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QUASI–CONVEX AND Q–CLASS FUNCTIONS

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Abstract. Convex functions and their variants have played a significant role in the literature. In this article, we investigate two important related classes, namely quasi-convex and Q-class functions. We will show that these two classes satisfy similar but different properties as those fulfilled by convex functions. Our discussion will include refinements of known inequalities, super-additivity behavior, Jensen-Mercer inequality, and other related results. Among many other results, we show that an increasing quasi-convex function $f:[0,\infty)\to\mathbb{R}$ satisfies the inequality

$$\frac{f(a)+f(b)}{2}\leqslant f\left(\frac{a+b}{2}\right)+\frac{1}{2}f(a+b),\quad (a,b>0),$$

while a Q-class function with $f(0)\leqslant 0$ satisfies the super-additive inequality

$$f(a)+f(b)\leqslant \frac{(a+b)^2}{ab}f(a+b),\quad (a,b>0)$$

similar to convex functions.

1. Introduction

Let J be a real interval. More than a century ago, Jensen [7] introduced the notion of convex functions as those functions $f:J\subseteq\mathbb{R}\to\mathbb{R}$ such that

$$f((1-t)a+tb)\leqslant(1-t)f(a)+tf(b) \tag{1.1}$$

for all $a,b\in J$ and all $0\leqslant t\leqslant 1$. A convex function defined on a closed interval is bounded above by the maximum of its values at the endpoints, but the converse needs not to be true. That is, a function bounded by the maximum of its values at the endpoints need not be convex. This fact motivates researchers to define quasi-convex functions (see, for example, [13]) as those functions $f:J\subseteq\mathbb{R}\to\mathbb{R}$ satisfying

$$f((1-t)a+tb)\leqslant\max\{f(a),f(b)\}, \tag{1.2}$$

for all $a,b\in J$ and $0\leqslant t\leqslant 1$. Clearly, any convex function is a quasi-convex function. On the other hand, there exist quasi-convex functions which are not convex. For example, the function $f(x)=\ln x$ for $x\in(0,\infty)$ is not convex, yet it is quasi-convex.

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It is obvious that a monotone function (increasing or decreasing) is necessarily quasi-convex.

Many properties of convex functions have equivalent properties for quasi-convex functions. We refer the reader to the excellent review on quasi-convex functions in [5], where an informative list is provided.

Notice that quasi-convex functions belong to another class of functions called Q-class. A function $f:J\to\mathbb{R}$ is said to be Q-class if for any $0<t<1$

$$f((1-t)a+tb)\leqslant\frac{1}{1-t}f(a)+\frac{1}{t}f(b) \tag{1.3}$$

for all $a,b\in J$. Godunova and Levin introduced this concept in [4].

Let $\mathscr{S}$ be a subset of $\mathbb{R}$ with at least three elements. A function $f:\mathscr{S}\to\mathbb{R}$ is called a Schur function if

$$f(t)(t-s)(t-u)+f(s)(s-t)(s-u)+f(u)(u-t)(u-s)\geqslant 0 \tag{1.4}$$

for all $s,t,u\in\mathscr{S}$. For $f(x)=x^r$, $(x\in[0,\infty),r>0)$, (1.4) is just the well-known inequality due to Schur [18]. In [4], Godunova and Levin demonstrated that the class of Schur functions and the Q-class functions overlap. Several properties of classical Q-class functions can be found in [12].

It is uncomplicated to notice that every non-negative monotone function or convex function is of Q-class. Thus functions of class Q emerge as an extension of these two important classes of function.

One of the most celebrated inequalities for convex functions is Jensen-Mercer's inequality [8]. This inequality is expressed as follows: Let $f:[m,M]\to\mathbb{R}$ be a convex function and let $w_1,w_2,\dots,w_n\geqslant 0$ with $\sum_{i=1}^n w_i=1$. Then

$$f\left(M+m-\sum_{i=1}^n w_it_i\right)\leqslant f(M)+f(m)-\sum_{i=1}^n w_if(t_i);\ \ m\leqslant t_i\leqslant M,\ \ i=1,2,\dots,n.$$

Finding further inequalities for convex functions with possible applications has been an emerging trend in mathematical inequalities. See [3] for example.

This article presents several new inequalities for quasi-convex and Q-class functions. These new inequalities will match the corresponding known inequalities for convex functions.

2. Quasi-convex functions

We begin with the following refinement of (1.2).

