frac12}\right\}.\end{aligned}$$

If $j\geqslant 0$, we have

$$\begin{aligned}&\min\left\{\left(2^nL|\eta|^{\frac12}\right)^{-2},2^nL|\eta|^{\frac12}\right\}\\&\quad\times\left|e^{-4^{j+n+1}L^2\left(|\xi|^2+|\eta|\right)}-e^{-4^{j+n}L^2\left(|\xi|^2+|\eta|\right)}\right|\\&\lesssim\min\left\{\left(2^nL|\eta|^{\frac12}\right)^{-1},\left(2^nL|\eta|^{\frac12}\right)^{\frac12}\right\}\\&\quad\times\left(2^nL|\eta|^{\frac12}\right)^{\frac12}\left|e^{-4^{j+n+1}L^2\left(|\xi|^2+|\eta|\right)}-e^{-4^{j+n}L^2\left(|\xi|^2+|\eta|\right)}\right|\\&\lesssim\min\left\{\left(2^nL|\eta|^{\frac12}\right)^{-1},\left(2^nL|\eta|^{\frac12}\right)^{\frac12}\right\}\left(2^nL|\eta|^{\frac12}\right)^{\frac12}e^{-4^{j+n}L^2|\eta|}\\&\lesssim 2^{-\frac{j}{2}}\min\left\{\left(2^nL|\eta|^{\frac12}\right)^{-1},\left(2^nL|\eta|^{\frac12}\right)^{\frac12}\right\}.\end{aligned}$$

If $j<0$, we have

$$\begin{aligned}&\min\left\{\left(2^nL|\eta|^{\frac12}\right)^{-2},2^nL|\eta|^{\frac12}\right\}\\&\quad\times\left|e^{-4^{j+n+1}\left((L|\xi|)^2+L^2|\eta|\right)}-e^{-4^{j+n}L^2\left(|\xi|^2+|\eta|\right)}\right|\\&\lesssim\min\left\{\left(2^nL|\eta|^{\frac12}\right)^{-1},\left(2^nL|\eta|^{\frac12}\right)^{\frac12}\right\}\left(2^nL|\eta|^{\frac12}\right)^{-1}e^{-4^{j+n}L^2\left(|\xi|^2+|\eta|\right)}\\&\quad\times\left|e^{-3\cdot4^{j+n}L^2\left(|\xi|^2+|\eta|\right)}-1\right|\\&\lesssim\min\left\{\left(2^nL|\eta|^{\frac12}\right)^{-1},\left(2^nL|\eta|^{\frac12}\right)^{\frac12}\right\}\\&\quad\times\left(2^nL|\eta|^{\frac12}\right)^{-1}e^{-4^{j+n}L^2\left(|\xi|^2+|\eta|\right)}4^{j+n}L^2\left(|\xi|^2+|\eta|\right)\\&\lesssim 2^j\min\left\{\left(2^nL|\eta|^{\frac12}\right)^{-\frac12},\left(2^nL|\eta|^{\frac12}\right)\right\}.\end{aligned}$$

The proof is completed. □

Using the same method as above, we obtain the following estimate.

LEMMA 2.5. *For every* $0<\varepsilon<1$, $|\xi|^2<|\eta|$ $0\leqslant l\leqslant n$ *and* $0\leqslant s\leqslant 2^l-1$, *we have*

$$\Big|m\Big(\Big(2^n+2^{n-l}(s+1)\Big)\xi,\Big(2^n+2^{n-l}(s+1)\Big)^2\eta\Big) \\ -m\Big(\Big(2^n+2^{n-l}s\Big)\xi,\Big(2^n+2^{n-l}s\Big)^2\eta\Big)\Big|^{\varepsilon}\Big|e^{-4^{j+n+1}(|\xi|^2+|\eta|)}-e^{4^{j+n}(|\xi|^2+|\eta|)}\Big| \\ \lesssim L\min\{2^{\varepsilon j},2^{-\frac{j}{2}\varepsilon}\}\min\Big\{\Big(2^nL|\eta|^{\frac12}\Big)^{-\varepsilon},\Big(2^nL|\eta|^{\frac12}\Big)^{\frac12\varepsilon}\Big\}.$$

