Journal of Mathematical Inequalities
Volume 17, Number 3 (2023), 867–883
doi:10.7153/jmi-2023-17-55

ON THE SPECTRAL NORMS OF r–CIRCULANT AND GEOMETRIC CIRCULANT MATRICES WITH THE BI–PERIODIC HYPER–HORADAM SEQUENCE

NASSIMA BELAGGOUN* AND HACÈNE BELBACHIR

(*Communicated by M. Sababheh*)

Abstract. In this paper, we define the bi-periodic hyper-Horadam sequence $\{w_n^{(k)}\}_{n\in\mathbb{N}}$ and present its combinatorial properties. Moreover, we obtain upper and lower bounds for the spectral norms of different forms of the r-circulant and geometric circulant matrices with the bi-periodic hyper-Horadam sequence. Then we give some bounds for the spectral norms of the Kronecker and Hadamard products of these matrices.

1. Introduction

The Fibonacci numbers are defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$ for any $n \geqslant 2$, with $F_0 = 0$ and $F_1 = 1$ as initial conditions. There have been many studies in the literature dealing with the generalized Fibonacci sequence. In 1965, Horadam [12] gave a generalization of this recurrence, called the Horadam sequence, which is defined as

$$H_n = xH_{n-1} + yH_{n-2}, \quad n \geqslant 2,$$

with initial values H_0 and H_1, where H_0, H_1, x, and y are arbitrary integers.

The hyper-Horadam numbers, denoted as $H_n^{(k)}(H_0,H_1;x,y)$, or briefly, $H_n^{(k)}$, are defined by the following recurrence relation

$$H_n^{(k)} = xH_{n-1}^{(k)} + yH_n^{(k-1)}, \quad n,k \geqslant 1,$$

with $H_0^{(k)} = y^k H_0$ and $H_n^{(0)} = H_n$, where H_n is the n-th Horadam number. This recurrence relation can be written as follows (see [6])

$$H_n^{(k)} = \sum_{j=0}^{n} yx^{n-j} H_j^{(k-1)}.$$

Mathematics subject classification (2020): 15A60, 15B05, 15B36, 11B39.
Keywords and phrases: Spectral norms, Euclidean norms, r-circulant matrix, geometric circulant matrix, bi-periodic hyper-Horadam sequence.

* Corresponding author.

Paper JMI-17-55

Another interesting generalization of the Fibonacci sequence, called the bi-periodic Horadam sequence, $w_n := w_n(w_0, w_1; a, b, t)$, was introduced by Edson and Yayenie in [10] as follows:

$$w_n = \begin{cases} aw_{n-1} + t\,w_{n-2}, & \text{if } n \text{ is even,} \\ bw_{n-1} + t\,w_{n-2}, & \text{if } n \text{ is odd,} \end{cases} \tag{1}$$

with a, b, t, w_0, and w_1 are arbitrary positive integers. Obviously, when $w_0 = 0$, $w_1 = 1$, and $w_0 = 2$, $w_1 = a$, these two sequences reduce to the well-known bi-periodic Fibonacci sequence [10] and bi-periodic Lucas sequence [7], respectively.

On the other hand, given a real or complex number r, an $n \times n$ r-circulant matrix, C_r, associated with the complex numbers $c_0, c_1, \ldots, c_{n-1}$, is of the form $c_{i,j} = c_{j-i}$ whenever $j \geqslant i$ and $c_{i,j} = rc_{n+j-i}$ for $j < i$, that is

$$C_r = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ rc_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ rc_{n-2} & rc_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ rc_2 & rc_3 & rc_4 & \cdots & c_0 & c_1 \\ rc_1 & rc_2 & rc_3 & \cdots & rc_{n-1} & c_0 \end{pmatrix}.$$

For brevity, we denote it as $C_r = Circ_{n,r}(c_0,\ c_1, \ldots, c_{n-1})$.

In [15], Kızılateş and Tuglu defined the $n \times n$ geometric circulant matrix, C_{r^*}, associated with the complex numbers $c_0, c_1, \ldots, c_{n-1}$, as

$$Cr^* = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ rc_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ r^2c_{n-2} & rc_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ r^{n-2}c_2 & r^{n-3}c_3 & r^{n-4}c_4 & \cdots & c_0 & c_1 \\ r^{n-1}c_1 & r^{n-2}c_2 & r^{n-3}c_3 & \cdots & rc_{n-1} & c_0 \end{pmatrix}.$$

For brevity, we denote the geometric circulant matrix with $C_{r^*} = Circ_{n,r^*}(c_0, c_1, \ldots, c_{n-1})$.

Note that for $r = 1$, the r-circulant and the geometric circulant matrices reduce to the circulant matrix $C = Circ_n(c_0, c_1, \ldots, c_{n-1})$. The circulant matrices are normal matrices [9], i.e., $AA^H = A^H A$, where A^H is the conjugate transpose matrix of A. The eigenvalues of C are computed as follows:

$$\lambda_s = \sum_{k=0}^{n-1} c_k \mu_s^{-k}, \quad s = 0, 1, \ldots, n-1, \tag{2}$$

where $\mu_s = \exp(\frac{2\pi i}{n} s)$ and $i^2 = -1$ (see [9, 14]).

In this study, we introduce the bi-periodic hyper-Horadam sequence and establish some combinatorial identities. Then, we compute the spectral and Euclidean norms of different forms of the circulant matrices associated with the bi-periodic hyper-Horadam sequence. Moreover, we use some relations concerning the spectral and Euclidean norms to give the upper and lower bounds of the spectral norms of the r-circulant and the geometric circulant matrices with the bi-periodic hyper-Horadam sequence and their Hadamard and Kronecker products.

Now, we will provide some definitions and lemmas related to our research.

DEFINITION 1. Let $A=(a_{ij})$ be any $m\times n$ matrix. The well-known Frobenious (or Euclidean) norm of A is

$$\|A\|_E=\left(\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2\right)^{1/2}. \tag{3}$$

DEFINITION 2. Let $A=(a_{ij})$ be any $m\times n$ matrix. The spectral norm of A is

$$\|A\|_2=\sqrt{\max_i \lambda_i(A^HA)}, \tag{4}$$

where $\lambda_i(A^HA)$ are eigenvalues of A^HA.

The connection between Frobenius norm and spectral norm is given by (see [13])

$$\frac{1}{\sqrt{n}}\|A\|_E\leqslant\|A\|_2\leqslant\|A\|_E \tag{5}$$

and

$$\|A\|_2\leqslant\|A\|_E\leqslant\sqrt{n}\|A\|_2. \tag{6}$$

LEMMA 1. ([13]) *Let A be a normal matrix with eigenvalues $\lambda_0,\lambda_1,\ldots,\lambda_{n-1}$. Then the spectral norm of A is*

$$\|A\|_2=\max_{0\leqslant i\leqslant n-1}\lambda_i. \tag{7}$$

DEFINITION 3. Let $A=(a_{ij})$ and $B=(b_{ij})$ be two $m\times n$ matrices. The Hadamard product of A and B is (see [19, 21, 26])

$$A\circ B=(a_{ij}b_{ij}).$$

The following inequalities involving the Hadamard product are valid.