LEMMA 2.1. *Let $f:J\to\mathbb{R}$ be a quasi-convex function. Then for all $a,b\in J$ and* $0\leqslant t\leqslant 1$,

$$f((1-t)a+tb)\leqslant\begin{cases}\max\left\{f(a),f\left(\frac{a+b}{2}\right)\right\}; & 0\leqslant t\leqslant\frac{1}{2}\\ \max\left\{f(b),f\left(\frac{a+b}{2}\right)\right\}; & \frac{1}{2}\leqslant t\leqslant 1\end{cases}.$$

Proof. We consider the case $0 \leqslant t \leqslant 1/2$. In this case, we have

$$f((1-t)a+tb) = f\left((1-2t)a + 2t\frac{a+b}{2}\right) \leqslant \max\left\{f(a), f\left(\frac{a+b}{2}\right)\right\},$$

where we have used (1.2) to obtain the inequality, noting that $0 \leqslant 2t \leqslant 1$.

For the case $1/2 \leqslant t \leqslant 1$, we can write

$$f((1-t)a+tb) = f\left((2t-1)b + (2-2t)\frac{a+b}{2}\right) \leqslant \max\left\{f(b), f\left(\frac{a+b}{2}\right)\right\}.$$

This completes the proof. □

REMARK 2.1. Since f is quasi-convex on $[a,b]$ in Lemma 2.1, we have $f\left(\frac{a+b}{2}\right) \leqslant \max\{f(a), f(b)\}$. Therefore the inequality in Lemma 2.1 gives an improvement of the definition of the quasi-convex function on $[a,b]$ in (1.2).

At this point, we should remark that Lemma 2.1 is the quasi-version of the inequality

$$f((1-t)a+tb) \leqslant (1-t)f(a) + tf(b) - 2r\left(\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right), \tag{2.1}$$

valid for the convex function $f : J \to \mathbb{R}$, where $a, b \in J$, $0 \leqslant t \leqslant 1$ and $r = \min\{t, 1-t\}$. This inequality was proved in [2]. Later, in [10], further discussion was made on the general case. In [14, 15], a more elaborated discussion with some geometric comprehension was made.

Now we use Lemma 2.1 to present the following upper bounds for quasi-convex functions.

COROLLARY 2.1. *Let $f : J \to \mathbb{R}$ be a quasi-convex function and let $a, b \in J$.*

(i) If $0 \leqslant t \leqslant 1/2$, then

$$f((1-t)a+tb) \leqslant \frac{3}{4}f(a) + \frac{1}{4}f(b) + \frac{1}{2}\left(\left|f(a) - f\left(\frac{a+b}{2}\right)\right| + \frac{1}{2}|f(a)-f(b)|\right).$$

(ii) If $1/2 \leqslant t \leqslant 1$,

$$f((1-t)a+tb) \leqslant \frac{3}{4}f(b) + \frac{1}{4}f(a) + \frac{1}{2}\left(\left|f(b) - f\left(\frac{a+b}{2}\right)\right| + \frac{1}{2}|f(a)-f(b)|\right).$$

Proof. First of all, notice that

$$\max\{x, y\} = \frac{x+y+|x-y|}{2}; \quad x, y \in \mathbb{R}.$$

If $0 \leqslant t \leqslant 1/2$, Lemma 2.1 implies

$$\begin{aligned} f((1-t)a+tb) &\leqslant \max\left\{f(a), f\left(\frac{a+b}{2}\right)\right\} \\ &= \frac{1}{2}\left(f(a)+f\left(\frac{a+b}{2}\right)+\left|f(a)-f\left(\frac{a+b}{2}\right)\right|\right) \\ &\leqslant \frac{1}{2}\left(f(a)+\max\{f(a), f(b)\}+\left|f(a)-f\left(\frac{a+b}{2}\right)\right|\right) \\ &= \frac{3}{4}f(a)+\frac{1}{4}f(b)+\frac{1}{2}\left(\left|f(a)-f\left(\frac{a+b}{2}\right)\right|+\frac{1}{2}|f(a)-f(b)|\right), \end{aligned}$$

where the first inequality follows from Lemma 2.1 and the second inequality is obtained from (1.2) by setting $t=1/2$. Therefore,

$$f((1-t)a+tb) \leqslant \frac{3}{4}f(a)+\frac{1}{4}f(b)+\frac{1}{2}\left(\left|f(a)-f\left(\frac{a+b}{2}\right)\right|+\frac{1}{2}|f(a)-f(b)|\right).$$

This proves (i). If $1/2 \leqslant t \leqslant 1$, then

$$\begin{aligned} f((1-t)a+tb) &\leqslant \max\left\{f(b), f\left(\frac{a+b}{2}\right)\right\} \\ &= \frac{1}{2}\left(f(b)+f\left(\frac{a+b}{2}\right)+\left|f(b)-f\left(\frac{a+b}{2}\right)\right|\right) \\ &\leqslant \frac{1}{2}\left(f(b)+\max\{f(a), f(b)\}+\left|f(b)-f\left(\frac{a+b}{2}\right)\right|\right) \\ &= \frac{3}{4}f(b)+\frac{1}{4}f(a)+\frac{1}{2}\left(\left|f(b)-f\left(\frac{a+b}{2}\right)\right|+\frac{1}{2}|f(a)-f(b)|\right). \end{aligned}$$