By inequality (2.4), it follows that

LEMMA 2.6. *For* $0\leqslant l\leqslant n$ *and* $0\leqslant s\leqslant 2^l-1$, *we have*

$$\Big|m\Big((2^n+2^{n-l}(s+1))\xi,(2^n+2^{n-l}(s+1))^2\eta\Big) \\ -m\Big((2^n+2^{n-l}s)\xi,(2^n+2^{n-l}s)^2\eta\Big)\Big|\lesssim\frac{2^{-l}}{1+2^{-l}s}. \tag{2.5}$$

Proof. Observe that

$$\Big|m\Big((2^n+2^{n-l}(s+1))\xi,(2^n+2^{n-l}(s+1))^2\eta\Big) \\ -m\Big((2^n+2^{n-l}s)\xi,(2^n+2^{n-l}s)^2\eta\Big)\Big| \\ =\Big|\int_{2^n+2^{n-l}s}^{2^n+2^{n-l}(s+1)}\frac{d}{dt}m(t\xi,t^2\eta)dt\Big| \\ \leqslant\int_{2^n+2^{n-l}s}^{2^n+2^{n-l}(s+1)}\Big|\frac{d}{dt}m(t\xi,t^2\eta)\Big|dt \\ =\int_{2^n+2^{n-l}s}^{2^n+2^{n-l}(s+1)}|\langle\nabla m(t\xi,t^2\eta),(t\xi,2t^2\eta)\rangle|\frac{dt}{t}. \tag{2.6}$$

By inequality (2.4), it can be deduced that

$$|\langle\nabla m(\xi,\eta),(\xi,2\eta)\rangle|\leqslant|\langle\nabla m(\xi,\eta),(\xi,\eta)\rangle|+\Big|\eta\frac{\partial}{\partial\eta}m(\xi,\eta)\Big|\leqslant C \tag{2.7}$$

Combining inequalities (2.6) and (2.7), we obtain the estimate (2.5). □

2.2. An almost orthogonality principle

In this subsection, we show an almost orthogonality principle from [9] which will be used to prove Theorem 1.2 and Theorem 1.3. We omit the proof here, since we can find the proof of its discrete version in [7].

PROPOSITION 2.7. *Let $(T_t : t \in U)$ be a family of linear operators defined on $\cup_{1\leqslant p\leqslant\infty} L^p(\mathbb{R}^{d+1})$ for some index set $U \subset (0,\infty)$. Suppose that $T_t = M_t - H_t$ for each $t \in U$, where M_t, H_t are positive linear operators. Assume that the following conditions are satisfied.*

(i) For every $p \in (1,2]$ we have

$$\sup_{n\in\mathbb{Z}} \|H_{*,U_n}\|_{L^p(\mathbb{R}^{d+1})\to L^p(\mathbb{R}^{d+1})} < \infty,$$

where $U_n = [a_n, a_{n+1}] \cap U$ and $(a_n : n \in \mathbb{Z}) \subset (0,\infty)$ is a lacunary sequence obeying

$$1 < a \leqslant \frac{a_n}{a_{n-1}} \leqslant a^2,$$

for some $a > 1$.

(ii) There is $p_0 \in (1,2)$ with the property that for every $p \in (p_0, 2]$ we have

$$\sup_{n\in\mathbb{Z}} \|T_{*,U_n}\|_{L^p(\mathbb{R}^{d+1})\to L^p(\mathbb{R}^{d+1})} < \infty$$

(iii) There exists a sequence $(b_j : j \in \mathbb{Z})$ of positive numbers so that $\sum_{j\in\mathbb{Z}} b_j^\rho = B_\rho < \infty$ for every $\rho > 0$. Moreover, for every $j \in \mathbb{Z}$ we have

$$\sup_{\|f\|_{L^2(\mathbb{R}^{d+1})}\leqslant 1} \left\| \left(\sum_{n\in\mathbb{Z}} \sup_{t\in U_n} \left| T_t S_{n+j} f \right|^2 \right)^{\frac{1}{2}} \right\|_{L^2(\mathbb{R}^{d+1})} \leqslant b_j,$$

where $(S_n : n \in \mathbb{Z})$ is the resolution of identity satisfying

$$f = \sum_{n\in\mathbb{Z}} S_n f$$

and

$$\left\| \left(\sum_j |S_j f|^2 \right)^{\frac{1}{2}} \right\|_{L^p\left(\mathbb{R}^{d+1}\right)} \leqslant C_p a \|f\|_{L^p(\mathbb{R}^{d+1})}$$

for all $p \in (1,\infty)$. Then for every $p \in (p_0, 2]$, there exists a constant C_p such that

$$\sup_{\|f\|_{L^p(\mathbb{R}^{d+1})}\leqslant 1} \left\| \left(\sum_{n\in\mathbb{Z}} \sup_{t\in U_n} |T_n f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^{d+1})} \leqslant C_p \|f\|_{L^p(\mathbb{R}^{d+1})}.$$

The key ingredient of the proof of main theorem will be the following inequality which can be found in [6].