LEMMA 2. ([13]) *Let $A=(a_{ij})$ and $B=(b_{ij})$ be any $m\times n$-matrices. Then*

$$\|A\circ B\|_2\leqslant\|A\|_2\|B\|_2. \tag{8}$$

LEMMA 3. ([19]) *Let $A=(a_{ij})$ and $B=(b_{ij})$ be any $m\times n$-matrices. Then*

$$\|A\circ B\|_2\leqslant r_1(A)c_1(B), \tag{9}$$

where

$$r_1(A)=\max_{1\leqslant i\leqslant m}\sqrt{\sum_{j=1}^{n}|a_{ij}|^2}\quad \text{and}\quad c_1(B)=\max_{1\leqslant j\leqslant n}\sqrt{\sum_{i=1}^{m}|b_{ij}|^2}.$$

DEFINITION 4. Let $A=(a_{ij})$ and $B=(b_{ij})$ be $m\times n$ and $p\times q$ matrices, respectively, the Kronecker product of A and B noted $A\otimes B$ is the $pm\times qn$ block matrix defined by

$$A\otimes B=(a_{ij}B)=\begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}.$$

It has the following property.

LEMMA 4. ([13, 19]) *Let $A=(a_{ij})$ and $B=(b_{ij})$ be $m\times n$ and $p\times q$ matrices, respectively. Then*

$$\|A\otimes B\|_2=\|A\|_2\|B\|_2. \tag{10}$$

2. Main results

We start by defining the bi-periodic hyper-Horadam sequence $(w_n^{(k)}(w_0,w_1;a,b,t))_n$, or briefly $(w_n^{(k)})_n$.

DEFINITION 5. Let a, b, t, w_0, and w_1 be arbitrary positive integers. The bi-periodic hyper-Horadam sequence is defined by

$$w_n^{(k)}=\begin{cases} aw_{n-1}^{(k)}+t\,w_n^{(k-1)}, & \text{if } n \text{ is even},\\ bw_{n-1}^{(k)}+t\,w_n^{(k-1)}, & \text{if } n \text{ is odd}.\end{cases} \tag{11}$$

with the initial values $w_0^{(k)}=t^kw_0$ and $w_n^{(0)}=w_n$, where w_n is n-th term of the bi-periodic Horadam sequence.

From the definition, we have the following recurrence relation

$$w_n^{(k+1)}=\sum_{j=0}^{n}a^{\xi(n+1)\xi(j)}b^{\xi(n)\xi(j+1)}(ab)^{\lfloor (n-j)/2\rfloor}tw_j^{(k)}, \tag{12}$$

where $\xi(n)=n-2\lfloor n/2\rfloor$, i.e., $\xi(n)=0$ when n is even and $\xi(n)=1$ when n is odd.

Some of the special cases are:

i. If $a=b=x$ and $t=y$, then $w_n^{(k)}(w_0,w_1;x,x,y)$ is the classical hyper-Horadam numbers, that is, $H_n^{(k)}$.

ii. If $w_n^{(0)}=w_n(0,1;a,b,1)=q_n$ and $w_0^{(k)}=q_0=0$, then $w_n^{(k)}$ is the bi-periodic hyper-Fibonacci numbers, that is, $w_n^{(k)}=q_n^{(k)}$ (see [5]).

iii. If $w_n^{(0)}=w_n(2,a;b,a,1)=l_n$ and $w_0^{(k)}=l_0=2$, then $w_n^{(k)}$ is the bi-periodic hyper-Lucas numbers, that is, $w_n^{(k)}=l_n^{(k)}$.

THEOREM 1. *For any integers $n \geqslant 0$, $k \geqslant 1$, and $l \geqslant 0$, we have*

$$w_n^{(l+k)} = \sum_{j=0}^{n} a^{\xi(n+1)\xi(j)} b^{\xi(n)\xi(j+1)} \binom{n+k-j-1}{k-1} (ab)^{\lfloor (n-j)/2 \rfloor} t^k w_j^{(l)}. \tag{13}$$

Proof. We prove this identity with the principle of mathematical induction on n. Since $w_0^{(l+k)} = t^k w_0^{(l)} = t^{l+k} w_0$, the formula works for $n = 0$. Now assume that the equation is true for $n \geqslant 0$. Then, we can verify it for $n+1$ as follows:

$$\begin{aligned}
&\sum_{j=0}^{n+1} a^{\xi(n)\xi(j)} b^{\xi(n+1)\xi(j+1)} \binom{n+k-j}{k-1} (ab)^{\lfloor \frac{n+1-j}{2} \rfloor} t^k w_j^{(l)} \\
&\quad = \sum_{j \geqslant 0} a^{\xi(n)\xi(j)} b^{\xi(n+1)\xi(j+1)} \left[\binom{n+k-j-1}{k-1} + \binom{n+k-j-1}{k-2} \right] (ab)^{\lfloor \frac{n+1-j}{2} \rfloor} t^k w_j^{(l)}.
\end{aligned}$$

Since $\lfloor \frac{n+1-j}{2} \rfloor = \lfloor \frac{n-j}{2} \rfloor + \xi(n-j)$ and $\xi(n-j) = \xi(n) + \xi(j) - 2\xi(n)\xi(j)$, we get

$$\begin{aligned}
&\sum_{j=0}^{n+1} a^{\xi(n)\xi(j)} b^{\xi(n+1)\xi(j+1)} \binom{n+k-j}{k-1} (ab)^{\lfloor \frac{n+1-j}{2} \rfloor} t^k w_j^{(l)} \\
&\quad = \sum_{j=0}^{n} a^{\xi(n+1)\xi(j)+\xi(n)} b^{\xi(n)\xi(j+1)+\xi(n+1)} \binom{n+k-j-1}{k-1} (ab)^{\lfloor \frac{n-j}{2} \rfloor} t^k w_j^{(l)} \\
&\qquad + \sum_{j=0}^{n+1} a^{\xi(n)\xi(j)} b^{\xi(n+1)\xi(j+1)} \binom{n+k-j-1}{k-2} (ab)^{\lfloor \frac{n+1-j}{2} \rfloor} t^k w_j^{(l)} \\
&\quad = a^{\xi(n)} b^{\xi(n+1)} w_n^{(l+k)} + t w_{n+1}^{(l+k-1)} \\
&\quad = w_{n+1}^{(l+k)},
\end{aligned}$$

which completes the proof. $\square$

Note that, if we take $l = 0$ in (13), we obtain the following result.

COROLLARY 1. *For $n \geqslant 0$ and $k \geqslant 1$, we have*

$$w_n^{(k)} = \sum_{j=0}^{n} a^{\xi(n+1)\xi(j)} b^{\xi(n)\xi(j+1)} \binom{n+k-j-1}{k-1} (ab)^{\lfloor (n-j)/2 \rfloor} t^k w_j. \tag{14}$$

In the following theorem, we give the sum formula for the bi-periodic hyper-Horadam sequence.

