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TWO LOWER BOUNDS FOR THE SMALLEST SINGULAR VALUE OF MATRICES

WUSHUANG LIU, XINGKAI HU*, YAOQUN WANG AND YUAN YI

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Abstract. In this paper, we present two lower bounds for the smallest singular value of nonsingular matrices. Moreover, we illustrate with numerical examples that these bounds are better than the existing bounds.

1. Introduction

Let M_n be the space of $n\times n$ complex matrices where its identity matrix is denoted by I_n. The singular value of $A\in M_n$ is $\sigma_i(i=1,\cdots,n)$ and $\sigma_1\geqslant\sigma_2\geqslant\cdots\geqslant\sigma_n\geqslant 0$. For $A=[a_{ij}]\in M_n$, the Frobenius norm of A is defined by

$$\|A\|_F=\left(\sum_{i,j=1}^{n}|a_{ij}|^2\right)^{\frac{1}{2}}=(\text{tr}A^*A)^{\frac{1}{2}},$$

where A^* is the conjugate transpose of A. A particular lower bound is determined by the determinant of the matrix and the Frobenius norm.

It is well known that lower bounds for the smallest singular value σ_n of a nonsingular matrix $A\in M_n$ have many potential theoretical and practical applications [1]–[3].

Yu and Gu [4] gave a lower bound for σ_n by showing that

$$\sigma_n\geqslant|\text{det}A|\left(\frac{n-1}{\|A\|_F^2}\right)^{\frac{n-1}{2}}=l>0. \tag{1.1}$$

Zou [5] refined the inequality (1.1) as follows:

$$\sigma_n\geqslant|\text{det}A|\left(\frac{n-1}{\|A\|_F^2-l^2}\right)^{\frac{n-1}{2}}=l_0. \tag{1.2}$$

Lin and Xie [6] obtained a lower bound for smallest singular value of matrices by showing that a is the smallest positive solution to the equation

$$x^2(\|A\|_F^2-x^2)^{n-1}=|\text{det}A|^2(n-1)^{n-1}$$

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* Corresponding author.

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and $\sigma_n \geqslant a > l_0$.

Recently, Shun [7] refined the inequality (1.2) as follows:

$$\sigma_n \geqslant \left(l_0^2 + |\det(l_0^2 I_n - A^*A)| \left(\frac{n-1}{\|A\|_F^2 - nl_0^2} \right)^{n-1} \right)^{\frac{1}{2}} = l_1.$$

At the same time, in [7], Shun improved a lower bound for smallest singular value of matrices by showing that b is the smallest positive solution to the equation

$$x^2 = l_0^2 + |\det(l_0^2 I_n - A^*A)| \left(\frac{n-1}{\|A\|_F^2 - x^2 - (n-1)l_0^2} \right)^{n-1}$$

and $\sigma_n \geqslant b > l_1$.

In this paper, following the idea of Lin et al. [6] and Shun [7], we establish new lower bounds l_2 and c for the smallest singular value of nonsingular matrices, which have not appeared in previous papers. Quite apart from that, specific examples are given to compare our results with existing results.

2. Main results

In the forthcoming, we are in a position to begin our main work. We give two new estimates for singular value of nonsingular matrices.

THEOREM 1. *Let $A \in M_n$ be nonsingular,*

$$\left(l_1^2 - l_0^2 + |\det((l_1^2 - l_0^2) I_n - A^*A)| \left(\frac{n-1}{\|A\|_F^2 - n(l_1^2 - l_0^2)} \right)^{n-1} \right)^{\frac{1}{2}} = l_2,$$

then $\sigma_n \geqslant l_2$, where

$$l_0 = |\det A| \left(\frac{n-1}{\|A\|_F^2 - l^2} \right)^{\frac{n-1}{2}},$$

$$l_1 = \left(l_0^2 + |\det(l_0^2 I_n - A^*A)| \left(\frac{n-1}{\|A\|_F^2 - nl_0^2} \right)^{n-1} \right)^{\frac{1}{2}}.$$