LEMMA 2.8. *For every* $n \in \mathbb{N}_0$, $r > 1$ *and every function* $a : [2^n, 2^{n+1}] \to \mathbb{C}$, *we have*

$$\sup_{2^n \leqslant t < 2^{n+1}} |a(t) - a(2^n)|$$
$$\leqslant 2^{1-\frac{1}{r}} \sum_{0 \leqslant l \leqslant n} \left(\sum_{k=0}^{2^l - 1} \left| a\left(2^n + 2^{n-l}(k+1)\right) - a\left(2^n + 2^{n-l}k\right) \right|^r \right)^{\frac{1}{r}}.$$

2.3. A diffusion semigroup and corresponding Littlewood-Paley theory

In [15], there is shown a Littewood-Paley inequality which has an dimension-free estimate.

LEMMA 2.9. *Let* $(X, \mathcal{B}(X), \mu)$ *be a* σ*-finite measure space, and* $(T_t)_{t \geqslant 0}$ *be a strongly continuous semigroup on* $L^2(X)$, *which maps* $L^1(X) + L^\infty(X)$ *to itself for every* $t \geqslant 0$. *We say that* $(T_t)_{t \geqslant 0}$ *is a symmetric diffusion semigroup, if it satisfies for all* $t \geqslant 0$ *the following conditions:*

(i) Contraction property: for all $p \in [1, \infty]$ *and* $f \in L^p(X)$ *we have* $\|T_t f\|_{L^p(X)} \leqslant \|f\|_{L^p(X)}$.

(ii) Symmetry property: each T_t *is a self-adjoint operator on* $L^2(X)$.

(iii) Positivity property: $T_t f \geqslant 0$, *if* $f \geqslant 0$.

(iv) Conservation property: $T_t 1 = 1$.

Then for $1 < p \leqslant \infty$, *we have*

$$\left\| \sup_{t>0} |T_t f| \right\|_{L^p(d\mu)} \leqslant C_p \|f\|_{L^p(d\mu)}.$$

Therefore, define $\widehat{G_t f}(\xi, \eta) = e^{-tL^2(|\xi|^2 + |\eta|)} \widehat{f}(\xi, \eta)$. It follows from Lemma 2.9 that

$$\left\| \sup_{t>0} |G_t f| \right\|_{L^p(\mathbb{R}^{d+1})} \leqslant C_p \|f\|_{L^p(\mathbb{R}^{d+1})}. \tag{2.8}$$

Moreover, if we let

$$\widehat{S_j f}(\xi, \eta) = \left(e^{-4^{j+1} L^2(|\xi|^2 + |\eta|)} - e^{-4^j L^2(|\xi|^2 + |\eta|)} \right) \widehat{f}(\xi, \eta).$$

Due to Lemma 2.9, it can be deduced that we can find a constant C_p independent of d such that

$$\left\| \left(\sum_{j \in Z} |S_j f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^{d+1})} \leqslant C_p \|f\|_{L^p(\mathbb{R}^{d+1})}, \tag{2.9}$$

for $1 < p < \infty$.

3. The proof of Theorem 1.2

By interpolation, we need only to prove Theorem 1.2 for $1 < p \leqslant 2$. Let $a_n = 2^n$ and $U = \{2^n : n \in \mathbb{Z}\}$. Then $U_n = [a_n, a_{n+1}) \cap U = \{2^n\}$. Set $T_t f = M_t f - G_t f$. It is enough to prove

$$\left\| \sup_{n \in \mathbb{Z}} |T_{2^n} f| \right\|_{L^p(\mathbb{R}^{d+1})} \leqslant C_p \|f\|_{L^p(\mathbb{R}^{d+1})}, \tag{3.1}$$

for $1 < p \leqslant \infty$.

It is easy to know the operators G_t satisfy condition (i). For $1 < p \leqslant \infty$, condition (ii) follows since

$$\left\| \sup_{t \in U_n} |T_t f| \right\|_{L^p(\mathbb{R}^{d+1})} = \|T_{2^n} f\|_{L^p(\mathbb{R}^{d+1})} \leqslant 2\|f\|_{L^p(\mathbb{R}^{d+1})}.$$

It remains to prove condition (iii) for us.

