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ASYMPTOTICS FOR RANDOM–TIME RUIN PROBABILITY OF A RISK MODEL WITH DIFFUSION, CONSTANT INTEREST FORCE AND NON–STATIONARY ARRIVALS

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Abstract. Consider an insurance risk model with diffusion, constant interest force and non-stationary arrivals, where the claims arrive according to a non-stationary process satisfying a large deviation principle. The asymptotic formula for the random-time ruin probability is obtained if the claim-size distribution is subexponential. Furthermore, with a certain dependence structure among claim sizes, the formula still holds if the claim-size distribution belongs to the class with long tails and dominatedly varying tails.

1. Introduction

In the section, we introduce the risk model with non-stationary arrivals, the heavy-tailed distribution classes, and the dependence structure among modelling components, respectively.

1.1. Risk model

Consider a non-stationary risk model perturbed by diffusion with constant interest force, in which the claim sizes $\{X_n, n \geqslant 1\}$, form a sequence of nonnegative, identically distributed, but not necessarily independent random variables (r.v.s) with common distribution F, and their inter-arrival times $\{\theta_n, n \geqslant 1\}$ form a sequence of positive and not necessarily identically distributed r.v.s, but θ_1 and $\{\theta_n, n \geqslant 2\}$ are mutually independent. The claim arrival times $\{\tau_n = \sum_{i=1}^{n} \theta_i, n \geqslant 1\}$, constitute a counting process

$$N(t) = \sum_{n=1}^{\infty} \mathbf{1}_{\{\tau_n \leqslant t\}}, \quad t \geqslant 0,$$

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with a finite mean function

$$\lambda(t) = EN(t) = \sum_{n=1}^{\infty} P(\tau_n \leqslant t), \quad t \geqslant 0.$$

Denote by $x \geqslant 0$ the insurer's initial surplus, $r \geqslant 0$ the constant interest force, $\{C(t), t \geqslant 0\}$ the income process of insurance premiums, and $\{B(t), t \geqslant 0\}$ the diffusion perturbation that is a standard Brownian motion with volatility parameter $\sigma \geqslant 0$. Then, the surplus process of an insurance company is expressed as

$$U(x,t) = e^{rt}x + \int_{0^-}^{t} e^{r(t-s)} dC(s) - \sum_{i=1}^{N(t)} X_i e^{r(t-\tau_i)} + \sigma \int_{0^-}^{t} e^{r(t-s)} dB(s), \ \ t \geqslant 0, \qquad (1.1)$$

where $\{X_n, n \geqslant 1\}$, $\{N(t), t \geqslant 0\}$, $\{C(t), t \geqslant 0\}$ and $\{B(t), t \geqslant 0\}$ are mutually independent, and the total discounted value of premium accumulated up to time $t > 0$ satisfies

$$0 \leqslant \widetilde{C}(t) := \int_{0^-}^{t} e^{-rs} dC(s) < \infty \ \text{ a.s., for any } \ 0 < t < \infty.$$

It is well-known that in the renewal risk model, the claim inter-arrival times $\{\theta_n, n \geqslant 1\}$ is a sequence of identically distributed r.v.s. An increasing attention of researchers has been paid to conduct risk analyses with independence or dependence structures, see Klüppelberg and Stadtimüller (1998), Konstantinides et al. (2002), Tang (2005, 2007), Chen and Ng (2007), Hao and Tang (2008), Wang (2008), Li et al. (2010), Gao and Liu (2013), Wang et al. (2013), Gao and Yang (2014), Peng and Wang (2018), Gao et al. (2019), Li (2017), Wang et al. (2020), and Liu and Gao (2022). However in most practical situations, the stationarity assumption on claim inter-arrival times is unrealistic, and then it limits the usefulness of the obtained results to some extent. So in this paper, we turn to consider the risk model introduced by (1.1) with non-stationary arrivals. In other words, $\{\theta_n, n \geqslant 1\}$ are not necessarily a stationary sequence, which means that the insurance claims arrivals according to a non-stationary process, see Zhu (2013), Fu and Li (2019), Fu and Liu (2022), and references therein. For example, in the field of finance and insurance, the occurrence of claim events has congregation effect and contagion effect, which means that the occurrence of one claim event may promote the occurrence of the next claim event faster and more continuously.

Motivated by Dembo and Zeitouni (1998), we assume that the probability measures $P(N(t)/t \in \cdot)$ on a topological space X satisfies the large deviation principle (LDP) with rate function $I : X \to [0, \infty]$ being lower semi-continuous, and for any measurable set Ω,

$$\begin{aligned} -\inf_{x \in \Omega^o} I(x) &\leqslant \liminf_{t \to \infty} \frac{1}{t} \log P\left(\frac{N(t)}{t} \in \Omega\right) \\ &\leqslant \limsup_{t \to \infty} \frac{1}{t} \log P\left(\frac{N(t)}{t} \in \Omega\right) \leqslant -\inf_{x \in \overline{\Omega}} I(x), \end{aligned} \qquad (1.2)$$

where Ω^o and $\overline{\Omega}$ are the interior and the closure of Ω, respectively. Note that the LDP could cover some counting processes such as the linear Hawkes process and Cox process with shot noise intensity, and then condition (1.2) is mild, see Fu et al. (2022).

In actuarial science, the random-time ruin probability provides a good risk measure for insurance business. For example, the cyclical fluctuation of economy is inevitable, and the cycle varies randomly. Practitioners in insurance company are usually care about the ruin probability of insurance company in the cycle, namely the random-time ruin probability. Assume that τ, independent of the other sources of randomness, is an arbitrary non-negative and proper r.v. with finite mean. Therefore for the initial surplus x, the time of ruin is defined by

$$\tau(x) = \inf\{t \geqslant 0 : U(x,t) < 0 \mid U(x,0) = x\},$$

and hence the random-time ruin probability is

$$\psi(x,\tau) = P(\tau(x) \leqslant \tau) = P\left(\inf_{0\leqslant s\leqslant \tau} U(x,s) < 0 \mid U(x,0) = x\right). \tag{1.3}$$

Particularly when $\tau \equiv t$, the random time ruin probability is exactly the finite-time ruin probability, namely that the random time ruin probability is the randomized version of the finite time ruin probability. For more details of the random-time ruin probability, the readers can refer to Wang et al. (2009), Bai and Song (2012), Bai et al. (2015), Gao et al. (2016) and Liu et al. (2022).