THEOREM 2. *For $n \geqslant 1$ and $k \geqslant 1$, we have*

$$\sum_{j=0}^{k} a^{\xi(n)} b^{\xi(n+1)} t^{k-j} w_n^{(j)} = w_{n+1}^{(k)} - t^{k+1} w_{n-1}. \tag{15}$$

Proof. From Corollary 1, we have

$$\begin{aligned}
&\sum_{j=1}^{k} a^{\xi(n)} b^{\xi(n+1)} t^{k-j} w_n^{(j)} \\
&\quad = \sum_{j=1}^{k} a^{\xi(n)} b^{\xi(n+1)} t^{k-j} \sum_{i=0}^{n} a^{\xi(n+1)\xi(i)} b^{\xi(n)\xi(i+1)} \binom{n+j-i-1}{j-1} (ab)^{\lfloor \frac{n-i}{2} \rfloor} t^j w_i \\
&\quad = \sum_{i=0}^{n} a^{\xi(n)\xi(i)} b^{\xi(n+1)\xi(i+1)} (ab)^{\lfloor \frac{n+1-i}{2} \rfloor} t^k w_i \sum_{j=1}^{k} \binom{n+j-i-1}{j-1}.
\end{aligned}$$

By the binomial identity (see [11])

$$\sum_{j=0}^{k} \binom{n+j}{j} = \binom{n+k+1}{k},$$

we have

$$\begin{aligned}
&\sum_{j=1}^{k} a^{\xi(n)} b^{\xi(n+1)} t^{k-j} w_n^{(j)} \\
&\quad = \sum_{i=0}^{n} a^{\xi(n)\xi(i)} b^{\xi(n+1)\xi(i+1)} \binom{n+k-i}{k-1} (ab)^{\lfloor (n+1-i)/2 \rfloor} t^k w_i \\
&\quad = \sum_{i=0}^{n+1} a^{\xi(n)\xi(i)} b^{\xi(n+1)\xi(i+1)} \binom{n+k-i}{k-1} (ab)^{\lfloor (n+1-i)/2 \rfloor} t^k w_i - t^k w_{n+1} \\
&\quad = w_{n+1}^{(k)} - t^k w_{n+1}.
\end{aligned}$$

Thus

$$\sum_{j=0}^{k} a^{\xi(n)} b^{\xi(n+1)} t^{k-j} w_n^{(j)} = w_{n+1}^{(k)} - t^{k+1} w_{n-1}. \quad \square$$

Let $\mathscr{Q}_r$ and $\mathscr{S}_r$ be the r-circulant matrices with the bi-periodic hyper-Horadam numbers defined as

$$\mathscr{Q}_r = Circ_{n,r}\left(b^{\xi(n+1)} (ab)^{\lfloor \frac{n-1}{2} \rfloor} t w_0^{(k)}, a^{\xi(n)} (ab)^{\lfloor \frac{n-2}{2} \rfloor} t w_1^{(k)}, \ldots, t w_{n-1}^{(k)}\right)$$

and

$$\mathscr{S}_r = Circ_{n,r}\left(a^{\xi(k)} b^{\xi(k+1)} t^{n-1} w_k^{(0)}, a^{\xi(k)} b^{\xi(k+1)} t^{n-2} w_k^{(1)}, \ldots, a^{\xi(k)} b^{\xi(k+1)} w_k^{(n-1)}\right).$$

In the following theorem, we evaluate the spectral norm of the circulant matrix $\mathscr{Q}_1$.

THEOREM 3. *For $n \geqslant 1$, the spectral norm of the matrix $\mathscr{Q}_1$ is*

$$\|\mathscr{Q}_1\|_2 = w_{n-1}^{(k+1)}.$$

Proof. According to (2), the eigenvalues of $\mathscr{Q}_1$ are of the form

$$\lambda_s=\sum_{j=0}^{n-1} a^{\xi(n)\xi(j)} b^{\xi(n+1)\xi(j+1)}(ab)^{\lfloor\frac{n-1-j}{2}\rfloor} t w_j^{(k)} \exp\left(-\frac{2\pi i}{n} sj\right), \quad \text{for all } 0 \leqslant s \leqslant n-1.$$

Then, for $s=0$, $\lambda_0=\sum_{j=0}^{n-1} a^{\xi(n)\xi(j)} b^{\xi(n+1)\xi(j+1)}(ab)^{\lfloor\frac{n-1-j}{2}\rfloor} t w_j^{(k)}$. From (12), we get $\lambda_0=w_{n-1}^{(k+1)}$. Hence, for $1 \leqslant s \leqslant n-1$, we have

$$\begin{aligned} |\lambda_s| &= \left|\sum_{j=0}^{n-1} a^{\xi(n)\xi(j)} b^{\xi(n+1)\xi(j+1)}(ab)^{\lfloor\frac{n-1-j}{2}\rfloor} t w_j^{(k)} \exp\left(-\frac{2\pi i}{n} sj\right)\right| \\ &\leqslant \sum_{j=0}^{n-1}\left|a^{\xi(n)\xi(j)} b^{\xi(n+1)\xi(j+1)}(ab)^{\lfloor\frac{n-1-j}{2}\rfloor} t w_j^{(k)}\right|\left|\exp\left(-\frac{2\pi i}{n} sj\right)\right| \\ &\leqslant \sum_{j=0}^{n-1} a^{\xi(n)\xi(j)} b^{\xi(n+1)\xi(j+1)}(ab)^{\lfloor\frac{n-1-j}{2}\rfloor} t w_j^{(k)}=\lambda_0. \end{aligned}$$

Since $\mathscr{Q}_1$ is a normal matrix, we have

$$\|\mathscr{Q}_1\|_2=w_{n-1}^{(k+1)}. \quad \square$$

COROLLARY 2. *The Euclidean norm of the matrix $\mathscr{Q}_1$ holds*

$$w_{n-1}^{(k+1)} \leqslant \|\mathscr{Q}_1\|_E \leqslant \sqrt{n} w_{n-1}^{(k+1)}. \tag{16}$$

Proof. The proof follows from Theorem 3 and the connection between the spectral norm and the Euclidean norm in (6). $\square$

COROLLARY 3. *We have*

$$\frac{1}{\sqrt{n}} w_{n-1}^{(k+1)} \leqslant \sqrt{\sum_{j=0}^{n-1}\left(a^{\xi(n)\xi(j)} b^{\xi(n+1)\xi(j+1)}(ab)^{\lfloor\frac{n-1-j}{2}\rfloor} t w_j^{(k)}\right)^2} \leqslant w_{n-1}^{(k+1)}. \tag{17}$$

Proof. The proof follows from the definition of Euclidean norm (3) and Corollary 2. $\square$

In the following theorem, we evaluate the spectral norm of the circulant matrix $\mathscr{S}_1$.

THEOREM 4. *For $n \geqslant 1$ and $k \geqslant 1$, the spectral norm of the matrix $\mathscr{S}_1$ is*

$$\|\mathscr{S}_1\|_2=w_{k+1}^{(n-1)}-t^n w_{k-1}.$$

Proof. According to (2), the eigenvalues of $\mathscr{S}_1$ are of the form

$$\lambda_s = \sum_{j=0}^{n-1} a^{\xi(k)} b^{\xi(k+1)} t^{n-1-j} w_k^{(j)} \exp\left(-\frac{2\pi i}{n} sj\right).$$