By Plancherel's theorem, we obtain

$$\begin{aligned} \left\| \left(\sum_{n \in \mathbb{Z}} |T_{2^n} S_{j+n} f|^2 \right)^{\frac{1}{2}} \right\|_{L^2(\mathbb{R}^{d+1})}^2 &= \int_{\mathbb{R}^{d+1}} \sum_{n \in \mathbb{Z}} \left| T_{2^n} S_{j+n} f(x,y) \right|^2 dx dy \\ = \int_{\mathbb{R}^{d+1}} \sum_{n \in \mathbb{Z}} \Big(m(2^n \xi, 2^{2n} \eta) &- e^{-4^n L^2 \left(|\xi|^2 + |\eta| \right)} \Big)^2 \\ \times \left(e^{-4^{(j+n+1)} L^2 \left(|\xi|^2 + |\eta| \right)} - e^{-4^{(j+n)} L^2 \left(|\xi|^2 + |\eta| \right)} \right)^2 &|\widehat{f}(\xi, \eta)|^2 d\xi d\eta. \end{aligned}$$

By Lemma 2.2 and Lemma 2.4, we have

$$\left\| \left(\sum_{n \in \mathbb{Z}} |T_{2^n} S_{j+n} f|^2 \right)^{\frac{1}{2}} \right\|_{L^2(\mathbb{R}^{d+1})}^2 \lesssim L^2 \int_{\mathbb{R}^{d+1}} |f(x,y)|^2 dx dy.$$

Therefore, By the Proposition 2.7, we have

$$\left\| \left(\sum_{n \in Z} |T_{2^n} f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^{d+1})} \lesssim C_p(L(G)) \|f\|_{L^p(\mathbb{R}^{d+1})}.$$

It implies that

$$\left\| \sup_{n \in Z} |T_{2^n} f| \right\|_{L^p(\mathbb{R}^{d+1})} \lesssim C_p(L(G)) \|f\|_{L^p(\mathbb{R}^{d+1})}.$$

The proof of Theorem 1.2 is completed.

4. The proof of Theorem 1.3

By interpolation, we need only to prove Theorem 1.3 for $\frac{3}{2} < p \leqslant 2$. Observe that

$$M_*^G f(x,y) \leqslant \lim_{k\to\infty} \sup_{m\in\mathbb{Z}} |M_{2^{-k}m} f(x,y)|.$$

Thus, it is enough to prove

$$\left\| \sup_{m\in\mathbb{Z}} |M_{2^{-k}m} f| \right\|_{L^p(\mathbb{R}^{d+1})} \leqslant C_p(L(G)) \|f\|_{L^p(\mathbb{R}^{d+1})} \tag{4.1}$$

for $\frac{3}{2} < p \leqslant 2$.

Set $H_t f = M_{2^n} f$ for $2^n \leqslant t < 2^{n+1}$ and take $T_t = M_t - H_t$. It follows from Theorem 1.2 that we have

$$\left\| \sup_{t>0} H_t f \right\|_{L^p(\mathbb{R}^{d+1})} = \left\| \sup_{n\in\mathbb{Z}} M_{2^n} f \right\|_{L^p(\mathbb{R}^{d+1})} \leqslant C_p(L(G)) \|f\|_{L^p(\mathbb{R}^{d+1})},$$

for $1 < p \leqslant \infty$. Thus inequality (4.1) follows from

$$\left\| \sup_{m\in\mathbb{Z}} |T_{2^{-k}m} f| \right\|_{L^p(\mathbb{R}^{d+1})} \leqslant C_p(L(G)) \|f\|_{L^p(\mathbb{R}^{d+1})} \tag{4.2}$$

for $\frac{3}{2} < p \leqslant 2$.

We will use Proposition 2.7 to prove inequality (4.2). Let $U = [2^{-k}m : m \in \mathbb{Z})$ and $a_n = 2^n$. Then we have

$$U_n = [a_n, a_{n+1}) \cap U = \left\{ 2^n + 2^{-k}m : 0 \leqslant m < 2^{n+k} \right\}.$$

By definition, we have

$$\left\| \sup_{t\in U_n} H_t f \right\|_{L^p(\mathbb{R}^{d+1})} = \|M_{2^n} f\|_{L^p(\mathbb{R}^{d+1})} \leqslant \|f\|_{L^p(\mathbb{R}^{d+1})},$$

for $1 < p \leqslant \infty$. Thus, we obtain condition (i).