Consequently,

$$f((1-t)a+tb) \leqslant \frac{3}{4}f(b)+\frac{1}{4}f(a)+\frac{1}{2}\left(\left|f(b)-f\left(\frac{a+b}{2}\right)\right|+\frac{1}{2}|f(a)-f(b)|\right),$$

which completes the proof. □

We notice that a convex function $f: J \to \mathbb{R}$ satisfies the mid-convexity condition $f\left(\frac{a+b}{2}\right) \leqslant \frac{f(a)+f(b)}{2}$. It is interesting that quasi-convex functions satisfy the following.

THEOREM 2.1. *Let $f:[0,\infty) \to \mathbb{R}$ be a quasi-convex function and let $a,b>0$. Then*

$$\begin{aligned} &\frac{f(a)+f(b)}{2} \\ &\leqslant f\left(\frac{a+b}{2}\right)+\frac{1}{2}\left(f(0)+\left|f(a+b)-f\left(\frac{a+b}{2}\right)\right|+\left|f(0)-f\left(\frac{a+b}{2}\right)\right|\right). \end{aligned}$$

Proof. Noting that $tx=(1-t)\times 0+tx$, Lemma 2.1 implies

$$f(tx)\leqslant\begin{cases}\max\left\{f(0),f\left(\frac{x}{2}\right)\right\}; & 0\leqslant t\leqslant\frac{1}{2}\\ \max\left\{f(x),f\left(\frac{x}{2}\right)\right\}; & \frac{1}{2}\leqslant t\leqslant 1\end{cases}. \tag{2.2}$$

First assume that $a\leqslant b$ (i.e., $a/(a+b)\leqslant 1/2$ and $1/2\leqslant b/(a+b)$). Then (2.2) implies

$$f(a)=f\left(\frac{a}{a+b}\cdot(a+b)\right)\leqslant\max\left\{f(0),f\left(\frac{a+b}{2}\right)\right\},$$

and

$$f(b)=f\left(\frac{b}{a+b}\cdot(a+b)\right)\leqslant\max\left\{f(a+b),f\left(\frac{a+b}{2}\right)\right\}.$$

In summary,

$$a\leqslant b\ \Rightarrow\ \begin{cases}f(a)\leqslant\max\left\{f(0),f\left(\frac{a+b}{2}\right)\right\}\\ f(b)\leqslant\max\left\{f(a+b),f\left(\frac{a+b}{2}\right)\right\}\end{cases}.$$

Thus,

$$\begin{aligned}&f(a)+f(b)\\&\leqslant\max\left\{f(0),f\left(\frac{a+b}{2}\right)\right\}+\max\left\{f(a+b),f\left(\frac{a+b}{2}\right)\right\}\\&=f\left(\frac{a+b}{2}\right)+\frac{1}{2}\left(f(a+b)+f(0)+\left|f(a+b)-f\left(\frac{a+b}{2}\right)\right|+\left|f(0)-f\left(\frac{a+b}{2}\right)\right|\right),\end{aligned}$$

which is equivalent to the desired inequality for the case $a\leqslant b$.

If $b\leqslant a$, interchanging a and b in the first case completes the proof. □

We notice that a concave function $f:[0,\infty)\to\mathbb{R}$ satisfies the inequality $\frac{f(a)+f(b)}{2}\leqslant f\left(\frac{a+b}{2}\right)$. It is interesting that a monotone quasi-convex function follows a similar behavior, as we state in the following remark.

REMARK 2.2. From Theorem 2.1, we find the following.

(i) If a quasi-convex function $f:[0,\infty)\to\mathbb{R}$ is increasing, then we have

$$\frac{f(a)+f(b)}{2}\leqslant f\left(\frac{a+b}{2}\right)+\frac{1}{2}f(a+b),\quad(a,b>0).$$

(ii) If a quasi-convex function $f:[0,\infty)\to\mathbb{R}$ is decreasing, then we have

$$\frac{f(a)+f(b)}{2}\leqslant f\left(\frac{a+b}{2}\right)+f(0)-\frac{1}{2}f(a+b),\quad(a,b>0).$$

For a convex function $f: J \to \mathbb{R}$, it is well known that if $t \geqslant 0$ or $t \leqslant -1$, then

$$f((1+t)a - tb) \geqslant (1+t)f(a) - tf(b), a, b \in J,$$

provided that $(1+t)a - tb \in J$. This inequality received some attention in the literature due to its applications in other fields, like operator theory. The reader is referred to [6, 11] for further related readings. In the following result, we present the quasi-version of this inequality.