1.2. Heavy-tailed distribution classes

In this paper, the claim sizes are assumed to be heavy-tailed r.v.s, which can model the large claims caused by natural and man-made disasters. For a proper distribution V on $(-\infty,\infty)$, we denote its tail by $\overline{V}(x) = 1 - V(x)$. By definition, a distribution V supported on $[0,\infty)$ belongs to the long-tailed class, denoted by $V \in \mathscr{L}$, if for any $y > 0$,

$$\lim_{x\to\infty} \overline{V}(x+y)/\overline{V}(x) = 1;$$

belongs to subexponential class, denoted by $V \in \mathscr{S}$, if

$$\lim_{x\to\infty} \overline{V^{*2}}(x)/\overline{V}(x) = 2,$$

where V^{*2} denotes the 2-fold convolution of V; belongs to the dominated variation class, denoted by $V \in \mathscr{D}$, if

$$\overline{V}_*(y) > 0 \text{ for any } y > 1, \text{ or equivalently, } \overline{V}^*(y) < \infty \text{ for any } 0 < y < 1$$

where $\overline{V}_*(y) := \liminf_{x\to\infty} \overline{V}(xy)/\overline{V}(x)$ and $\overline{V}^*(y) := \limsup_{x\to\infty} \overline{V}(xy)/\overline{V}(x)$.

More generally, we say that a distribution V supported on $(-\infty,\infty)$ belongs to a distribution class if $V(x)\mathbf{1}_{\{x\geqslant 0\}}$ belongs to the same class, where $\mathbf{1}_A$ denotes the indicator function of event A. In conclusion,

$$\mathscr{L} \cap \mathscr{D} \subset \mathscr{S} \subset \mathscr{L}.$$

For more details of heavy-tailed distributions and their applications to insurance and finance, we refer the readers to Bingham et al. (1987) and Embrechts et al. (1997).

1.3. Dependence structures

Adopting the term of Liu et al. (2012), we present a dependence structure as that r.v.s $\{\xi_n, n \geqslant 1\}$ are said to be upper tail asymptotically independent (UTAI), if $P(\xi_n > x) > 0$ for all $x \in (-\infty, \infty)$, $n \geqslant 1$, and

$$\lim_{\min\{x_i, x_j\} \to \infty} P(\xi_i > x_i | \xi_j > x_j) = 0 \qquad \text{for all} \quad 1 \leqslant i \neq j < \infty.$$

Note that the UTAI structure allows both negative dependence and positive dependence.

The rest part of this paper is organized as follows: we present two theorems in Section 2, and prove them in Sections 4 and 5, respectively. And in Section 3, we give two examples of non-stationary arrival process and some corollaries concerning the two theorems.

2. Main results

All limit relationships in the paper are for $x \to \infty$ unless mentioned otherwise. For two positive functions $a(\cdot)$ and $b(\cdot)$, we write $a(x) \lesssim b(x)$ if $\limsup a(x)/b(x) \leqslant 1$, write $a(x) \gtrsim b(x)$ if $\liminf a(x)/b(x) \geqslant 1$, write $a(x) \sim b(x)$ if both, write $a(x) = o(1)b(x)$ if $\lim a(x)/b(x) = 0$. For a proper distribution V on $(-\infty, \infty)$, we denote its upper and lower Matuszewska indices by, respectively,

$$J_V^+ = -\lim_{y \to \infty} \frac{\log \overline{V}_*(y)}{\log y} \quad \text{and} \quad J_V^- = -\lim_{y \to \infty} \frac{\log \overline{V}^*(y)}{\log y}.$$

Define $\Lambda = \{t : \lambda(t) > 0\}$ with $\underline{t} = \inf\{t : \lambda(t) > 0\} = \inf\{t : P(\tau_1 \leqslant t) > 0\}$. Clearly,

$$\Lambda = \begin{cases} [\underline{t}, \infty], & \text{if } P(\tau_1 = \underline{t}) > 0; \\ (\underline{t}, \infty], & \text{if } P(\tau_1 = \underline{t}) = 0. \end{cases}$$

For notational convenience, we write $\Lambda_T = \Lambda \cap [0, T]$ for any fixed $T \in \Lambda$.

Consider the risk model introduced by (1.1) with non-stationary arrivals, namely that the claim inter-arrival times $\{\theta_n, n \geqslant 1\}$ are not necessarily stationary, we establish the LDP for the probability measure $P(N(t)/t \in \cdot)$, where the LDP implies some useful exponential inequalities, see Dembo and Zeitouni (1998). Before giving the main results, we introduce the assumptions for the non-stationary arrival process $\{N(t), t \geqslant 0\}$ as follows:

A1. $(N(t)/t \in \cdot)$ *satisfies the LDP defined by* (1.2) *with rate function* $I(\cdot)$ *such that* $I(x) = 0$ *if and only if* $x = x_0$*;*

A2. $I(x)$ *is increasing on* $[x_0, \infty)$ *and decreasing on* $[0, x_0]$.

Remark here that the assumptions above come from Assumptions 1(i) and 1(ii) of Zhu (2013), respectively. In fact, there do exist a lot of counting processes such as the renewal process, the Hawkes process and Cox process satisfying Assumptions **A1** and **A2**, and in general, $x_0 = \lim_{t \to \infty} \frac{N(t)}{t}$, see Section 3 below for details.

The following two theorems are the main results of this paper, among which the first one is concerned with the asymptotics of the random-time ruin probability with independent and subexponential claim sizes.

THEOREM 2.1. *Consider the risk model introduced by* (1.1) *with Assumptions* ***A1*** *and* ***A2****, in which the claim sizes* $\{X_n, n \geqslant 1\}$ *are independent, identically distributed r.v.s with common distribution* $F \in \mathscr{S}$*. Then it holds for* $\tau \in \Lambda_T$ *that*

$$\psi(x,\tau) \sim \int_{0^-}^{T} \overline{F}(xe^{rs}) P(s < \tau < T) d\lambda(s). \tag{2.1}$$

In the second theorem, we turn to discuss the case when the claim sizes satisfy a certain dependence structure, i.e. the UTAI structure.