Next, we try to prove T_t is satisfies with condition (ii). That is

$$\left\| \sup_{t\in U_n} |T_t f| \right\|_{L^p(\mathbb{R}^{d+1})} \leqslant C_p(L(G)) \|f\|_{L^p(\mathbb{R}^{d+1})}$$

for $\frac{3}{2} < p \leqslant \infty$. By Lemma 2.8, it follows that

$$\sup_{2^n \leqslant t < 2^{n+1}} |T_t f| \leqslant 2^{\frac{1}{2}} \sum_{l=0}^{n+k} \left(\sum_{s=0}^{2^l-1} \left| M_{2^n+2^{n-l}(s+1)} f - M_{2^n+2^{n-l}s} f \right|^2 \right)^{\frac{1}{2}}.$$

Therefore, it enough to prove

$$\left\|\sum_{l=0}^{n+k}\left(\sum_{s=0}^{2^l-1}\left|M_{2^n+2^{n-l}(s+1)}f-M_{2^n+2^{n-l}s}f\right|^2\right)^{\frac{1}{2}}\right\|_{L^p\left(\mathbb{R}^{d+1}\right)} \leqslant C_p\left\|f\right\|_{L^p\left(\mathbb{R}^{d+1}\right)}, \tag{4.3}$$

for $\frac{3}{2}<p\leqslant\infty$.

We will try to estimate

$$\left\|\left(\sum_{s=0}^{2^l-1}\left|M_{2^n+2^{n-l}(s+1)}f-M_{2^n+2^{n-l}s}f\right|^2\right)^{\frac{1}{2}}\right\|_{L^p(\mathbb{R}^{d+1})}.$$

When $p=1$, using triangle inequality, we have

$$\begin{aligned}&\left\|\left(\sum_{s=0}^{2^l-1}\left|M_{2^n+2^{n-l}(s+1)}f-M_{2^n+2^{n-l}s}f\right|^2\right)^{\frac{1}{2}}\right\|_{L^1(\mathbb{R}^{d+1})}\\ \leqslant&\left\|\sum_{s=0}^{2^l-1}\left|M_{2^n+2^{n-l}(s+1)}f-M_{2^n+2^{n-l}s}f\right|\right\|_{L^1(\mathbb{R}^{d+1})}\leqslant 2^{l+1}\left\|f\right\|_{L^1(\mathbb{R}^{d+1})}.\end{aligned}$$

When $p=2$, by inequality (2.5) and the Plancherel theorem, we obtain

$$\begin{aligned}&\left\|\left(\sum_{s=0}^{2^l-1}\left|M_{2^n+2^{n-l}(s+1)}f-M_{2^n+2^{n-l}s}f\right|^2\right)^{\frac{1}{2}}\right\|_{L^2(\mathbb{R}^{d+1})}\\ =&\left(\int_{\mathbb{R}^{d+1}}\sum_{s=0}^{2^l-1}\left(m\left((2^n+2^{n-l}(s+1))\xi,(2^n+2^{n-l}(s+1))^2\eta\right)\right.\right.\\ &\left.\left.-m\left((2^n+2^{n-l}s)\xi,\left(2^n+2^{n-l}s\right)^2\eta\right)\right)^2|\widehat{f}(\xi,\eta)|^2d\xi d\eta\right)^{\frac{1}{2}}\\ \lesssim&\left(\int_{\mathbb{R}^{d+1}}\sum_{s=0}^{2^l-1}\left|\frac{2^{-l}}{1+2^{-l}s}\right|^2\widehat{f}(\xi,\eta)|^2d\xi d\eta\right)^{\frac{1}{2}}\\ \lesssim&\,2^{-\frac{l}{2}}\|f\|_{L^2(\mathbb{R}^{d+1})}.\end{aligned}$$