THEOREM 2.2. *Let $f: J \to \mathbb{R}$ be a quasi-convex function. Then for all $a, b \in J$,*

$$\min\{f(a), f(b)\} \leqslant \begin{cases} \max\{f((1+t)a - tb), f(b)\}; & t \geqslant 0 \\ \max\{f((1+t)a - tb), f(a)\}; & t \leqslant -1 \end{cases},$$

provided that $(1+t)a - tb \in J$.

Proof. Notice that

$$\min\{x, y\} = \frac{x + y - |x - y|}{2}; \quad x, y \in \mathbb{R}.$$

If $t \geqslant 0$, we have

$$\begin{aligned} \min\{f(a), f(b)\} &\leqslant f(a) \\ &= f\left(\frac{1}{1+t}((1+t)a - tb) + \frac{t}{1+t}b\right) \\ &\leqslant \max\{f((1+t)a - tb), f(b)\} \end{aligned}$$

where the second inequality is obtained from (1.2). If $t \leqslant -1$, we get

$$\begin{aligned} \min\{f(a), f(b)\} &\leqslant f(b) \\ &= f\left(-\frac{1}{t}((1+t)a - tb) + \frac{1+t}{t}a\right) \\ &\leqslant \max\{f((1+t)a - tb), f(a)\}, \end{aligned}$$

which completes the proof. □

REMARK 2.3. A quasi-concave function is a function whose negative is quasi-convex. Equivalently a function f is quasi-concave if

$$f((1-t)a + tb) \geqslant \min\{f(a), f(b)\}.$$

Applying the same method as in the proof of Lemma 2.1, we get that

$$f((1-t)a + tb) \geqslant \begin{cases} \min\left\{f(a), f\left(\frac{a+b}{2}\right)\right\}; & 0 \leqslant t \leqslant \frac{1}{2} \\ \min\left\{f(b), f\left(\frac{a+b}{2}\right)\right\}; & \frac{1}{2} \leqslant t \leqslant 1 \end{cases},$$

which can be reduced to

$$f(tx) \geqslant \begin{cases} \min\left\{f(0), f\left(\frac{x}{2}\right)\right\}; & 0 \leqslant t \leqslant \frac{1}{2} \\ \min\left\{f(x), f\left(\frac{x}{2}\right)\right\}; & \frac{1}{2} \leqslant t \leqslant 1 \end{cases}.$$

Now, utilizing the same strategy used in the proof of Theorem 2.1, we infer that

$$a \leqslant b \Rightarrow \begin{cases} f(a) \geqslant \min\left\{f(0), f\left(\frac{a+b}{2}\right)\right\} \\ f(b) \geqslant \min\left\{f(a+b), f\left(\frac{a+b}{2}\right)\right\} \end{cases}$$

and

$$b \leqslant a \Rightarrow \begin{cases} f(a) \geqslant \min\left\{f(a+b), f\left(\frac{a+b}{2}\right)\right\} \\ f(b) \geqslant \min\left\{f(0), f\left(\frac{a+b}{2}\right)\right\} \end{cases}.$$

Accordingly,

$$\begin{aligned} & f(a)+f(b) \\ & \geqslant f\left(\frac{a+b}{2}\right)+\frac{1}{2}\left(f(a+b)+f(0)-\left|f(a+b)-f\left(\frac{a+b}{2}\right)\right|-\left|f(0)-f\left(\frac{a+b}{2}\right)\right|\right). \end{aligned}$$

We end this section by presenting Jensen-Mercer's inequality for quasi-convex functions.

THEOREM 2.3. *Let $f : [m,M] \to \mathbb{R}$ be a quasi-convex function, let $t_i \in [m,M]$ $(i = 1,2,\dots,n)$ and let $w_1, w_2, \dots, w_n$ be positive scalars such that $\sum_{i=1}^{n} w_i = 1$. Then*

$$f\left(M+m-\sum_{i=1}^{n} w_i t_i\right) \leqslant 2\max\{f(m), f(M)\} - \sum_{i=1}^{n} w_i f(t_i).$$

Proof. The inequality (1.2) is equivalent to

$$f((1-t)a+tb) \leqslant \frac{1}{2}\left(f(a)+f(b)+|f(a)-f(b)|\right), \tag{2.3}$$

where $a,b \in [m,M]$ and $0 \leqslant t \leqslant 1$. If we put $1-t = \frac{t_i-m}{M-m}$, $t = \frac{M-t_i}{M-m}$, $a = M$, and $b = m$, in (2.3), we get

$$f(t_i) \leqslant \frac{1}{2}\left(f(M)+f(m)+|f(M)-f(m)|\right) \tag{2.4}$$

for any $m \leqslant t_i \leqslant M$ $(i = 1,2,\dots,n)$. Multiplying inequality (2.4) by w_i $(i = 1,2,\dots,n)$ and adding, we get

$$\sum_{i=1}^{n} w_i f(t_i) \leqslant \frac{1}{2}\left(f(M)+f(m)+|f(M)-f(m)|\right). \tag{2.5}$$