THEOREM 2.2. *Let the conditions of Theorem* 2.1 *be true, and further the claim sizes* $\{X_n, n \geqslant 1\}$ *be a sequence of UTAI r.v.s distributed by* $F \in \mathscr{L} \cap \mathscr{D}$*, then relation* (2.1) *still holds for* $\tau \in \Lambda_T$*.*

3. Examples of non-stationary arrival process

In this section, we provide two examples of non-stationary arrival process that satisfy Assumptions **A1** and **A2**, and give some corollaries concerning Theorems 2.1 and 2.2, respectively.

3.1. Hawkes process

The Hawkes process proposed by Hawkes (1971) is a simple point process with self-exciting property, clustering effect and long memory, and thus is extensively applied in events arrival modeling with clustering or contagion effects in many fields such as finance, seismology, neuroscience, and DNA modeling. A simple point process $N(t)$, which is a linear Hawkes process, possesses an intensity

$$\mu(t) = \nu + \sum_{\tau_i < t} h(t - \tau_i),$$

where ν is a positive constant, τ_i, $i \geqslant 1$, are the arrival times of $N(t)$, and $h(\cdot) : [0,\infty) \to (0,\infty)$ is integrable function with $\|h\|_{L^1} := \int_0^\infty h(t)dt < 1$. Assume also that the Hawkes process starts with empty past history. By our definition, the linear Hawkes process is non-stationary and in general even non-Markovian (unless $h(\cdot)$ is an exponential function). Note that the linear Hawkes process calculates the number of arrivals, but unlike the Poisson process, it is self-exciting.

For the linear Hawkes process $N(t)$, we know that $(N(t)/t \in \cdot)$ satisfies a LDP with rate function

$$I(x) = \begin{cases} x \log\left(\frac{x}{\nu + x\|h\|_{L^1}}\right) - x + x\|h\|_{L^1} + \nu, & x \in [0,\infty), \\ +\infty, & \text{otherwise,} \end{cases}$$

and $\lim_{t\to\infty}\frac{N(t)}{t}=\frac{\nu}{1-\|h\|_{L^1}}$, which ensures the linear Hawkes process satisfies Assumptions **A1** and **A2** with $x_0=\frac{\nu}{1-\|h\|_{L^1}}$. See, for example, Bordenave and Torrisi (2007), Zhu (2011), and Daley and Vere-Jones (2003). Also, the linear Hawkes process has a property as

$$\lambda(t)=\frac{\nu t}{1-\|h\|_{L^1}}-\int_0^t H(t-y)E(\mu(y))dy,$$

where $H(t)=\int_t^\infty h(y)dy,$ see Zhu (2013). Now we present the following two corollaries.

COROLLARY 3.1. *Consider the risk model introduced by* (1.1) *with* $N(t)$ *a linear Hawkes process defined as above. If the claim sizes* $\{X_n, n\geqslant 1\}$ *are a sequence of independent r.v.s identically distributed by* $F\in\mathscr{S}$, *then it holds for* $\tau\in\Lambda_T$ *that*

$$\psi(x,\tau)\sim\int_{0^-}^T \overline{F}(xe^{rs})P(s<\tau<T)d\left(\frac{\nu s}{1-\|h\|_{L^1}}-\int_0^s H(s-y)E(\mu(y))dy\right). \quad (3.1)$$

COROLLARY 3.2. *Let the conditions of Corollary* (3.1) *be true, and further the claim sizes* $\{X_n, n\geqslant 1\}$ *be a sequence of UTAI r.v.s identically distributed by* $F\in\mathscr{L}\cap\mathscr{D}$, *then relation* (3.1) *still holds for* $\tau\in\Lambda_T$.

3.2. Cox process with shot noise intensity

Let $N(t)$ be a Cox process with shot noise intensity of the from

$$\mu(t)=\nu(t)+\sum_{\tau'<t} g(t-\tau_i'),$$

where τ_i', $i\geqslant 1$, are the arrival times of an external homogeneous Poisson process $N'(t)$ with the intensity of an arbitrary positive constant ρ, and $g(\cdot):[0,\infty)\to[0,\infty)$ is integrable and $\nu(t)$ is a positive, continuous and deterministic function satisfying $\nu(t)\to\nu$ as $t\to\infty$. The ruin probability with the claim arrival process being the shot noise Cox process are well-known, see, e.g., Albrecher and Asmussen (2006). Zhu (2013) proved that $(N(t)/t\in\cdot)$ satisfies a LDP with rate function

$$I(x)=\begin{cases}\sup_{\kappa\in(-\infty,\infty)}\left\{\kappa x-(e^\kappa-1)\nu-\rho\left(e^{(e^\kappa-1)\|g\|_{L^1}}-1\right)\right\}, & x\in[0,\infty),\\ +\infty, & \text{otherwise,}\end{cases}$$

and $\lim_{t\to\infty}\frac{N(t)}{t}=\nu+\rho\|g\|_{L^1}$, which ensures that the shot noise Cox process satisfies Assumptions **A1** and **A2** with $x_0=\nu+\rho\|g\|_{L^1}$. Recalling the properties of Cox process in Zhu (2013), it follows that

$$\lambda(t)=\int_0^t \nu(y)dy+\rho\int_0^t\int_0^u g(y)dydu.$$

Hence we get two corollaries as follows:

COROLLARY 3.3. *Consider the risk model introduced by* (1.1) *with* $N(t)$ *the shot noise Cox process defined as above. If the claim sizes* $\{X_n, n \geqslant 1\}$ *are a sequence of independent r.v.s identically distributed by* $F \in \mathscr{S}$, *then it holds for* $\tau \in \Lambda_T$ *that*

$$\psi(x,\tau) \sim \int_{0-}^{T} \overline{F}(xe^{rs}) P(s < \tau < T) d\left(\int_0^s \nu(y)dy + \rho \int_0^s \int_0^u g(y)dydu\right). \quad (3.2)$$

COROLLARY 3.4. *Let the conditions of Corollary* (3.3) *be true, and further the claim sizes* $\{X_n, n \geqslant 1\}$ *be a sequence of UTAI r.v.s identically distributed by* $F \in \mathscr{L} \cap \mathscr{D}$, *then relation* (3.2) *still holds for* $\tau \in \Lambda_T$.