By interpolation, we have

$$\left\|\left(\sum_{s=0}^{2^l-1}\left|M_{2^n+2^{n-l}(s+1)}f-M_{2^n+2^{n-l}s}f\right|^2\right)^{\frac{1}{2}}\right\|_{L^p(\mathbb{R}^{d+1})}\lesssim 2^{l\theta}2^{-\frac{l(1-\theta)}{2}}\|f\|_{L^p(\mathbb{R}^{d+1})},$$

where $0 \leqslant \theta \leqslant 1$ and $\frac{1}{p} = \frac{\theta}{1} + \frac{1-\theta}{2}$. Therefore when $p > \frac{3}{2}$, we have $\delta = \frac{1}{2} - \frac{3}{2}\theta > 0$. It follows that

$$\left\| \left(\sum_{s=0}^{2^l-1} \left| M_{2^n+2^{n-l}(s+1)} f - M_{2^n+2^{n-l}s} f \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^{d+1})} \lesssim 2^{-\delta l} \|f\|_{L^p(\mathbb{R}^{d+1})}.$$

Thus we obtain inequality (4.3) for $\frac{3}{2} < p \leqslant \infty$. Then, we have condition (ii).

At last, we consider condition (iii). Using Lemma 2.8 again, we obtain

$$\begin{aligned} &\sup_{t \in U_n} \left| T_t S_{j+n} f(x,y) \right| \\ &\lesssim \sum_{l=0}^{n+k} \left(\sum_{s=0}^{2^l-1} \left| M_{2^n+2^{n-l}(s+1)} S_{j+n} f(x,y) - M_{2^n+2^{n-l}s} S_{j+n} f(x,y) \right|^2 \right)^{\frac{1}{2}} \end{aligned}$$

It follows that

$$\begin{aligned} &\left\| \left(\sup_{t \in U_n} \left| T_t S_{j+n} f \right|^2 \right)^{\frac{1}{2}} \right\|_{L^2(\mathbb{R}^{d+1})} \\ \lesssim &\left\| \left\{ \sum_{n \in \mathbb{Z}} \left[\sum_{l=0}^{n+k} \left(\sum_{s=0}^{2^l-1} \left| M_{2^n+2^{n-l}(s+1)} S_{j+n} f - M_{2^n+2^{n-l}s} S_{j+n} f \right|^2 \right)^{\frac{1}{2}} \right]^2 \right\}^{\frac{1}{2}} \right\|_{L^2(\mathbb{R}^{d+1})} \\ \leqslant &\left\| \sum_{l=0}^{\infty} \left(\sum_{n \geqslant l-k} \sum_{s=0}^{2^l-1} \left| M_{2^n+2^{n-l}(s+1)} S_{j+n} f - M_{2^n+2^{n-l}s} S_{j+n} f \right|^2 \right)^{\frac{1}{2}} \right\|_{L^2(\mathbb{R}^{d+1})} \end{aligned}$$

Using triangle inequality and the Plancherel theorem, we deduce that

$$\left\| \left(\sup_{t \in U_n} \left| T_t S_{j+n} f \right|^2 \right)^{\frac{1}{2}} \right\|_{L^2(\mathbb{R}^{d+1})}$$

can be controlled by

$$\begin{aligned} &\sum_{l=0}^{\infty} \left(\int_{\mathbb{R}^{d+1}} \sum_{n \geqslant l-k} \sum_{s=0}^{2^l-1} \left(m\left((2^n + 2^{n-l}(s+1))\xi, \left(2^n + 2^{n-l}(s+1)\right)^2 \eta \right) \right.\right. \\ &\left. - m\left((2^n + 2^{n-l}s)\xi, \left(2^n + 2^{n-l}s\right)^2 \eta \right) \right)^2 \left(e^{-4^{j+n+1}(|\xi|^2+|\eta|)} - e^{-4^{j+n}(|\xi|^2+|\eta|)} \right)^2 \\ &\left. \times |\widehat{f}(\xi,\eta)|^2 d\xi d\eta \right)^{\frac{1}{2}}. \end{aligned}$$