On the other hand, $m \leqslant t_i \leqslant M$ implies $m \leqslant M+m-t_i \leqslant M$. Thus, $m \leqslant M+m-\sum_{i=1}^n w_it_i \leqslant M$ for $i=1,2,\dots,n$. From (2.4), we conclude that

$$f\left(M+m-\sum_{i=1}^n w_it_i\right) \leqslant \frac{1}{2}\left(f(M)+f(m)+|f(M)-f(m)|\right). \tag{2.6}$$

Adding (2.5) and (2.6) implies

$$f\left(M+m-\sum_{i=1}^n w_it_i\right)+\sum_{i=1}^n w_if(t_i) \leqslant f(M)+f(m)+|f(M)-f(m)|,$$

which completes the proof. □

REMARK 2.4. It follows from (2.5) and (2.6)

$$\max\left\{\sum_{i=1}^n w_if(t_i), f\left(M+m-\sum_{i=1}^n w_it_i\right)\right\} \leqslant \frac{1}{2}\left(f(M)+f(m)+|f(M)-f(m)|\right).$$

This implies

$$\begin{aligned}&\frac{1}{2}\left(\sum_{i=1}^n w_if(t_i)+f\left(M+m-\sum_{i=1}^n w_it_i\right)+\left|\sum_{i=1}^n w_if(t_i)-f\left(M+m-\sum_{i=1}^n w_it_i\right)\right|\right)\\ &\quad\leqslant \max\{f(m),f(M)\},\end{aligned}$$

where we have used the formula $\max\{x,y\}=\frac{x+y+|x-y|}{2}$, when $x,y\in\mathbb{R}$, to obtain the above inequality. This shows that

$$\begin{aligned}f\left(M+m-\sum_{i=1}^n w_it_i\right)&+\left|\sum_{i=1}^n w_if(t_i)-f\left(M+m-\sum_{i=1}^n w_it_i\right)\right|\\ &\leqslant 2\max\{f(m),f(M)\}-\sum_{i=1}^n w_if(t_i).\end{aligned}$$

This provides a refinement of the result in Theorem 2.3.

3. Further properties on Q-class functions

We start this section by showing the supplemental inequality to (1.3). This simulates Theorem 2.2, which we proved for quasi-convex functions.

THEOREM 3.1. *Let $f: J \to \mathbb{R}$ be a Q-class function and let $t<0$ or $t>1$. Then*

$$f((1-t)a+tb) \geqslant \frac{1}{1-t}f(a)+\frac{1}{t}f(b) \tag{3.1}$$

provided that $(1-t)a+tb \in J$.

Proof. In the case when $t<0$, we notice that $0<\frac{1}{1-t}<1$ and $0<\frac{-t}{1-t}<1$. Now (1.3) implies

$$f(a)=f\left(\frac{1}{1-t}((1-t)a+tb)+\left(\frac{-t}{1-t}\right)b\right)\leqslant(1-t)f((1-t)a+tb)+\frac{1-t}{-t}f(b)$$

which yields (3.1).

In the case when $t>1$, we notice that $0<\frac{1}{t}<1$ and $0<\frac{t-1}{t}<1$. Again, using (1.3), we infer that

$$f(b)=f\left(\frac{1}{t}((1-t)a+tb)+\frac{t-1}{t}a\right)\leqslant tf((1-t)a+tb)+\frac{t}{t-1}f(a),$$

which completes the proof. □

We have seen how (2.1) refines (1.1) for convex functions and how Corollary 2.1 refines (1.2) for quasi-convex functions. We present a similar approach for Q-class functions in the following result.

PROPOSITION 3.1. *Let $f:J\to\mathbb{R}$ be a Q-class function, $a,b\in J$ and let $0<t<1$.*

(i) If $0<t<1/2$, then

$$\frac{1-2t}{1-t}f((1-2t)a+(1-t)b)+\frac{1}{1-t}\left(f(a)+f(b)-\frac{1}{2}f\left(\frac{a+b}{2}\right)\right)\leqslant\frac{1}{1-t}f(a)+\frac{1}{t}f(b).$$

(ii) If $1/2<t<1$, then

$$\frac{2t-1}{t}f(ta+(2t-1)b)+\frac{1}{t}\left(f(a)+f(b)-\frac{1}{2}f\left(\frac{a+b}{2}\right)\right)\leqslant\frac{1}{1-t}f(a)+\frac{1}{t}f(b).$$

Proof. We compute

$$\begin{aligned}&\frac{1}{1-t}f(a)+\frac{1}{t}f(b)-\frac{1}{1-t}\left(f(a)+f(b)-\frac{1}{2}f\left(\frac{a+b}{2}\right)\right)\\&=\frac{1-2t}{1-t}\left(\frac{1}{t}f(b)+\frac{1}{2(1-2t)}f\left(\frac{a+b}{2}\right)\right)\\&\geqslant\frac{1-2t}{1-t}f((1-2t)a+(1-t)b)\quad\text{(by (1.3))}\end{aligned}$$

which implies (i). The second desired inequality can be shown similarly. □

The following theorem shows the Jensen-Mercer inequality for Q-class functions.