4. Proof of Theorem 2.1

In the section, we give the proof of Theorem 2.1. To this end, we firstly prepare some lemmas, among which the first one is due to Embrechts et al. (1997).

LEMMA 4.1. *For a distribution* V *supported on* $(-\infty,\infty)$,

(i) $V \in \mathscr{L}$ *if and only if there exists a function* $h(\cdot) : [0,\infty) \mapsto [0,\infty)$ *such that* $h(x) \to \infty$, $h(x) = o(x)$ *and*

$$\overline{V}(x+y) \sim \overline{V}(x)$$

holds uniformly for all $|y| \leqslant h(x)$;

(ii) if $F \in \mathscr{S}$, *then for every* $\varepsilon > 0$, *it holds that*

$$e^{-\varepsilon x} = o(1)\overline{V}(x).$$

The second lemma follows from Lemma 5.1 of Tang and Yuan (2014) and Lemma 2.2 of Wang et al. (2018).

LEMMA 4.2. *(i) Let* $\{\xi_i, 1 \leqslant i \leqslant n\}$ *be* n *real-valued and independent r.v.s with distributions* $V_i \in \mathscr{L}$ *satisfying* $\overline{V_i}(x) \asymp \overline{V}(x)$ *for some* $V \in \mathscr{S}$, $1 \leqslant i \leqslant n$, *then for every fixed* $0 < a \leqslant b < \infty$, *it holds uniformly for all* $(c_1, c_2, \ldots, c_n) \in [a,b]^n$ *that*

$$P\left(\sum_{i=1}^{n} c_i\xi_i > x\right) \sim \sum_{i=1}^{n} P(c_i\xi_i > x). \quad (4.1)$$

(ii) Let the conditions of (i) be true, and further there exists a real-valued r.v. ξ, *independent of* $\{\xi_i, 1 \leqslant i \leqslant n\}$, *such that* $P(\xi > x) = o(1)\overline{V}(x/c)$ *for some* $c > 0$, *then for any fixed* $a < b$, *it holds uniformly for all* $(c_1, c_2, \ldots, c_n) \in [a,b]^n$ *that*

$$P\left(\sum_{i=1}^{n} c_i\xi_i + \xi > x\right) \sim \sum_{i=1}^{n} P(c_i\xi_i > x). \quad (4.2)$$

The third lemma is a restatement of Lemma 1.3.5 (c) of Embrechts et al. (1997).

LEMMA 4.3. *Let $\{\xi_i, 1 \leqslant i \leqslant n\}$ be n non-negative and i.i.d. r.v.s with common distribution $V \in \mathscr{S}$, then for any fixed $\varepsilon \in (0,1)$, there exists a finite constant $K > 0$ such that for all $n \geqslant 2$,*

$$P\left(\sum_{i=1}^{n} \xi_i > x\right) \leqslant K(1+\varepsilon)^n \overline{V}(x).$$

The fourth lemma can extend Theorem 2.1 of Hao and Tang (2008) to the case with non-stationary arrival process.

LEMMA 4.4. *Under the conditions of Theorem 2.1, it holds for all $t \in \Lambda_T$ that*

$$P\left(\sum_{i=1}^{N(t)} X_i e^{-r\tau_i} > x\right) \sim \int_{0^-}^{t} \overline{F}(xe^{rs}) d\lambda(s). \tag{4.3}$$

Proof. Note that, for an arbitrarily fixed integer $N \geqslant 1$,

$$\begin{aligned} P\left(\sum_{i=1}^{N(t)} X_i e^{-r\tau_i} > x\right) &= \left(\sum_{n=N+1}^{\infty} + \sum_{n=1}^{N}\right) P\left(\sum_{i=1}^{n} X_i e^{-r\tau_i} > x, N(t) = n\right) \\ &:= J_1(x,t) + J_2(x,t). \end{aligned} \tag{4.4}$$

For $J_1(x,t)$, by $F \in \mathscr{S}$ and Lemma 4.3, we prove that for all $t \in \Lambda_T$,

$$\begin{aligned} J_1(x,t) &= \sum_{n=N+1}^{\infty} P\left(\sum_{i=1}^{n} X_i e^{-r\tau_i} > x, N(t) = n\right) \\ &\leqslant \sum_{n=N+1}^{\infty} P\left(\sum_{i=1}^{n} X_i e^{-r\tau_1} > x\right) P(\theta_2 + \theta_3 + \ldots + \theta_n \leqslant t) \\ &\leqslant \sum_{n=N+1}^{\infty} \int_{0^-}^{t} P\left(\sum_{i=1}^{n} X_i > xe^{rs}\right) P(\tau_1 \in ds) P(N_1(t) \geqslant n-1) \\ &\leqslant \sum_{n=N+1}^{\infty} P(N_1(t) \geqslant n-1) \int_{0^-}^{t} P\left(\sum_{i=1}^{n} X_i > xe^{rs}\right) d\lambda(s) \\ &= \sum_{n=N}^{\infty} P(N_1(t) \geqslant n) \int_{0^-}^{t} P\left(\sum_{i=1}^{n+1} X_i > xe^{rs}\right) d\lambda(s) \\ &\leqslant K \sum_{n=N}^{\infty} (1+\varepsilon)^{n+1} P(N_1(t) \geqslant n) \int_{0^-}^{t} \overline{F}(xe^{rs}) d\lambda(s), \end{aligned} \tag{4.5}$$

where $N_1(t) = \sum_{n=2}^{\infty} \mathbf{1}_{\{\theta_2+\theta_3+\ldots+\theta_n \leqslant t\}}$, $t \geqslant 0$. Note that $\{N_1(t), t \geqslant 0\}$ is another counting process generated by $\{\theta_n, n \geqslant 2\}$, and then is still a non-stationary process fulfilling Assumptions **A1** and **A2**. Therefore, for any fixed $x_1 > x_0$, there exists some $\delta_1 > 0$ such that $I(x_1) - \delta_1 > 0$, and for sufficiently large t,

$$P(N_1(t)/t \geqslant x_1) \leqslant e^{-t(I(x_1)-\delta_1)}. \tag{4.6}$$

By Assumptions **A1** and **A2**, for any $\varepsilon \in (0,1)$, there exist some small $\delta_2 > 0$, $\varepsilon \in (0,1)$ and sufficiently large $N > 2x_0 T$ such that $I(2x_0) - \delta_2 > \ln(1+\varepsilon)^{4x_0}$. Hence by (4.6), we derive that for all $t \in \Lambda_T$,