Proof. Define $0 < \zeta < \lambda < \sigma_n^2$, we have

$$\begin{aligned} &|(\lambda - \zeta - \sigma_1^2)(\lambda - \zeta - \sigma_2^2) \cdots (\lambda - \zeta - \sigma_{n-1}^2)| \\ &\leqslant \left(\frac{\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_{n-1}^2 - (n-1)(\lambda - \zeta)}{n-1} \right)^{n-1}, \end{aligned}$$

and so

$$\begin{aligned}&|(\lambda-\zeta-\sigma_1^2)(\lambda-\zeta-\sigma_2^2)\cdots(\lambda-\zeta-\sigma_{n-1}^2)|\\&=\frac{|(\lambda-\zeta-\sigma_1^2)(\lambda-\zeta-\sigma_2^2)\cdots(\lambda-\zeta-\sigma_n^2)|}{\sigma_n^2-(\lambda-\zeta)}\\&=\frac{|\det((\lambda-\zeta)I_n-A^*A)|}{\sigma_n^2-(\lambda-\zeta)}.\end{aligned}$$

Then

$$\frac{|\det((\lambda-\zeta)I_n-A^*A)|}{\sigma_n^2-(\lambda-\zeta)}\leqslant\left(\frac{\sigma_1^2+\sigma_2^2+\cdots+\sigma_{n-1}^2-(n-1)(\lambda-\zeta)}{n-1}\right)^{n-1}.$$

Consequently

$$\sigma_n^2\geqslant\lambda-\zeta+|\det((\lambda-\zeta)I_n-A^*A)|\left(\frac{n-1}{\sigma_1^2+\cdots+\sigma_{n-1}^2-(n-1)(\lambda-\zeta)}\right)^{n-1}.$$

That is

$$\sigma_n\geqslant\left(\lambda-\zeta+|\det((\lambda-\zeta)I_n-A^*A)|\left(\frac{n-1}{\sigma_1^2+\cdots+\sigma_{n-1}^2-(n-1)(\lambda-\zeta)}\right)^{n-1}\right)^{\frac{1}{2}}.$$

Let $\zeta=l_0^2,\lambda=l_1^2$, we have

$$\sigma_n\geqslant\left(l_1^2-l_0^2+|\det((l_1^2-l_0^2)I_n-A^*A)|\left(\frac{n-1}{\|A\|_F^2-\sigma_n^2-(n-1)(l_1^2-l_0^2)}\right)^{n-1}\right)^{\frac{1}{2}}.\tag{2.1}$$

Hence

$$\sigma_n\geqslant\left(l_1^2-l_0^2+|\det((l_1^2-l_0^2)I_n-A^*A)|\left(\frac{n-1}{\|A\|_F^2-n(l_1^2-l_0^2)}\right)^{n-1}\right)^{\frac{1}{2}}.$$

This completes the proof. □

THEOREM 2. *Let $A\in M_n$ be nonsingular,*

$$c_{k+1}=\left(l_1^2-l_0^2+|\det((l_1^2-l_0^2)I_n-A^*A)|\left(\frac{n-1}{\|A\|_F^2-(n-1)(l_1^2-l_0^2)-c_k^2}\right)^{n-1}\right)^{\frac{1}{2}},$$

where $l_0=|\det A|\left(\frac{n-1}{\|A\|_F^2-l^2}\right)^{\frac{n-1}{2}}, l_1=\left(l_0^2+|\det(l_0^2I_n-A^*A)|\left(\frac{n-1}{\|A\|_F^2-nl_0^2}\right)^{n-1}\right)^{\frac{1}{2}},$

$$c_1=\left(l_1^2-l_0^2+|\det((l_1^2-l_0^2)I_n-A^*A)|\left(\frac{n-1}{\|A\|_F^2-(n-1)(l_1^2-l_0^2)}\right)^{n-1}\right)^{\frac{1}{2}},$$

then, $0 < c_k < c_{k+1} \leqslant \sigma_n, k = 1, 2, \cdots, \lim_{k \longrightarrow \infty} c_k$ *exists.*