THEOREM 3.2. *Let $f : J \to \mathbb{R}$ be a Q-class function, let $m < t_i < M$ for $i = 1,2,\dots,n$, and let $w_1, w_2, \dots, w_n > 0$ with $\sum_{i=1}^n w_i = 1$. Then*

$$f\left(M+m-\sum_{i=1}^n w_i t_i\right) \leqslant \frac{(M-m)^2 (f(M)+f(m))}{(M+m)\sum_{i=1}^n t_i/w_i - Mm - \sum_{i=1}^n t_i^2/w_i} - \sum_{i=1}^n \frac{f(t_i)}{w_i}.$$

Proof. Notice that if $m < t < M$, then $0 < \frac{M-t}{M-m}, \frac{t-m}{M-m} < 1$. Put $1-t = \frac{t-m}{M-m}$, $t = \frac{M-t}{M-m}$, $a = M$, and $b = m$, in (1.3). Then

$$\begin{aligned} f(t) &= f\left(\frac{t-m}{M-m}M + \frac{M-t}{M-m}m\right) \\ &\leqslant \frac{1}{\frac{t-m}{M-m}} f(M) + \frac{1}{\frac{M-t}{M-m}} f(m) \\ &= \frac{M-m}{t-m} f(M) + \frac{M-m}{M-t} f(m). \end{aligned} \tag{3.2}$$

Since $m < t < M$, then $m < M+m-t < M$. Thus, we can substitute t by $M+m-t$, in (3.2). This yields

$$f(M+m-t) \leqslant \frac{M-m}{M-t} f(M) + \frac{M-m}{t-m} f(m). \tag{3.3}$$

Adding the two inequalities (3.2) and (3.3), we get

$$f(M+m-t) \leqslant \frac{(M-m)^2}{(t-m)(M-t)} (f(M)+f(m)) - f(t).$$

Hence,

$$f(M+m-t_i) \leqslant \frac{(M-m)^2}{(M+m)t_i - Mm - t_i^2} (f(M)+f(m)) - f(t_i),$$

provided that $m < t_i < M$ for $i = 1,2,\dots,n$.

Multiplying this inequality with $\frac{1}{w_i}$ and adding, we have

$$\begin{aligned} &\sum_{i=1}^n \frac{f(M+m-t_i)}{w_i} \\ &\leqslant \sum_{i=1}^n \frac{1}{w_i}\left(\frac{(M-m)^2 (f(M)+f(m))}{(M+m)t_i - Mm - t_i^2} - f(t_i)\right) \\ &= \frac{(M-m)^2 (f(M)+f(m))}{(M+m)\sum_{i=1}^n t_i/w_i - Mm - \sum_{i=1}^n t_i^2/w_i} - \sum_{i=1}^n \frac{f(t_i)}{w_i}. \end{aligned} \tag{3.4}$$

On the other hand, we know that [9]

$$f\left(\sum_{i=1}^n w_i t_i\right) \leqslant \sum_{i=1}^n \frac{f(t_i)}{w_i}, \tag{3.5}$$

which implies

$$f\left(M+m-\sum_{i=1}^{n} w_i t_i\right) = f\left(\sum_{i=1}^{n} w_i \left(M+m-t_i\right)\right) \leqslant \sum_{i=1}^{n} \frac{f\left(M+m-t_i\right)}{w_i}. \tag{3.6}$$

Noting the two inequalities (3.4) and (3.6), we get

$$f\left(M+m-\sum_{i=1}^{n} w_i t_i\right) \leqslant \frac{(M-m)^2\left(f(M)+f(m)\right)}{(M+m)\sum_{i=1}^{n} t_i/w_i \ -Mm-\sum_{i=1}^{n} t_i^2/w_i} - \sum_{i=1}^{n} \frac{f(t_i)}{w_i},$$

as desired. □

We provide a reverse for the inequality (3.5) in the following result.