$$\begin{aligned}\sum_{n=N}^{\infty}(1+\varepsilon)^{n+1}P(N_1(t) \geqslant n) &\leqslant \sum_{n=N}^{\infty}(1+\varepsilon)^{n+1}P\left(N_1\left(\frac{n}{2x_0}\right) \geqslant n\right)\\ &\leqslant \sum_{n=N}^{\infty}(1+\varepsilon)^{n+1}e^{-\frac{n}{2x_0}(I(2x_0)-\delta_2)}\\ &< \frac{\varepsilon}{K},\end{aligned}\tag{4.7}$$

which, along with (4.5), proves that for all $t \in \Lambda_T$,

$$J_1(x,t) \leqslant \varepsilon \int_{0^-}^{t} \overline{F}(xe^{rs})d\lambda(s).\tag{4.8}$$

For $J_2(x,t)$, we denote the joint distribution of random vector $(\tau_1, \tau_2, \ldots, \tau_{n+1})$ by $H(t_1,t_2,\ldots,t_{n+1})$, and write $\Omega_{n,t} = \{0 \leqslant t_1 \leqslant \ldots \leqslant t_n \leqslant t, t_{n+1} > t\}$ for any $t \in \Lambda_T$. Hence by Lemma 4.2(i), we prove that for all $t \in \Lambda_T$,

$$\begin{aligned}J_2(x,t) &= \sum_{n=1}^{N} P\left(\sum_{i=1}^{n} X_i e^{-r\tau_i} > x, N(t) = n\right)\\ &= \sum_{n=1}^{N}\int_{\Omega_{n,t}} P\left(\sum_{i=1}^{n} X_i e^{-rt_i} > x\right) dH(t_1,t_2,\ldots,t_{n+1})\\ &\sim \sum_{n=1}^{N}\sum_{i=1}^{n}\int_{\Omega_{n,t}} P\left(X_i e^{-rt_i} > x\right) dH(t_1,t_2,\ldots,t_{n+1})\\ &= \left(\sum_{n=1}^{\infty} - \sum_{n=N+1}^{\infty}\right)\sum_{i=1}^{n} P\left(X_i e^{-r\tau_i} > x, N(t) = n\right)\\ &:= J_{21}(x,t) - J_{22}(x,t).\end{aligned}\tag{4.9}$$

Clearly, for all $t \in \Lambda_T$,

$$\begin{aligned}J_{21}(x,t) &= \sum_{i=1}^{\infty} P\left(X_i e^{-r\tau_i} > x, \tau_i \leqslant t\right)\\ &= \int_{0^-}^{t} \overline{F}(xe^{rs})d\lambda(s),\end{aligned}\tag{4.10}$$

and

$$\begin{aligned}J_{22}(x,t) &\leqslant \sum_{n=N+1}^{\infty}\sum_{i=1}^{n} P\left(X_i e^{-r\tau_1} > x, N(t) = n\right)\\ &= \sum_{n=N+1}^{\infty}\sum_{i=1}^{n}\int_{0^-}^{t} P(X_i > xe^{rs})P(\tau_1 \in ds)P(N_1(t) \geqslant n-1)\end{aligned}$$

$$\leqslant \sum_{n=N}^{\infty}(n+1)P(N_1(t)\geqslant n)\int_{0^-}^{t}\overline{F}(xe^{rs})d\lambda(s)$$
$$\leqslant \sum_{n=N}^{\infty}(1+\varepsilon)^{n+1}P(N_1(t)\geqslant n)\int_{0^-}^{t}\overline{F}(xe^{rs})d\lambda(s)$$
$$\leqslant \varepsilon\int_{0^-}^{t}\overline{F}(xe^{rs})d\lambda(s), \tag{4.11}$$

where in the last step we used the similar method to that in (4.8). Hence, we substitute (4.10) and (4.11) into (4.9) to obtain that for all $t\in\Lambda_T$,

$$(1-\varepsilon)\int_{0^-}^{t}\overline{F}(xe^{rs})d\lambda(s)\lesssim J_2(x,t)\lesssim\int_{0^-}^{t}\overline{F}(xe^{rs})d\lambda(s). \tag{4.12}$$

Now, we conclude from (4.4), (4.8) and (4.12) that for all $t\in\Lambda_T$,

$$(1-\varepsilon)\int_{0^-}^{t}\overline{F}(xe^{rs})d\lambda(s)\lesssim P\left(\sum_{i=1}^{N(t)}X_ie^{-r\tau_i}>x\right)\lesssim(1+\varepsilon)\int_{0^-}^{t}\overline{F}(xe^{rs})d\lambda(s),$$

which, along with the arbitrariness of ε, completes the proof of Lemma 4.4. □

Recall that in risk model (1.1), the diffusion perturbation $\{B(t),t\geqslant 0\}$ is a standard Brownian motion. For $t\geqslant 0$, we write

$$\mathbb{B}(t)=\sigma\int_{0^-}^{t}e^{-rs}dB(s),$$

$$\mathbb{B}_*(t)=\inf_{0\leqslant s\leqslant t}\mathbb{B}(s)\quad\text{and}\quad\mathbb{B}^*(t)=\sup_{0\leqslant s\leqslant t}\mathbb{B}(s),$$

where $\mathbb{B}_*(t)\leqslant 0$, $\mathbb{B}^*(t)\geqslant 0$, and $\{\mathbb{B}(t),t\geqslant 0\}$ is a Gaussian process with mean 0 and variance function

$$\sigma^2(t)=\sigma^2\int_{0^-}^{t}e^{-2rs}ds=\frac{\sigma^2(1-e^{-2rt})}{2r},\quad t\geqslant 0.$$

Obviously, if $\sigma>0$, then $\sigma^2(t)$ is strictly increasing and hence get its maximum at the unique point T on any interval $[0,T]$ for $T>0$. And by Theorem D.3(ii) of Piterbarg (1996), it follows that for every $t>0$,

$$P(\mathbb{B}_*(t)<-x)=P(\mathbb{B}^*(t)>x)\sim 2\overline{\Phi}\left(\frac{\sqrt{2r}}{\sigma\sqrt{1-e^{-2rt}}}x\right), \tag{4.13}$$

where Φ is the standard Gaussian distribution function, and the second relation shows that for every $t>0$, $\mathbb{B}^*(t)$ either degenerates at 0 as $\sigma=0$ or has a Gaussian-type tail. So by Lemma 4.1(ii), the tail of $\mathbb{B}^*(t)$ is negligible asymptotically in comparison to any subexponential tail, and then satisfies all the conditions imposed on ξ in Lemma 4.2(ii). Hence, we get the following lemma, which is an extension of Lemma 4.4.