Then, for $s=0$, $\lambda_0 = \sum_{j=0}^{n-1} a^{\xi(k)} b^{\xi(k+1)} t^{n-1-j} w_k^{(j)}$. From (15), we have $\lambda_0 = w_{k+1}^{(n-1)} - t^n w_{k-1}$. Hence, for $1 \leqslant s \leqslant n-1$, we have

$$\begin{aligned}
|\lambda_s| &= \left|\sum_{j=0}^{n-1} a^{\xi(k)} b^{\xi(k+1)} t^{n-1-j} w_k^{(j)} \exp\left(-\frac{2\pi i}{n} sj\right)\right| \\
&\leqslant \sum_{j=0}^{n-1} \left|a^{\xi(k)} b^{\xi(k+1)} t^{n-1-j} w_k^{(j)}\right| \left|\exp\left(-\frac{2\pi i}{n} sj\right)\right| \\
&\leqslant \sum_{j=0}^{n-1} a^{\xi(k)} b^{\xi(k+1)} t^{n-1-j} w_k^{(j)} = \lambda_0.
\end{aligned}$$

Since $\mathscr{S}_1$ is a normal matrix, we get

$$\|\mathscr{S}_1\|_2 = w_{k+1}^{(n-1)} - t^n w_{k-1}. \quad \square$$

COROLLARY 4. *The Euclidean norm of the matrix $\mathscr{S}_1$ holds*

$$w_{k+1}^{(n-1)} - t^n w_{k-1} \leqslant \|\mathscr{S}_1\|_E \leqslant \sqrt{n}\left(w_{k+1}^{(n-1)} - t^n w_{k-1}\right). \tag{18}$$

Proof. The proof follows from Theorem 4 and the connection between the spectral norm and the Euclidean norm in (6). $\square$

COROLLARY 5. *We have*

$$\frac{1}{\sqrt{n}}\left(w_{k+1}^{(n-1)} - t^n w_{k-1}\right) \leqslant \sqrt{\sum_{j=0}^{n-1}\left(a^{\xi(k)} b^{\xi(k+1)} t^{n-1-j} w_k^{(j)}\right)^2} \leqslant w_{k+1}^{(n-1)} - t^n w_{k-1}. \tag{19}$$

Proof. It follows from the definition of the Euclidean norm (3) and Corollary 4. $\square$

COROLLARY 6. *The spectral norm of the Hadamard product of $\mathscr{Q}_1$ and $\mathscr{S}_1$ satisfies*

$$\|\mathscr{Q}_1 \circ \mathscr{S}_1\|_2 \leqslant w_{n-1}^{(k+1)}\left(w_{k+1}^{(n-1)} - t^n w_{k-1}\right).$$

The spectral norm of the Kronecker product of $\mathscr{Q}_1$ and $\mathscr{S}_1$ satisfies

$$\|\mathscr{Q}_1 \otimes \mathscr{S}_1\|_2 = w_{n-1}^{(k+1)}\left(w_{k+1}^{(n-1)} - t^n w_{k-1}\right).$$

In the following theorem, we give the upper and lower bounds for the spectral norm of the r-circulant matrix $\mathscr{Q}_r$.

THEOREM 5. *Let $r \in \mathbb{C}$ and $\mathscr{Q}_r$ be an $n \times n$ r-circulant matrix. Then*

(i) For $|r| \geqslant 1$, we have

$$\frac{1}{\sqrt{n}} w_{n-1}^{(k+1)} \leqslant \|\mathscr{Q}_r\|_2 \leqslant \sqrt{(n-1)|r|^2+1}\, w_{n-1}^{(k+1)}.$$

(ii) For $|r| < 1$, we have

$$\frac{|r|}{\sqrt{n}} w_{n-1}^{(k+1)} \leqslant \|\mathscr{Q}_r\|_2 \leqslant \sqrt{n}\, w_{n-1}^{(k+1)}.$$

Proof. Let

$$\mathscr{Q}_r := \begin{pmatrix} b^{\xi(n+1)}(ab)^{\lfloor \frac{n-1}{2} \rfloor} t w_0^{(k)} & a^{\xi(n)}(ab)^{\lfloor \frac{n-2}{2} \rfloor} t w_1^{(k)} & \cdots & t w_{n-1}^{(k)} \\ r t w_{n-1}^{(k)} & b^{\xi(n+1)}(ab)^{\lfloor \frac{n-1}{2} \rfloor} t w_0^{(k)} & \cdots & a^{\xi(n)} b^{\xi(n+1)} t w_{n-2}^{(k)} \\ \vdots & \vdots & \ddots & \vdots \\ r b^{\xi(n+1)}(ab)^{\lfloor \frac{n-3}{2} \rfloor} t w_2^{(k)} & r a^{\xi(n)}(ab)^{\lfloor \frac{n-4}{2} \rfloor} t w_3^{(k)} & \cdots & a^{\xi(n)}(ab)^{\lfloor \frac{n-2}{2} \rfloor} t w_1^{(k)} \\ r a^{\xi(n)}(ab)^{\lfloor \frac{n-2}{2} \rfloor} t w_1^{(k)} & r b^{\xi(n+1)}(ab)^{\lfloor \frac{n-3}{2} \rfloor} t w_2^{(k)} & \cdots & b^{\xi(n+1)}(ab)^{\lfloor \frac{n-1}{2} \rfloor} t w_0^{(k)} \end{pmatrix}.$$

From the definition of the Euclidean norm, we have

$$\begin{aligned} \|\mathscr{Q}_r\|_E &= \sqrt{\sum_{j=0}^{n-1} (n-j)|c_j|^2 + \sum_{j=0}^{n-1} j|r|^2|c_j|^2} \\ &= \sqrt{\sum_{j=0}^{n-1} ((n-j)+j|r|^2) \left(a^{\xi(n)\xi(j)} b^{\xi(n+1)\xi(j+1)} (ab)^{\lfloor \frac{n-1-j}{2} \rfloor} t w_j^{(k)} \right)^2}. \end{aligned}$$

(i) For $|r| \geqslant 1$, using (17), we get

$$\begin{aligned} \|\mathscr{Q}_r\|_E &\geqslant \sqrt{\sum_{j=0}^{n-1} ((n-j)+j) \left(a^{\xi(n)\xi(j)} b^{\xi(n+1)\xi(j+1]} (ab)^{\lfloor \frac{n-j-1}{2} \rfloor} t w_i^{(k)} \right)^2} \\ &= \sqrt{\sum_{j=0}^{n-1} n \left(a^{\xi(n)\xi(j)} b^{\xi(n+1)\xi(j+1)} (ab)^{\lfloor \frac{n-1-j}{2} \rfloor} t w_j^{(k)} \right)^2} \\ &\geqslant w_{n-1}^{(k+1)}. \end{aligned}$$

From Inequality (5), we obtain

$$\|\mathscr{Q}_r\|_2 \geqslant \frac{1}{\sqrt{n}} w_{n-1}^{(k+1)}.$$

On the other hand, let $\mathscr{Q}_r = A \circ B$, where

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ r & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ r & r & \cdots & 1 & 1 \\ r & r & \cdots & r & 1 \end{pmatrix}$$

and

$$B = \begin{pmatrix} b^{\xi(n+1)}(ab)^{\lfloor\frac{n-1}{2}\rfloor}tw_0^{(k)} & a^{\xi(n)}(ab)^{\lfloor\frac{n-2}{2}\rfloor}tw_1^{(k)} & \cdots & tw_{n-1}^{(k)} \\ tw_{n-1}^{(k)} & b^{\xi(n+1)}(ab)^{\lfloor\frac{n-1}{2}\rfloor}tw_0^{(k)} & \cdots & a^{\xi(n)}b^{\xi(n+1)}tw_{n-2}^{(k)} \\ \vdots & \vdots & \ddots & \vdots \\ b^{\xi(n+1)}(ab)^{\lfloor\frac{n-3}{2}\rfloor}tw_2^{(k)} & a^{\xi(n)}(ab)^{\lfloor\frac{n-4}{2}\rfloor}tw_3^{(k)} & \cdots & a^{\xi(n)}(ab)^{\lfloor\frac{n-2}{2}\rfloor}tw_1^{(k)} \\ a^{\xi(n)}(ab)^{\lfloor\frac{n-2}{2}\rfloor}tw_1^{(k)} & b^{\xi(n+1)}(ab)^{\lfloor\frac{n-3}{2}\rfloor}tw_2^{(k)} & \cdots & b^{\xi(n+1)}(ab)^{\lfloor\frac{n-1}{2}\rfloor}tw_0^{(k)} \end{pmatrix}.$$