PROPOSITION 3.2. *Let $f : J \to \mathbb{R}$ be a Q-class function, let $m < t_i < M$ for $i = 1,2,\ldots,n$, and let $w_1, w_2, \ldots, w_n > 0$ with $\sum_{i=1}^{n} w_i = 1$. Then for any $\alpha \geqslant 0$,*

$$\sum_{i=1}^{n} \frac{f(t_i)}{w_i} \leqslant \beta + \alpha f\left(\sum_{i=1}^{n} w_i t_i\right)$$

where $\beta = \max_{m<x<M} \left\{ \frac{M-m}{x-m} f(M) + \frac{M-m}{M-x} f(m) - \alpha f(x) \right\}$.

Proof. Multiplying (3.4) by $\frac{1}{w_i}$ $(i = 1,2,\ldots,n)$, then adding over i from 1 to n, we have

$$\sum_{i=1}^{n} \frac{f(t_i)}{w_i} \leqslant \frac{M-m}{\sum_{i=1}^{n} w_i t_i - m} f(M) + \frac{M-m}{M-\sum_{i=1}^{n} w_i t_i} f(m).$$

Therefore,

$$\begin{aligned}
&\sum_{i=1}^{n} \frac{f(t_i)}{w_i} - \alpha f\left(\sum_{i=1}^{n} w_i t_i\right) \\
&\leqslant \frac{M-m}{\sum_{i=1}^{n} w_i t_i - m} f(M) + \frac{M-m}{M-\sum_{i=1}^{n} w_i t_i} f(m) - \alpha f\left(\sum_{i=1}^{n} w_i t_i\right) \\
&\leqslant \max_{m<x<M} \left\{ \frac{M-m}{x-m} f(M) + \frac{M-m}{M-x} f(m) - \alpha f(x) \right\}.
\end{aligned}$$

This completes the proof. □

REMARK 3.1. Let the assumptions of Proposition 3.2 hold.

- If we put $\beta = 0$, then
$$\sum_{i=1}^{n} \frac{f(t_i)}{w_i} \leqslant \alpha f\left(\sum_{i=1}^{n} w_i t_i\right)$$
where $\alpha = \max_{m \leqslant x \leqslant M} \left\{ \frac{1}{f(x)} \left(\frac{M-m}{x-m} f(M) + \frac{M-m}{M-x} f(m) \right) \right\}$.

- If we put $\alpha = 1$, then
$$\sum_{i=1}^{n} \frac{f(t_i)}{w_i} \leqslant \beta + f\left(\sum_{i=1}^{n} w_i t_i\right)$$
where $\beta = \max_{m \leqslant x \leqslant M} \left\{ \frac{M-m}{x-m} f(M) + \frac{M-m}{M-x} f(m) - f(x) \right\}$.

It is well known that if $f : [0,\infty) \to \mathbb{R}$ is a convex function such that $f(0) \leqslant 0$, then $f(a) + f(b) \leqslant f(a+b)$. Usually, this is referred to as the super-additivity of convex functions. Interestingly, Q-class functions satisfy the following super-additive behavior.

THEOREM 3.3. *Let $f : [0,\infty) \to \mathbb{R}$ be a Q-class function. If $f(0) \leqslant 0$, then*
$$f(a) + f(b) \leqslant \frac{(a+b)^2}{ab} f(a+b)$$
for any $a, b > 0$.

Proof. It follows from (1.3) that for any $0 < t < 1$ and $x \in [0,\infty)$,
$$f(tx) \leqslant \frac{1}{1-t} f(0) + \frac{1}{t} f(x) \leqslant \frac{1}{t} f(x) \tag{3.7}$$
where the second inequality follows from the hypothesis $f(0) \leqslant 0$. Utilizing (1.3) and (3.7), we have
$$f(a) = f\left(\frac{a}{a+b}(a+b)\right) \leqslant \frac{a+b}{a} f(a+b).$$
Likewise,
$$f(b) \leqslant \frac{a+b}{b} f(a+b).$$
Adding the two inequalities above implies
$$f(a) + f(b) \leqslant \frac{(a+b)^2}{ab} f(a+b),$$
which completes the proof. □

We show a Shur-Jensen-type inequality for Q-class functions in the following.

THEOREM 3.4. *Let $f : J \to \mathbb{R}$ be a Q-class function, let $s_i, t_i \in J$ $(i = 1, 2, \ldots, n)$, and let $w_1, w_2, \ldots, w_n > 0$ with $\sum_{i=1}^{n} w_i = 1$. If $f(0) = 0$, then*

$$\left(\sum_{i=1}^{n} w_i f(s_i) s_i\right) \sum_{i=1}^{n} w_i t_i - \sum_{i=1}^{n} w_i f(s_i) s_i^2 \leqslant \sum_{i=1}^{n} w_i f(t_i) t_i^2 - \left(\sum_{i=1}^{n} w_i f(t_i) t_i\right) \sum_{i=1}^{n} w_i s_i.$$