LEMMA 4.5. *Under the conditions of Theorem 2.1, it holds for all $t \in \Lambda_T$ that*

$$P\left(\sum_{i=1}^{N(t)} X_i e^{-r\tau_i} \pm \mathbb{B}^*(t) > x\right) \sim \int_{0^-}^t \overline{F}(xe^{rs})d\lambda(s). \tag{4.14}$$

Proof. In order to establish relation (4.14), it suffices to prove that for all $t \in \Lambda_T$,

$$P\left(\sum_{i=1}^{N(t)} X_i e^{-r\tau_i} + \mathbb{B}^*(t) > x\right) \sim \int_{0^-}^t \overline{F}(xe^{rs})d\lambda(s) \tag{4.15}$$

and

$$P\left(\sum_{i=1}^{N(t)} X_i e^{-r\tau_i} - \mathbb{B}^*(t) > x\right) \sim \int_{0^-}^t \overline{F}(xe^{rs})d\lambda(s). \tag{4.16}$$

Firstly, we prove relation (4.15) with the similar method to that in the proof of Lemma 4.4. Note that, for an arbitrarily fixed integer $N \geqslant 1$,

$$\begin{aligned} P\left(\sum_{i=1}^{N(t)} X_i e^{-r\tau_i} + \mathbb{B}^*(t) > x\right) &= \left(\sum_{n=N+1}^{\infty} + \sum_{n=1}^{N}\right) P\left(\sum_{i=1}^{n} X_i e^{-r\tau_i} + \mathbb{B}^*(t) > x, N(t) = n\right) \\ &:= J_3(x,t) + J_4(x,t). \end{aligned} \tag{4.17}$$

For $J_3(x,t)$, we obtain that for all $t \in \Lambda_T$,

$$\begin{aligned} J_3(x,t) &= \sum_{n=N+1}^{\infty} P\left(\sum_{i=1}^{n} X_i e^{-r\tau_i} + \mathbb{B}^*(t) > x, N(t) = n\right) \\ &\leqslant \sum_{n=N+1}^{\infty} P\left(\sum_{i=1}^{n} X_i e^{-r\tau_1} + \mathbb{B}^*(t)\mathbf{1}_{\{\mathbb{B}^*(t)\leqslant x\}} > x, \tau_n \leqslant t\right) + P(\mathbb{B}^*(t) > x) \\ &:= J_{31}(x,t) + J_{32}(x,t). \end{aligned} \tag{4.18}$$

By $F \in \mathscr{S}$, Lemmas 4.2(ii) and 4.3, it follows that for all $t \in \Lambda_T$,

$$\begin{aligned} J_{31}(x,t) = &\sum_{n=N+1}^{\infty} \int_{0^-}^t \int_{0^-}^x P\left(\sum_{i=1}^{n} X_i > (x-y)e^{rs}\right) \\ &P(\mathbb{B}^*(t) \in dy)P(\tau_1 \in ds)P(N_1(t) \geqslant n-1) \\ \leqslant &\sum_{n=N}^{\infty} P(N_1(t) \geqslant n) \int_{0^-}^t \int_{0^-}^x P\left(\sum_{i=1}^{n+1} X_i > (x-y)e^{rs}\right) P(\mathbb{B}^*(t) \in dy)P(\tau_1 \in ds) \\ \leqslant &K \sum_{n=N}^{\infty} (1+\varepsilon)^{n+1} P(N_1(t) \geqslant n) \int_{0^-}^t \int_{0^-}^x \overline{F}((x-y)e^{rs})P(\mathbb{B}^*(t) \in dy)P(\tau_1 \in ds) \\ \leqslant &K \sum_{n=N}^{\infty} (1+\varepsilon)^{n+1} P(N_1(t) \geqslant n) P\left(X_1 e^{-r\tau_1}\mathbf{1}_{\{\tau_1 \leqslant t\}} + \mathbb{B}^*(t) > x\right) \end{aligned}$$

$$\begin{aligned}
&\leqslant K\sum_{n=N}^{\infty}(1+\varepsilon)^{n+1}P(N_1(t)\geqslant n)\int_{e^{-rT}}^{1}P\left(yX_1+\mathbb{B}^*(t)>x\right)P(e^{-r\tau_1}\in dy)\\
&\sim K\sum_{n=N}^{\infty}(1+\varepsilon)^{n+1}P(N_1(t)\geqslant n)P\left(X_1e^{-r\tau_1}\mathbf{1}_{\{\tau_1\leqslant t\}}>x\right)\\
&\leqslant K\sum_{n=N}^{\infty}(1+\varepsilon)^{n+1}P(N_1(t)\geqslant n)\int_{0^-}^{t}\overline{F}(xe^{rs})d\lambda(s),
\end{aligned}$$

which, similarly to the derivation of (4.8), implies that for all $t\in\Lambda_T$,

$$J_{31}(x,t)\lesssim\varepsilon\int_{0^-}^{t}\overline{F}(xe^{rs})d\lambda(s). \tag{4.19}$$

Since $F\in\mathscr{S}$ and $e^{-r\tau_1}\mathbf{1}_{\{\tau_1\leqslant t\}}\in[0,1]$, we can know by Theorem 1.1 of Tang (2006) that $X_1e^{-r\tau_1}\mathbf{1}_{\{\tau_1\leqslant t\}}$ has a subexponential tail. Thus by (4.13) and Lemma 4.1(ii), we get that for all $t>0$ and sufficiently large x,

$$J_{32}(x,t)=P(\mathbb{B}^*(t)>x)<\varepsilon P(X_1e^{-r\tau_1}\mathbf{1}_{\{\tau_1\leqslant t\}}>x)\leqslant\varepsilon\int_{0^-}^{t}\overline{F}(xe^{rs})d\lambda(s),$$

which, along with (4.18) and (4.19), yields that for all $t\in\Lambda_T$,

$$J_3(x,t)\lesssim 2\varepsilon\int_{0^-}^{t}\overline{F}(xe^{rs})d\lambda(s). \tag{4.20}$$