Then

$$r_1(A) = \max_{1\leqslant i\leqslant n}\sqrt{\sum_{j=1}^{n}|a_{ij}|^2} = \sqrt{\sum_{j=1}^{n}|a_{nj}|^2} = \sqrt{(n-1)|r|^2+1}$$

and

$$\begin{aligned} c_1(B) &= \max_{1\leqslant j\leqslant n}\sqrt{\sum_{i=1}^{n}|b_{ij}|^2} = \sqrt{\sum_{i=1}^{n}|b_{in}|^2} \\ &= \sqrt{\sum_{i=0}^{n-1}\left(a^{\xi(n)\xi(j)}b^{\xi(n+1)\xi(j+1)}(ab)^{\lfloor\frac{n-j-1}{2}\rfloor}tw_i^{(k)}\right)^2}. \end{aligned}$$

Using (9) and (17), we obtain

$$\|\mathscr{Q}_r\|_2 \leqslant r_1(A)c_1(B) \leqslant \sqrt{(n-1)|r|^2+1}\,w_{n-1}^{(k+1)}.$$

The proof is completed for the first part.

(ii) For $|r|<1$, using (17), we get

$$\begin{aligned} \|\mathscr{Q}_r\|_E &\geqslant \sqrt{\sum_{j=0}^{n-1}((n-j)|r|^2+j|r|^2)\left(a^{\xi(n)\xi(j)}b^{\xi(n+1)\xi(j+1]}(ab)^{\lfloor\frac{n-j-1}{2}\rfloor}tw_i^{(k)}\right)^2} \\ &= |r|\sqrt{\sum_{j=0}^{n-1}n\left(a^{\xi(n)\xi(j)}b^{\xi(n+1)\xi(j+1]}(ab)^{\lfloor\frac{n-j-1}{2}\rfloor}tw_i^{(k)}\right)^2} \\ &\geqslant |r|w_{n-1}^{(k+1)}. \end{aligned}$$

From Inequality (5), we get

$$\|\mathscr{Q}_r\|_2 \geqslant \frac{|r|}{\sqrt{n}} w_{n-1}^{(k+1)}.$$

On the other hand, let $\mathscr{Q}_r = A \circ B$, where

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ r & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ r & r & \cdots & 1 & 1 \\ r & r & \cdots & r & 1 \end{pmatrix}$$

and

$$B = \begin{pmatrix} b^{\xi(n+1)}(ab)^{\lfloor \frac{n-1}{2} \rfloor} t w_0^{(k)} & a^{\xi(n)}(ab)^{\lfloor \frac{n-2}{2} \rfloor} t w_1^{(k)} & \cdots & t w_{n-1}^{(k)} \\ t w_{n-1}^{(k)} & b^{\xi(n+1)}(ab)^{\lfloor \frac{n-1}{2} \rfloor} t w_0^{(k)} & \cdots & a^{\xi(n)} b^{\xi(n+1)} t w_{n-2}^{(k)} \\ \vdots & \vdots & \ddots & \vdots \\ b^{\xi(n+1)}(ab)^{\lfloor \frac{n-3}{2} \rfloor} t w_2^{(k)} & a^{\xi(n)}(ab)^{\lfloor \frac{n-4}{2} \rfloor} t w_3^{(k)} & \cdots & a^{\xi(n)}(ab)^{\lfloor \frac{n-2}{2} \rfloor} t w_1^{(k)} \\ a^{\xi(n)}(ab)^{\lfloor \frac{n-2}{2} \rfloor} t w_1^{(k)} & b^{\xi(n+1)}(ab)^{\lfloor \frac{n-3}{2} \rfloor} t w_2^{(k)} & \cdots & b^{\xi(n+1)}(ab)^{\lfloor \frac{n-1}{2} \rfloor} t w_0^{(k)} \end{pmatrix}.$$

Then

$$r_1(A) = \max_{1 \leqslant i \leqslant n} \sqrt{\sum_{j=1}^{n} |a_{ij}|^2} = \sqrt{\sum_{j=1}^{n} |a_{1j}|^2} = \sqrt{n}$$

and

$$c_1(B) = \max_{1 \leqslant j \leqslant n} \sqrt{\sum_{i=1}^{n} |b_{ij}|^2} = \sqrt{\sum_{i=1}^{n} |b_{in}|^2}$$

$$= \sqrt{\sum_{i=0}^{n-1} \left(a^{\xi(n)\xi(j)} b^{\xi(n+1)\xi(j+1]} (ab)^{\lfloor \frac{n-j-1}{2} \rfloor} t w_i^{(k)} \right)^2}.$$

Using (9) and (17), we obtain the second part of the proof

$$\|\mathscr{Q}_r\|_2 \leqslant r_1(A) c_1(B) \leqslant \sqrt{n} w_{n-1}^{(k+1)}.$$

Therefore, the proof is completed. □

In the following theorem,we give the upper and lower bounds for the spectral norm of the r-circulant matrix $\mathscr{S}_r$.

THEOREM 6. *Let $r \in \mathbb{C}$ and $\mathscr{S}_r$ be an $n \times n$ r-circulant matrix. Then*

(i) For $|r| \geqslant 1$, we have

$$\frac{1}{\sqrt{n}} \left(w_{k+1}^{(n-1)} - t^n w_{k-1} \right) \leqslant \|\mathscr{S}_r\|_2 \leqslant \sqrt{(n-1)|r|^2 + 1} \left(w_{k+1}^{(n-1)} - t^n w_{k-1} \right).$$

(ii) For $|r|<1$, we have

$$\frac{|r|}{\sqrt{n}}\left(w_{k+1}^{(n-1)}-t^{n}w_{k-1}\right)\leqslant\|\mathscr{S}_r\|_2\leqslant\sqrt{n}\left(w_{k+1}^{(n-1)}-t^{n}w_{k-1}\right).$$

Proof. The same method is used to prove the theorem. □

From (8) and (10), we get the following results.