Proof. We illustrate by induction on k that

$$\sigma_n \geqslant c_{k+1} > c_k > 0.$$

By (2.1), we have

$$\begin{aligned}
\sigma_n &\geqslant \left(l_1^2 - l_0^2 + |\det((l_1^2 - l_0^2)I_n - A^*A)| \left(\frac{n-1}{\|A\|_F^2 - \sigma_n^2 - (n-1)(l_1^2 - l_0^2)} \right)^{n-1} \right)^{\frac{1}{2}} \\
&\geqslant \left(l_1^2 - l_0^2 + |\det((l_1^2 - l_0^2)I_n - A^*A)| \left(\frac{n-1}{\|A\|_F^2 - (n-1)(l_1^2 - l_0^2)} \right)^{n-1} \right)^{\frac{1}{2}} = c_1.
\end{aligned}$$

So $\sigma_n \geqslant c_1$, we obtain

$$\begin{aligned}
\sigma_n &\geqslant \left(l_1^2 - l_0^2 + |\det((l_1^2 - l_0^2)I_n - A^*A)| \left(\frac{n-1}{\|A\|_F^2 - \sigma_n^2 - (n-1)(l_1^2 - l_0^2)} \right)^{n-1} \right)^{\frac{1}{2}} \\
&\geqslant \left(l_1^2 - l_0^2 + |\det((l_1^2 - l_0^2)I_n - A^*A)| \left(\frac{n-1}{\|A\|_F^2 - (n-1)(l_1^2 - l_0^2) - c_1^2} \right)^{n-1} \right)^{\frac{1}{2}} = c_2 \\
&> \left(l_1^2 - l_0^2 + |\det((l_1^2 - l_0^2)I_n - A^*A)| \left(\frac{n-1}{\|A\|_F^2 - (n-1)(l_1^2 - l_0^2)} \right)^{n-1} \right)^{\frac{1}{2}} = c_1 > 0.
\end{aligned}$$

When $k = 1$, we have

$$\sigma_n \geqslant c_2 > c_1 > 0.$$

Suppose that our claim is true for $k = m$, that is $\sigma_n \geqslant c_{m+1} > c_m > 0$. Now we consider the case when $k = m+1$. By (2.1), we get

$$\begin{aligned}
\sigma_n &\geqslant \left(l_1^2 - l_0^2 + |\det((l_1^2 - l_0^2)I_n - A^*A)| \left(\frac{n-1}{\|A\|_F^2 - \sigma_n^2 - (n-1)(l_1^2 - l_0^2)} \right)^{n-1} \right)^{\frac{1}{2}} \\
&\geqslant \left(l_1^2 - l_0^2 + |\det((l_1^2 - l_0^2)I_n - A^*A)| \left(\frac{n-1}{\|A\|_F^2 - (n-1)(l_1^2 - l_0^2) - c_{m+1}^2} \right)^{n-1} \right)^{\frac{1}{2}} \\
&= c_{m+2} \\
&> \left(l_1^2 - l_0^2 + |\det((l_1^2 - l_0^2)I_n - A^*A)| \left(\frac{n-1}{\|A\|_F^2 - (n-1)(l_1^2 - l_0^2) - c_m^2} \right)^{n-1} \right)^{\frac{1}{2}} \\
&= c_{m+1} > 0.
\end{aligned}$$

Hence $\sigma_n \geqslant c_{m+2} > c_{m+1} > 0$. This proves $\sigma_n \geqslant c_{k+1} > c_k > 0, k = 1, 2, \cdots$. It follows from the monotone convergence theorem, we can get that $\lim_{k \longrightarrow \infty} c_k$ exists. This completes the proof. $\square$

THEOREM 3. *Let $A \in M_n$ be nonsingular and $c = \lim_{k \longrightarrow \infty} c_k$,*

$$f(x) = \left(l_1^2 - l_0^2 + |\det((l_1^2 - l_0^2)I_n - A^*A)| \left(\frac{n-1}{\|A\|_F^2 - x^2 - (n-1)(l_1^2 - l_0^2)} \right)^{n-1} \right)^{\frac{1}{2}},$$

then c is the smallest positive solution to the equation $x = f(x)$ and $\sigma_n \geqslant c$.