Proof. If we set $u = 0$ and use the assumption $f(0) = 0$, we get from (1.4),

$$f(s)\left(st_i - s^2\right) \leqslant f(t_i)\left(t_i^2 - st_i\right) \tag{3.8}$$

for $i = 1, 2, \ldots, n$. Multiplying (3.8) by w_i $(i = 1, 2, \ldots, n)$ and adding over i from 1 to n, we infer

$$f(s)\left(s \sum_{i=1}^{n} w_i t_i - s^2\right) \leqslant \sum_{i=1}^{n} w_i f(t_i) t_i^2 - s \sum_{i=1}^{n} w_i f(t_i) t_i. \tag{3.9}$$

If we apply (3.9) for the selection $s = s_i$ $(i = 1, 2, \ldots, n)$, we may write

$$\left(\sum_{i=1}^{n} w_i t_i\right) f(s_i) s_i - f(s_i) s_i^2 \leqslant \sum_{i=1}^{n} w_i f(t_i) t_i^2 - \left(\sum_{i=1}^{n} w_i f(t_i) t_i\right) s_i. \tag{3.10}$$

Multiplying (3.10) by w_i $(i = 1, 2, \ldots, n)$ and adding over i from 1 to n, we get

$$\left(\sum_{i=1}^{n} w_i f(s_i) s_i\right) \sum_{i=1}^{n} w_i t_i - \sum_{i=1}^{n} w_i f(s_i) s_i^2 \leqslant \sum_{i=1}^{n} w_i f(t_i) t_i^2 - \left(\sum_{i=1}^{n} w_i f(t_i) t_i\right) \sum_{i=1}^{n} w_i s_i$$

as desired. □

In Theorem 3.4, letting $t_i = s_i$ $(i = 1, 2, \ldots, n)$, we get the following.

COROLLARY 3.1. *Let $f : J \to \mathbb{R}$ be a Q-class function, let $t_i \in J$ $(i = 1, 2, \ldots, n)$, and let $w_1, w_2, \ldots, w_n > 0$ with $\sum_{i=1}^{n} w_i = 1$. If $f(0) = 0$, then*

$$\left(\sum_{i=1}^{n} w_i f(t_i) t_i\right) \sum_{i=1}^{n} w_i t_i \leqslant \sum_{i=1}^{n} w_i f(t_i) t_i^2.$$

For the rest of our results, we present some mean-type inequalities for Q-class functions.

THEOREM 3.5. *Let $f : [a, b] \to \mathbb{R}$ be a continuous Q-class function. If $f(0) = 0$, then*

$$\frac{a+b}{2} \int_a^b t f(t)\, dt \leqslant \int_a^b t^2 f(t)\, dt.$$

Proof. Since f is Q-class function and $f(0)=0$, it fulfills the inequality

$$f(s)\left(st-s^2\right)\leqslant f(t)\left(t^2-st\right)$$

for any $s,t\in[a,b]$. Upon integration, this implies

$$\left(\frac{b^2-a^2}{2}\right)f(s)s-(b-a)f(s)s^2\leqslant\int_a^b t^2f(t)\,dt-s\int_a^b tf(t)\,dt.$$

Integration, again, implies

$$\left(\frac{b^2-a^2}{2}\right)\int_a^b tf(t)\,dt-(b-a)\int_a^b t^2f(t)\,dt$$

$$\leqslant(b-a)\int_a^b t^2f(t)\,dt-\left(\frac{b^2-a^2}{2}\right)\int_a^b tf(t)\,dt,$$

which yields

$$\frac{b^2-a^2}{2(b-a)}\int_a^b tf(t)\,dt\leqslant\int_a^b t^2f(t)\,dt$$

as desired. □

COROLLARY 3.2. *Let $f:[a,b]\to\mathbb{R}$ be a continuous Q-class function. If $f(0)=0$, then*

$$\int_0^1 tf(t)\,dt\leqslant 2\int_0^1 t^2f(t)\,dt.$$

PROPOSITION 3.3. *Let $f:[a,b]\to\mathbb{R}$ be a continuous Q-class function. Then*

$$\frac{2}{3}\left(b^2-a^2\right)\int_a^b tf(t)\,dt\leqslant(b-a)\int_a^b t^2f(t)\,dt+\frac{b^3-a^3}{3}\int_a^b f(t)\,dt.$$

In particular,

$$2\int_0^1 tf(t)\,dt\leqslant 3\int_0^1 t^2f(t)\,dt+\int_0^1 f(t)\,dt.$$

Proof. Setting $s=t$ in (1.4), we infer that

$$f(u)(u-t)^2\geqslant 0,$$

which is equivalent to

$$2tf(u)u\leqslant f(u)u^2+f(u)t^2.$$

We get the desired result by applying the same procedure as in the proof of Theorem 3.5. □

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