COROLLARY 7. *The spectral norm of Hadamard product of $\mathscr{Q}_r$ and $\mathscr{S}_r$ is given by*

(i) For $|r|\geqslant 1$, we have

$$\|\mathscr{Q}_r\circ\mathscr{S}_r\|_2\leqslant\left((n-1)|r|^2+1\right)w_{n-1}^{(k+1)}\left(w_{k+1}^{(n-1)}-t^{n}w_{k-1}\right).$$

(ii) For $|r|<1$, we have

$$\|\mathscr{Q}_r\circ\mathscr{S}_r\|_2\leqslant nw_{n-1}^{(k+1)}\left(w_{k+1}^{(n-1)}-t^{n}w_{k-1}\right).$$

COROLLARY 8. *The spectral norms of the Kronecker product of $\mathscr{Q}_r$ and $\mathscr{S}_r$ is given by*

(i) For $|r|\geqslant 1$, we have

$$\begin{aligned}\frac{1}{n}w_{n-1}^{(k+1)}\left(w_{k+1}^{(n-1)}-t^{n}w_{k-1}\right)&\leqslant\|\mathscr{Q}_r\otimes\mathscr{S}_r\|_2\\&\leqslant\left((n-1)|r|^2+1\right)w_{n-1}^{(k+1)}\left(w_{k+1}^{(n-1)}-t^{n}w_{k-1}\right).\end{aligned}$$

(ii) For $|r|<1$, we have

$$\frac{|r|^2}{n}w_{n-1}^{(k+1)}\left(w_{k+1}^{(n-1)}-t^{n}w_{k-1}\right)\leqslant\|\mathscr{Q}_r\otimes\mathscr{S}_r\|_2\leqslant nw_{n-1}^{(k+1)}\left(w_{k+1}^{(n-1)}-t^{n}w_{k-1}\right).$$

Let $\mathscr{Q}_{r^*}$ and $\mathscr{S}_{r^*}$ be the geometric circulant matrices with the bi-periodic hyper-Horadam's numbers defined as

$$\mathscr{Q}_{r^*}=Circ_{n,r^*}\left(b^{\xi(n+1)}(ab)^{\lfloor\frac{n-1}{2}\rfloor}tw_0^{(k)},a^{\xi(n)}(ab)^{\lfloor\frac{n-2}{2}\rfloor}tw_1^{(k)},\dots,tw_{n-1}^{(k)}\right)$$

and

$$\mathscr{S}_{r^*}=Circ_{n,r^*}\left(a^{\xi(k)}b^{\xi(k+1)}w_k^{(0)}t^{n-1},a^{\xi(k)}b^{\xi(k+1)}t^{n-2}w_k^{(1)},\dots,a^{\xi(k)}b^{\xi(k+1)}w_k^{(n-1)}\right).$$

In the following theorem, we give the upper and lower bounds for the spectral norm of the geometric circulant matrix $\mathscr{Q}_{r^*}$.

THEOREM 7. *Let $r \in \mathbb{C}$ and $\mathcal{Q}_{r^*}$ be an $n \times n$ geometric circulant matrix. Then*

(i) For $|r| \geqslant 1$, we have

$$\frac{1}{\sqrt{n}} w_{n-1}^{(k+1)} \leqslant \|\mathcal{Q}_{r^*}\|_2 \leqslant \sqrt{\frac{1-|r|^{2n}}{1-|r|^2}} w_{n-1}^{(k+1)}. \tag{20}$$

(ii) For $|r| < 1$, we have

$$\frac{|r|^n}{\sqrt{n}} w_{n-1}^{(k+1)} \leqslant \|\mathcal{Q}_{r^*}\|_2 \leqslant \sqrt{n} w_{n-1}^{(k+1)}. \tag{21}$$

Proof. Let

$$\mathcal{Q}_{r^*} := \begin{pmatrix} b^{\xi(n+1)}(ab)^{\lfloor\frac{n-1}{2}\rfloor} tw_0^{(k)} & a^{\xi(n)}(ab)^{\lfloor\frac{n-2}{2}\rfloor} tw_1^{(k)} & \cdots & tw_{n-1}^{(k)} \\ rtw_{n-1}^{(k)} & b^{\xi(n+1)}(ab)^{\lfloor\frac{n-1}{2}\rfloor} tw_0^{(k)} & \cdots & a^{\xi(n)} b^{\xi(n+1)} tw_{n-2}^{(k)} \\ \vdots & \vdots & \ddots & \vdots \\ r^{n-2} b^{\xi(n+1)}(ab)^{\lfloor\frac{n-3}{2}\rfloor} tw_2^{(k)} & r^{n-3} a^{\xi(n)}(ab)^{\lfloor\frac{n-4}{2}\rfloor} tw_3^{(k)} & \cdots & a^{\xi(n)}(ab)^{\lfloor\frac{n-2}{2}\rfloor} tw_1^{(k)} \\ r^{n-1} a^{\xi(n)}(ab)^{\lfloor\frac{n-2}{2}\rfloor} tw_1^{(k)} & r^{r-2} b^{\xi(n+1)}(ab)^{\lfloor\frac{n-3}{2}\rfloor} tw_2^{(k)} & \cdots & b^{\xi(n+1)}(ab)^{\lfloor\frac{n-1}{2}\rfloor} tw_0^{(k)} \end{pmatrix}.$$

From the definition of the Euclidean norm, we have

$$\begin{aligned} \|\mathcal{Q}_{r^*}\|_E &= \sqrt{\sum_{j=0}^{n-1} (n-j)|c_j|^2 + \sum_{j=0}^{n-1} j|r^{n-j}|^2 |c_j|^2} \\ &= \sqrt{\sum_{j=0}^{n-1} ((n-j) + j|r^{n-j}|^2) \left(a^{\xi(n)\xi(j)} b^{\xi(n+1)\xi(j+1)} (ab)^{\lfloor\frac{n-1-j}{2}\rfloor} tw_j^{(k)} \right)^2}. \end{aligned}$$

(i) For $|r| \geqslant 1$, using (17), we have

$$\begin{aligned} \|\mathcal{Q}_{r^*}\|_E &\geqslant \sqrt{\sum_{j=0}^{n-1} ((n-j)+j) \left(a^{\xi(n)\xi(j)} b^{\xi(n+1)\xi(j+1)} (ab)^{\lfloor\frac{n-1-j}{2}\rfloor} tw_j^{(k)} \right)^2} \\ &= \sqrt{\sum_{j=0}^{n-1} n \left(a^{\xi(n)\xi(j)} b^{\xi(n+1)\xi(j+1)} (ab)^{\lfloor\frac{n-1-j}{2}\rfloor} tw_j^{(k)} \right)^2} \\ &\geqslant w_{n-1}^{(k+1)}. \end{aligned}$$

From Inequality (5), we obtain

$$\|\mathcal{Q}_{r^*}\|_2 \geqslant \frac{1}{\sqrt{n}} w_{n-1}^{(k+1)}.$$

On the other hand, let $\mathcal{Q}_{r^*} = A \circ B$, where

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ r & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ r^{n-2} & r^{n-3} & \cdots & 1 \\ r^{n-1} & r^{n-2} & \cdots & 1 \end{pmatrix}$$

and

$$B = \begin{pmatrix} b^{\xi(n+1)}(ab)^{\lfloor \frac{n-1}{2} \rfloor} tw_0^{(k)} & a^{\xi(n)}(ab)^{\lfloor \frac{n-2}{2} \rfloor} tw_1^{(k)} & \cdots & tw_{n-1}^{(k)} \\ tw_{n-1}^{(k)} & b^{\xi(n+1)}(ab)^{\lfloor \frac{n-1}{2} \rfloor} tw_0^{(k)} & \cdots & a^{\xi(n)} b^{\xi(n+1)} tw_{n-2}^{(k)} \\ \vdots & \vdots & \ddots & \vdots \\ b^{\xi(n+1)}(ab)^{\lfloor \frac{n-3}{2} \rfloor} tw_2^{(k)} & a^{\xi(n)}(ab)^{\lfloor \frac{n-4}{2} \rfloor} tw_3^{(k)} & \cdots & a^{\xi(n)}(ab)^{\lfloor \frac{n-2}{2} \rfloor} tw_1^{(k)} \\ a^{\xi(n)}(ab)^{\lfloor \frac{n-2}{2} \rfloor} tw_1^{(k)} & b^{\xi(n+1)}(ab)^{\lfloor \frac{n-3}{2} \rfloor} tw_2^{(k)} & \cdots & b^{\xi(n+1)}(ab)^{\lfloor \frac{n-1}{2} \rfloor} tw_0^{(k)} \end{pmatrix}.$$