Note that when $|\eta| \leqslant |\xi|^2$, by Lemma 2.3 and Lemma 2.6, we have

$$\begin{aligned}
&\sum_{n\geqslant l-k}\sum_{s=0}^{2^l-1}\Bigg(m\bigg((2^n+2^{n-l}(s+1))\xi,\Big(2^n+2^{n-l}(s+1)\Big)^2\eta\bigg)\\
&\quad-m\bigg((2^n+2^{n-l}s)\xi,\Big(2^n+2^{n-l}s\Big)^2\eta\bigg)\Bigg)^2\Big(e^{-4^{j+n+1}(|\xi|^2+|\eta|)}-e^{-4^{j+n}(|\xi|^2+|\eta|)}\Big)^2\\
&=\sum_{n\geqslant l-k}\sum_{s=0}^{2^l-1}\Bigg(m\bigg((2^n+2^{n-l}(s+1))\xi,\Big(2^n+2^{n-l}(s+1)\Big)^2\eta\bigg)\\
&\quad-m\bigg((2^n+2^{n-l}s)\xi,\Big(2^n+2^{n-l}s\Big)^2\eta\bigg)\Bigg)^{2-2\varepsilon}\\
&\quad\times\Bigg(m\bigg((2^n+2^{n-l}(s+1))\xi,\Big(2^n+2^{n-l}(s+1)\Big)^2\eta\bigg)\\
&\quad-m\bigg((2^n+2^{n-l}s)\xi,\Big(2^n+2^{n-l}s\Big)^2\eta\bigg)\Bigg)^{2\varepsilon}\Big(e^{-4^{j+n+1}(|\xi|^2+|\eta|)}-e^{-4^{j+n}(|\xi|^2+|\eta|)}\Big)^2\\
&\lesssim\sum_{n\geqslant l-k}\sum_{s=0}^{2^l-1}\left(\frac{2^{-l}}{1+2^{-l}s}\right)^{2-2\varepsilon}2^{-\varepsilon|j|}\min\big\{(2^nL|\xi|)^{-\varepsilon},(2^nL|\xi|)^{\varepsilon}\big\}\lesssim 2^{-\varepsilon|j|}.
\end{aligned}$$

When $|\xi|^2 \leqslant |\eta|$, by Lemma 2.5 and Lemma 2.6, we have

$$\begin{aligned}
&\sum_{n\geqslant l-k}\sum_{s=0}^{2^l-1}\Bigg(m\bigg((2^n+2^{n-l}(s+1))\xi,\Big(2^n+2^{n-l}(s+1)\Big)^2\eta\bigg)\\
&\quad-m\bigg((2^n+2^{n-l}s)\xi,\Big(2^n+2^{n-l}s\Big)^2\eta\bigg)\Bigg)^2\Big(e^{-4^{j+n+1}(|\xi|^2+|\eta|)}-e^{-4^{j+n}(|\xi|^2+|\eta|)}\Big)^2\\
&=\sum_{n\geqslant l-k}\sum_{s=0}^{2^l-1}\Bigg(m\bigg((2^n+2^{n-l}(s+1))\xi,\Big(2^n+2^{n-l}(s+1)\Big)^2\eta\bigg)\\
&\quad-m\bigg((2^n+2^{n-l}s)\xi,\Big(2^n+2^{n-l}s\Big)^2\eta\bigg)\Bigg)^{2-2\varepsilon}\\
&\quad\times\Bigg(m\bigg((2^n+2^{n-l}(s+1))\xi,\Big(2^n+2^{n-l}(s+1)\Big)^2\eta\bigg)\\
&\quad-m\bigg((2^n+2^{n-l}s)\xi,\Big(2^n+2^{n-l}s\Big)^2\eta\bigg)\Bigg)^{2\varepsilon}\Big(e^{-4^{j+n+1}(|\xi|^2+|\eta|)}-e^{-4^{j+n}(|\xi|^2+|\eta|)}\Big)^2\\
&\lesssim L^2\sum_{n\geqslant l-k}\sum_{s=0}^{2^l-1}\left(\frac{2^{-l}}{1+2^{-l}s}\right)^{2-2\varepsilon}\min\{2^{2\varepsilon j},2^{-\varepsilon}\}\\
&\quad\times\min\left\{\Big(2^nL|\eta|^{\frac{1}{2}}\Big)^{-\varepsilon},\Big(2^nL|\eta|^{\frac{1}{2}\varepsilon}\Big)^{\frac{1}{2}}\right\}\\
&\lesssim L^2\min\{2^{2\varepsilon j},2^{-\varepsilon}\}.
\end{aligned}$$

Thus, we have

$$\left\| \left(\sup_{t\in U_n} \left| T_t S_{j+n} f \right|^2 \right)^{\frac{1}{2}} \right\|_{L^2(\mathbb{R}^{d+1})} \lesssim L \max \left\{ 2^{-\varepsilon|j|}, \min\{2^{\varepsilon j}, 2^{-\varepsilon \frac{j}{2}}\} \right\} \|f\|_{L^2(\mathbb{R}^{d+1})}.$$

Thus we have proved Theorem 1.3.

Declarations

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