Proof. Let x_0 be the smallest positive solution to the equation $x = f(x)$, we illustrate by induction on k that $x_0 > c_k$, $k = 1, 2, \cdots$. When $k = 1$, we get

$$\begin{aligned} x_0 &= \left(l_1^2 - l_0^2 + |\det((l_1^2 - l_0^2)I_n - A^*A)| \left(\frac{n-1}{\|A\|_F^2 - x_0^2 - (n-1)(l_1^2 - l_0^2)} \right)^{n-1} \right)^{\frac{1}{2}} \\ &> \left(l_1^2 - l_0^2 + |\det((l_1^2 - l_0^2)I_n - A^*A)| \left(\frac{n-1}{\|A\|_F^2 - (n-1)(l_1^2 - l_0^2)} \right)^{n-1} \right)^{\frac{1}{2}} = c_1. \end{aligned}$$

Suppose that our claim is true for $k = m$, that is $x_0 > c_m$. Now we consider the case, when $k = m + 1$, we have

$$\begin{aligned} x_0 &= \left(l_1^2 - l_0^2 + |\det((l_1^2 - l_0^2)I_n - A^*A)| \left(\frac{n-1}{\|A\|_F^2 - x_0^2 - (n-1)(l_1^2 - l_0^2)} \right)^{n-1} \right)^{\frac{1}{2}} \\ &> \left(l_1^2 - l_0^2 + |\det((l_1^2 - l_0^2)I_n - A^*A)| \left(\frac{n-1}{\|A\|_F^2 - c_m^2 - (n-1)(l_1^2 - l_0^2)} \right)^{n-1} \right)^{\frac{1}{2}} \\ &= c_{m+1}. \end{aligned}$$

Hence $x_0 > c_{m+1}$. This proves $x_0 > c_k$, $k = 1, 2, \cdots$. Since c is a positive solution to the equation $x = f(x)$ and $x_0 > c_k$, $k = 1, 2, \cdots$, then $c = x_0$. Therefore c is the smallest positive solution to the equation $x = f(x)$ and $\sigma_n \geqslant c$. This completes the proof. □

Thus, we obtain two new lower bounds l_2 and c for the smallest singular value of nonsingular matrices.

3. Numerical examples

In what follows, we use three examples to compare the values of l_0, l_1 and l_2.

EXAMPLE 1. Let

$$A = \begin{bmatrix} 4 & -4 & -3 \\ -4 & 9 & 4 \\ -1 & 7 & 9 \end{bmatrix}.$$

Then $\sigma_{min} = 1.8798$, and

$$l_0 = 0.9928845, \ l_1 = 1.292744.$$

Our result:

$$l_2 = 1.503161.$$

EXAMPLE 2. Let

$$A = \begin{bmatrix} 4 & 5 & 6 \\ -1 & 5 & 0 \\ 0 & 5 & 4 \end{bmatrix}.$$

Then $\sigma_{min} = 1.4065$, and

$$l_0 = 0.9786461, \quad l_1 = 1.199039.$$

Our result:

$$l_2 = 1.231601.$$

EXAMPLE 3. Let

$$A = \begin{bmatrix} 4 & 2 & 0 \\ 5 & 8 & 7 \\ 7 & 5 & 8 \end{bmatrix}.$$

Then $\sigma_{min} = 2.5249$, and

$$l_0 = 0.9079198, \quad l_1 = 1.220167.$$

Our result:

$$l_2 = 1.399112.$$

The following example is used to compare the values of b, c and l_2.

EXAMPLE 4. [7] Let

$$A = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 9 & 5 \\ 0 & 5 & 7 \end{bmatrix}.$$

Then

$$b = 1.3455.$$

Our result:

$$c = 1.6123, \quad l_2 = 1.61175.$$

These indicate that for such examples the bounds obtained by our results are better than that of Zou and Shun.

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Wushuang Liu
Faculty of Science
Kunming University of Science and Technology
Kunming, Yunnan 650500, P. R. China

Xingkai Hu
Faculty of Science
Kunming University of Science and Technology
Kunming, Yunnan 650500, P. R. China
e-mail: huxingkai84@163.com

Yaoqun Wang
Faculty of Science
Kunming University of Science and Technology
Kunming, Yunnan 650500, P. R. China

Yuan Yi
Faculty of Science
Kunming University of Science and Technology
Kunming, Yunnan 650500, P. R. China