For $J_4(x,t)$, by Lemma 4.2(ii), we prove that for all $t\in\Lambda_T$,

$$\begin{aligned}
J_4(x,t)&=\sum_{n=1}^{N}P\left(\sum_{i=1}^{n}X_ie^{-r\tau_i}+\mathbb{B}^*(t)>x,N(t)=n\right)\\
&=\sum_{n=1}^{N}\int_{\Omega_{n,t}}P\left(\sum_{i=1}^{n}X_ie^{-rt_i}+\mathbb{B}^*(t)>x\right)dH(t_1,t_2,\ldots,t_{n+1})\\
&\sim\sum_{n=1}^{N}\sum_{i=1}^{n}\int_{\Omega_{n,t}}P\left(X_ie^{-rt_i}>x\right)dH(t_1,t_2,\ldots,t_{n+1})\\
&\leqslant\sum_{n=1}^{\infty}\sum_{i=1}^{n}P\left(X_ie^{-r\tau_i}>x,N(t)=n\right)\\
&=\sum_{i=1}^{\infty}P\left(X_ie^{-r\tau_i}>x,\tau_i\leqslant t\right)\\
&=\int_{0^-}^{t}\overline{F}(xe^{rs})d\lambda(s).
\end{aligned} \tag{4.21}$$

Therefore by (4.17), (4.20), (4.21) and the arbitrariness of ε, we derive that relation (4.15) holds for all $t\in\Lambda_T$.

Secondly, we turn to prove relation (4.16). On the one hand, we get by Lemma 4.4 that for all $t\in\Lambda_T$,

$$P\left(\sum_{i=1}^{N(t)}X_ie^{-r\tau_i}-\mathbb{B}^*(t)>x\right)\leqslant P\left(\sum_{i=1}^{N(t)}X_ie^{-r\tau_i}>x\right)\sim\int_{0^-}^{t}\overline{F}(xe^{rs})d\lambda(s). \tag{4.22}$$

On the other hand, by $F \in \mathscr{S} \subset \mathscr{L}$, and Lemmas 4.1(i) and 4.4, we obtain that for all $t \in \Lambda_T$,

$$P\left(\sum_{i=1}^{N(t)} X_i e^{-r\tau_i} - \mathbb{B}^*(t) > x\right) \geqslant \int_{0^-}^{h(x)} P\left(\sum_{i=1}^{N(t)} X_i e^{-r\tau_i} > x+y\right) P(\mathbb{B}^*(t) \in dy)$$
$$\sim \int_{0^-}^{h(x)} \int_{0^-}^{t} \overline{F}((x+y)e^{rs}) d\lambda(s) P(\mathbb{B}^*(t) \in dy)$$
$$\sim \int_{0^-}^{t} \overline{F}(xe^{rs}) d\lambda(s). \tag{4.23}$$

Consequently, we prove by (4.22) and (4.23) that relation (4.16) holds for all $t \in \Lambda_T$. This completes the proof of Lemma 4.5. □

Proof of Theorem 2.1. Firstly, we prove the upper asymptotic bound of relation (2.1). For a sufficiently large N, by Lemma 4.5, (1.3) and (4.13), it follows that for $\tau \in \Lambda_T$,

$$\psi(x,\tau) \leqslant \int_{0^-}^{T} P\left(\sum_{i=1}^{N(t)} X_i e^{-r\tau_i} - \mathbb{B}_*(t) > x\right) P(\tau \in dt)$$
$$\leqslant \int_{0^-}^{T} P\left(\sum_{i=1}^{N(t)} X_i e^{-r\tau_i} + \mathbb{B}^*(t) > x\right) P(\tau \in dt)$$
$$\sim \int_{0^-}^{T} \int_{0^-}^{t} \overline{F}(xe^{rs}) d\lambda(s) P(\tau \in dt)$$
$$= \int_{0^-}^{T} \int_{s}^{T} \overline{F}(xe^{rs}) P(\tau \in dt) d\lambda(s)$$
$$= \int_{0^-}^{T} \overline{F}(xe^{rs}) P(s \leqslant \tau \leqslant T) d\lambda(s). \tag{4.24}$$

Secondly, we prove the lower asymptotic bound of relation (2.1). Note that from (1.3) and Lemma 4.5, it follows that for $\tau \in \Lambda_T$ and $h(x)$ in Lemma 4.1(i),

$$\psi(x,\tau) \geqslant \int_{0^-}^{T} P\left(\sum_{i=1}^{N(t)} X_i e^{-r\tau_i} - \mathbb{B}^*(t) > x + \widetilde{C}(t)\right) P(\tau \in dt)$$
$$\geqslant \int_{0^-}^{T} \int_{0}^{h(x)} P\left(\sum_{i=1}^{N(t)} X_i e^{-r\tau_i} - \mathbb{B}^*(t) > x+y\right) P(\widetilde{C}(t) \in dy) P(\tau \in dt)$$
$$\sim \int_{0^-}^{T} \int_{0}^{h(x)} \int_{0^-}^{t} \overline{F}((x+y)e^{rs}) d\lambda(s) P(\widetilde{C}(t) \in dy) P(\tau \in dt)$$
$$\sim \int_{0^-}^{T} \int_{0^-}^{t} \overline{F}(xe^{rs}) d\lambda(s) P(\tau \in dt)$$
$$= \int_{0^-}^{T} \overline{F}(xe^{rs}) P(s \leqslant \tau \leqslant T) d\lambda(s),$$

which, along with (4.24), implies that relation (2.1) holds for $\tau \in \Lambda_T$. This completes the proof of Theorem 2.1. □

5. Proof of Theorem 2.2

In this section, we are ready to prove Theorem 2.2. Now we should give some lemmas, among which the first one is the counterpart of Lemma 4.2 with the underlying r.v.s satisfying UTAI structures, and comes from Lemma 2.1 of Liu et al. (2012) and Lemma 3.2 of Li (2017) with slight modifications.