Then

$$r_1(A) = \max_{1 \leqslant i \leqslant n} \sqrt{\sum_{j=1}^{n} |a_{ij}|^2} = \sqrt{\sum_{j=1}^{n} |a_{nj}|^2}$$

$$= \sqrt{1 + |r|^2 + \cdots + |r^{n-1}|^2} = \sqrt{\frac{1 - |r|^{2n}}{1 - |r|^2}}$$

and

$$c_1(B) = \max_{1 \leqslant j \leqslant n} \sqrt{\sum_{i=1}^{n} |b_{ij}|^2} = \sqrt{\sum_{i=1}^{n} |b_{in}|^2}$$

$$= \sqrt{\sum_{i=0}^{n-1} \left(a^{\xi(n)\xi(j)} b^{\xi(n+1)\xi(j+1)} (ab)^{\lfloor \frac{n-1-j}{2} \rfloor} tw_j^{(k)} \right)^2}.$$

Using (9) and (17), we obtain

$$\|\mathcal{Q}_{r^*}\|_2 \leqslant r_1(A) c_1(B) \leqslant \sqrt{\frac{1 - |r|^{2n}}{1 - |r|^2}} w_{n-1}^{(k+1)}.$$

The proof is completed for the first part.

(ii) For $|r| < 1$, using (17), we have

$$\|\mathcal{Q}_{r^*}\|_E \geqslant \sqrt{\sum_{j=0}^{n-1} ((n-j)|r^{n-j}|^2 + j|r^{n-j}|^2) \left(a^{\xi(n)\xi(j)} b^{\xi(n+1)\xi(j+1)} (ab)^{\lfloor \frac{n-1-j}{2} \rfloor} tw_j^{(k)} \right)^2}$$

$$\geqslant \sqrt{\sum_{j=0}^{n-1} n|r|^{2n} \left(a^{\xi(n)\xi(j)} b^{\xi(n+1)\xi(j+1)} (ab)^{\lfloor \frac{n-1-j}{2} \rfloor} tw_j^{(k)} \right)^2}$$

$$\geqslant |r|^n w_{n-1}^{(k+1)}.$$

From Inequality(5), we get

$$\|\mathscr{Q}_{r^*}\|_2 \geqslant \frac{|r|^n}{\sqrt{n}} w_{n-1}^{(k+1)}.$$

On the other hand, let $\mathscr{Q}_{r^*} = A \circ B$, where

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ r & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ r^{n-2} & r^{n-3} & \cdots & 1 \\ r^{n-1} & r^{n-2} & \cdots & 1 \end{pmatrix}$$

and

$$B = \begin{pmatrix} b^{\xi(n+1)}(ab)^{\lfloor \frac{n-1}{2} \rfloor} t w_0^{(k)} & a^{\xi(n)}(ab)^{\lfloor \frac{n-2}{2} \rfloor} t w_1^{(k)} & \cdots & t w_{n-1}^{(k)} \\ t w_{n-1}^{(k)} & b^{\xi(n+1)}(ab)^{\lfloor \frac{n-1}{2} \rfloor} t w_0^{(k)} & \cdots & a^{\xi(n)} b^{\xi(n+1)} t w_{n-2}^{(k)} \\ \vdots & \vdots & \ddots & \vdots \\ b^{\xi(n+1)}(ab)^{\lfloor \frac{n-3}{2} \rfloor} t w_2^{(k)} & a^{\xi(n)}(ab)^{\lfloor \frac{n-4}{2} \rfloor} t w_3^{(k)} & \cdots & a^{\xi(n)}(ab)^{\lfloor \frac{n-2}{2} \rfloor} t w_1^{(k)} \\ a^{\xi(n)}(ab)^{\lfloor \frac{n-2}{2} \rfloor} t w_1^{(k)} & b^{\xi(n+1)}(ab)^{\lfloor \frac{n-3}{2} \rfloor} t w_2^{(k)} & \cdots & b^{\xi(n+1)}(ab)^{\lfloor \frac{n-1}{2} \rfloor} t w_0^{(k)} \end{pmatrix}.$$

Then

$$r_1(A) = \max_{1 \leqslant i \leqslant n} \sqrt{\sum_{j=1}^{n} |a_{ij}|^2} = \sqrt{\sum_{j=1}^{n} |a_{1j}|^2} = \sqrt{n}$$

and

$$\begin{aligned} c_1(B) &= \max_{1 \leqslant j \leqslant n} \sqrt{\sum_{i=1}^{n} |b_{ij}|^2} = \sqrt{\sum_{i=1}^{n} |b_{in}|^2} \\ &= \sqrt{\sum_{i=0}^{n-1} \left(a^{\xi(n)\xi(j)} b^{\xi(n+1)\xi(j+1)} (ab)^{\lfloor \frac{n-1-j}{2} \rfloor} t w_j^{(k)} \right)^2}. \end{aligned}$$

Using (9) and (17), we obtain the second part of the proof

$$\|\mathscr{Q}_{r^*}\|_2 \leqslant r_1(A) c_1(B) \leqslant \sqrt{n} w_{n-1}^{(k+1)}.$$

Therefore the proof is completed. □

In the following theorem, we give the upper and lower bounds for the spectral norm of the geometric circulant matrix $\mathscr{S}_{r^*}$.

THEOREM 8. *Let $r \in \mathbb{C}$ and $\mathscr{S}_{r^*}$ be an $n \times n$ geometric circulant matrix. Then*

(i) For $|r| \geqslant 1$, we have

$$\frac{1}{\sqrt{n}} \left(w_{k+1}^{(n-1)} - t^n w_{k-1} \right) \leqslant \|\mathscr{S}_{r^*}\|_2 \leqslant \sqrt{\frac{1 - |r|^{2n}}{1 - |r|^2}} \left(w_{k+1}^{(n-1)} - t^n w_{k-1} \right).$$

(ii) For $|r|<1$, we have

$$\frac{|r|^n}{\sqrt{n}}\left(w_{k+1}^{(n-1)}-t^n w_{k-1}\right)\leqslant\|\mathscr{S}_{r^*}\|_2\leqslant\sqrt{n}\left(w_{k+1}^{(n-1)}-t^n w_{k-1}\right).$$

Proof. The same method is used to prove the theorem. □

From (8) and (10), we get the following results.

COROLLARY 9. *The spectral norms of the Hadamard product of $\mathscr{Q}_{r^*}$ and $\mathscr{S}_{r^*}$ satisfies*

(i) For $|r|\geqslant 1$, we have $\|\mathscr{Q}_{r^*}\circ\mathscr{S}_{r^*}\|_2\leqslant\frac{1-|r|^{2n}}{1-|r|^2}w_{n-1}^{(k+1)}\left(w_{k+1}^{(n-1)}-t^n w_{k-1}\right)$.