LEMMA 5.1. *(i) Let* $\{\xi_i, 1 \leqslant i \leqslant n\}$ *be n nonnegative and UTAI r.v.s with common distribution* $V \in \mathscr{L} \cap \mathscr{D}$*, then for every fixed* $0 < a \leqslant b < \infty$*, relation* (4.1) *still holds uniformly for all* $(c_1, c_2, \ldots, c_n) \in [a,b]^n$.

(ii) Let the conditions of (i) be true, and ξ *be a real-valued r.v., independent of* $\{\xi_i, 1 \leqslant i \leqslant n\}$*, such that* $P(\xi > x) = o(1)\overline{V}(x/c)$ *for some* $c > 0$*, then relation* (4.2) *still holds uniformly for all* $(c_1, c_2, \ldots, c_n) \in [a,b]^n$.

The second lemma is a restatement of Lemma 3.4 of Liu and Gao (2022), where neither independence, nor a special dependence structure, is required among the underlying r.v.s.

LEMMA 5.2. *Let* $\{\xi_n, n \geqslant 1\}$ *be a sequence of real-valued r.v.s with common distribution* $V \in \mathscr{D}$*. Then for any* $\beta > J_V^+$*, there exists some constant* $C > 0$ *such that for all* $x \geqslant 0$ *and* $n \geqslant 1$,

$$P\left(\sum_{i=1}^{n} \xi_i > x\right) \leqslant C n^{\beta+1} \overline{V}(x).$$

LEMMA 5.3. *Under the conditions of Theorem* 2.2*, relation* (4.3) *holds for all* $t \in \Lambda_T$.

Proof. Following the proof of Lemma 4.4 with only change that we use Lemma 5.1(i) instead of Lemma 4.2(i), we will achieve the proof if we show that relations (4.8) and (4.11) still hold for all $t \in \Lambda_T$ under the conditions of Theorem 2.2.

Firstly, we establish relation (4.8) under the conditions of Theorem 2.2. By Assumptions **A1** and **A2**, we obtain that for $\varepsilon \in (0,1)$ as above, there exist some small $\delta_3 > 0$, $\varepsilon \in (0,1)$ and sufficiently large $N > 2x_0 T$ such that $I(2x_0) - \delta_3 > \ln(1+\varepsilon)^{4x_0}$. Then, similarly to the proof of (4.7), we derive by (4.6) that for all $t \in \Lambda_T$,

$$\sum_{n=N}^{\infty} (n+1)^{\beta+1} P(N_1(t) \geqslant n) \leqslant \sum_{n=N}^{\infty} (1+\varepsilon)^{n+1} P(N_1(t) \geqslant n) < \varepsilon / C. \tag{5.1}$$

Similarly to the proof of (4.5), we also derive by Lemma 5.2, (4.6) and (5.1) that for all $t \in \Lambda_T$,

$$J_1(x,t) \leqslant \varepsilon \int_{0^-}^{t} \overline{F}(xe^{rs}) d\lambda(s).$$

Subsequently, we establish relation (4.11) under the conditions of Theorem 2.2. Similarly to the proof of (5.1), for $\varepsilon \in (0,1)$ as above, there exist some small $\delta_4 > 0$, $\varepsilon \in (0,1)$ and sufficiently large $N > 2x_0T$ such that $I(2x_0) - \delta_4 > \ln(1+\varepsilon)^{4x_0}$ and $\sum_{n=N}^{\infty}(n+1)e^{-\frac{n}{2x_0}(I(2x_0)-\delta_4)} < \varepsilon$. Then it follows that for all $t \in \Lambda_T$,

$$\begin{aligned} J_{22}(x,t) &\leqslant \sum_{n=N}^{\infty}(n+1)P(N_1(t)\geqslant n)\int_{0-}^{t}\overline{F}(xe^{rs})d\lambda(s) \\ &\leqslant \sum_{n=N}^{\infty}(n+1)e^{-\frac{n}{2x_0}(I(2x_0)-\delta_4)}\int_{0-}^{t}\overline{F}(xe^{rs})d\lambda(s) \\ &\leqslant \varepsilon\int_{0-}^{t}\overline{F}(xe^{rs})d\lambda(s). \end{aligned}$$

This completes the proof of Lemma 5.3. □

LEMMA 5.4. *Under the conditions of Theorem* 2.2*, relation* (4.14) *holds for all* $t \in \Lambda_T$.

Proof. By mimicking the proof of Lemma 4.5 with only change that we use Lemmas 5.1(ii) and 5.3 instead of Lemmas 4.2(ii) and 4.4, respectively, this proof can be accomplished if we prove that the relation (4.19) holds for all $t \in \Lambda_T$ under the conditions of Theorem 2.2. In fact, similarly to the proof of (4.19), by Lemmas 5.1(ii) and 5.2, we know that for all $t \in \Lambda_T$,

$$\begin{aligned} J_{31}(x,t) &\leqslant \sum_{n=N}^{\infty}P(N_1(t)\geqslant n)\int_{0-}^{t}\int_{0-}^{x}P\left(\sum_{i=1}^{n+1}X_i > (x-y)e^{rs}\right)P(\mathbb{B}^*(t)\in dy)P(\tau_1\in ds) \\ &\leqslant C\sum_{n=N}^{\infty}(n+1)^{\beta+1}P(N_1(t)\geqslant n)\int_{0-}^{t}\int_{0-}^{x}\overline{F}((x-y)e^{rs})P(\mathbb{B}^*(t)\in dy)P(\tau_1\in ds) \\ &\lesssim C\sum_{n=N}^{\infty}(n+1)^{\beta+1}P(N_1(t)\geqslant n)\int_{0-}^{t}\overline{F}(xe^{rs})d\lambda(s), \end{aligned}$$

which, along with (5.1), proves that for all $t \in \Lambda_T$,

$$J_{31}(x,t) \lesssim \varepsilon\int_{0-}^{t}\overline{F}(xe^{rs})d\lambda(s).$$

This completes the proof of Lemma 5.4. □

Proof of Theorem 2.2. By going along the same lines of the proof of Theorem 2.1, we conclude by Lemma 5.4 that relation (2.1) still holds for $\tau \in \Lambda_T$, and hence complete the proof of Theorem 2.2. □

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