(ii) For $|r|<1$, we have $\|\mathscr{Q}_{r^*}\circ\mathscr{S}_{r^*}\|_2\leqslant n w_{n-1}^{(k+1)}\left(w_{k+1}^{(n-1)}-t^n w_{k-1}\right)$.

COROLLARY 10. *The upper and lower bounds for spectral norms of the Kronecker product of $\mathscr{Q}_{r^*}$ and $\mathscr{S}_{r^*}$ are obtained as*

(i) For $|r|\geqslant 1$, we have

$$\frac{1}{n}w_{n-1}^{(k+1)}\left(w_{k+1}^{(n-1)}-t^n w_{k-1}\right)\leqslant\|\mathscr{Q}_{r^*}\otimes\mathscr{S}_{r^*}\|_2\leqslant\frac{1-|r|^{2n}}{1-|r|^2}w_{n-1}^{(k+1)}\left(w_{k+1}^{(n-1)}-t^n w_{k-1}\right).$$

(ii) For $|r|<1$, we have

$$\frac{|r|^{2n}}{n}w_{n-1}^{(k+1)}\left(w_{k+1}^{(n-1)}-t^n w_{k-1}\right)\leqslant\|\mathscr{Q}_{r^*}\otimes\mathscr{S}_{r^*}\|_2\leqslant n w_{n-1}^{(k+1)}\left(w_{k+1}^{(n-1)}-t^n w_{k-1}\right).$$

Acknowledgements. The authors thank the referee for the helpful suggestions and comments. This paper is partially supported by DGRSDT Grant n° C0656701.

REFERENCES

[1] M. BAHŞI, I. MEZŐ AND S. SOLAK, *A symmetric algorithm for hyper-Fibonacci and hyper-lucas numbers*, Ann. Math. Inform., **43**, (2014), 19–27.

[2] M. BAHŞI AND S. SOLAK, *On the norms of r-circulant matrices with the hyper-Fibonacci and Lucas numbers*, J. Math. Inequal, **8**, 4 (2014), 693–705.

[3] M. BAHŞI, *On the norms of circulant matrices with the generalized Fibonacci and Lucas numbers*, TWMS J. Pure Appl. Math., **6**, 1 (2015), 84–92.

[4] M. BAHŞI, *On the norms of r-circulant matrices with the hyperharmonic numbers*, J. Math. Inequal., **10**, 2 (2015), 445–458.

[5] N. BELAGGOUN AND H. BELBACHIR, *Bi-periodic hyper-Fibonacci numbers*, Kragujevac J. Math., **49**, 4 (2025), 603–614.

[6] H. BELBACHIR AND A. BELKHIR, *On some generalizations of Horadam numbers*, Filomat, **32**, 14 (2018), 5037–5052.

[7] G. BILGICI, *Two generalizations of lucas sequence*, Appl. Math. Comput., **245**, (2014), 526–538.

[8] M. DAĞLI, E. TAN AND O. ÖLMEZ, *On r-circulant matrices with generalized bi-periodic Fibonacci numbers*, J. Appl. Math. Comput., **68**, 3 (2022), 2003–2014.

[9] P.J. DAVIS, *Circulant matrices, wiley*, National Bureau of Standards, Applied Mathematics Series **55**, New York, Washington, 1979.

[10] M. EDSON AND O. YAYENIE, *A new generalization of Fibonacci sequence and extended binet's formula*, Integers, **9**, (2009), 639–654.

[11] H. W. GOULD, *Combinatorial Identities: A Standardized Set of Tables Listing 500 Binomial Coefficient Summations*, Morgantown, West Virginia, 1972.

[12] A. F. HORADAM, *Basic Properties of a Certain Generalized Sequence of Numbers*, Messenger Math., **3**, (1965), 161–176.

[13] R. A. HORN AND C. R. JOHNSON, *Topics in matrix analysis, 1991*, Cambridge University Presss, Cambridge, **37**, 39 (1991).

[14] H. KARNER, J. SCHNEID AND C. W. UEBERHUBER, *Spectral decomposition of real circulant matrices*, Linear Algebra Appl., **367**, (2003), 301–311.

[15] C. KIZILATEŞ AND N. TUGLU, *On the bounds for the spectral norms of geometric circulant matrices*, J. Inequal. Appl., **2016**, 1 (2016), 1–15.

[16] C. KIZILATEŞ AND N. TUGLU, *On the Norms of Geometric and Symmetric Geometric Circulant Matrices with the Tribonacci Number*, Gazi Univ. J. Sci., **31**, 2 (2018), 555–567.

[17] E. G. KOCER, T. MANSOUR AND N. TUĞLU, *Norms of circulant and semicirculant matrices with Horadams's numbers*, Ars Comb., **85**, 1 (2018), 353–359.

[18] C. KÖMEE AND Y. YAZLIK, *On the spectral norms of r-circulant matrices with the biperiodic Fibonacci and Lucas numbers*, J. Inequal. Appl., **2017**, 1 (2017), 1–12.

[19] R. MATHIAS, *The spectral norm of a nonnegative matrix*, Linear Algebra Appl., **139**, (1990), 269–284.

[20] E. POLATLI, *On geometric circulant matrices whose entries are bi-periodic Fibonacci and bi-periodic Lucas numbers*, Univers. J. Math. Appl., **3**, 3 (2020), 102–108.

[21] R. REAMS, *Hadamard inverses, square roots and products of almost semidefinite matrices*, Linear Algebra Appl., **288**, (1999), 35–43.

[22] S. UYGUN, *On the bounds for the spectral norms of geometric and r-circulant matrices with bi-periodic Jacobsthal numbers*, J. Appl. Math. Inform., **38**, 1_2 (2020), 99–112.

[23] O. YAYENIE, *A note on generalized Fibonacci sequence*, Appl. Math. Comput., **217**, (2011), 5603–5611.

[24] Y. YAZLIK AND N. TASKARA, *A note on generalized k-Horadam sequence*, Comput. Math. Appl., **63**, 1 (2012), 36–41.

[25] Y. YAZLIK AND N. TASKARA, *On the norms of an r-circulant matrix with the generalized k-Horadam numbers*, J. Inequal. Appl., **2013**, 1 (2013), 1–8.

[26] G. ZIELKE, *Some remarks on matrix norms, condition numbers, and error estimates for linear equations*, Linear Algebra Appl., **110**, (1988), 29–41.

(Received August 24, 2022)

Nassima Belaggoun
USTHB, Faculty of Mathematics
RECITS Laboratory
Po. Box 32, El Alia 16111, Bab Ezzouar, Algiers, Algeria
and
CERIST, Research Center on Scientific and Technical Information
05, Rue des 3 frères Aissou, Ben Aknoun, Algiers, Algeria
e-mail: belaggounmanassil@gmail.com

Hacène Belbachir
USTHB, Faculty of Mathematics
RECITS Laboratory
Po. Box 32, El Alia 16111, Bab Ezzouar, Algiers, Algeria
and
CERIST, Research Center on Scientific and Technical Information
05, Rue des 3 frères Aissou, Ben Aknoun, Algiers, Algeria
e-mail: hacenebelbachir@gmail.com