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INEQUALITY FOR THE VARIANCE OF AN ASYMMETRIC LOSS

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Abstract. We assume that the forecast error follows a probability distribution which is symmetric and monotonically non-increasing on non-negative real numbers, and if there is a mismatch between observed and predicted value, then we suffer a loss. Under the assumptions, we solve a minimization problem with an asymmetric loss function. In addition, we give an inequality for the variance of the loss.

1. Introduction

Let $\hat{y}$ be a predicted value of an observed value y. In this paper, we make the assumptions (I) and (II):

(I) The prediction error $z := \hat{y} - y$ is the realized value of a random variable Z, whose probability density function $f(z)$ satisfies $f(x) = f(-x)$ for $x \in \mathbb{R}$ and $f(x) \geqslant f(y)$ for $0 \leqslant x \leqslant y$.

(II) Let k_1, $k_2 \in \mathbb{R}_{>0}$. If there is a mismatch between y and $\hat{y}$, then we suffer a loss

$$L(z) := \begin{cases} k_1 z, & z \geqslant 0, \\ -k_2 z, & z < 0. \end{cases}$$

Under the assumptions (I) and (II), we solve the minimization problem for the expected value of $L(Z+c)$:

$$C = \arg\min_{c} \{\mathrm{E}[L(Z+c)]\}.$$

In addition, we give the following theorem.

THEOREM 1. *We have*

$$\mathrm{V}[L(Z+C)] \leqslant \mathrm{V}[L(Z)],$$

where equality holds only when $C = 0$*; that is, when* $k_1 = k_2$.

Theorem 1 is obtained by the following lemma.

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LEMMA 2. *Suppose that a probability density function $f(t)$ is monotonically non-increasing on $\mathbb{R}_{\geqslant 0}$ and satisfies $\int_0^\infty f(t)dt = \frac{1}{2}$. Then, for any $x \geqslant 0$, we have*

$$\alpha(x) := 4\int_0^x f(t)dt \int_x^\infty t f(t)dt - \frac{x}{2} + 2x\left(\int_0^x f(t)dt\right)^2 \geqslant 0.$$

If $f(t)$ is strictly decreasing, then $\alpha(x) > 0$ holds for $x > 0$. Also, $\alpha(x) = 0$ holds for $x \geqslant 0$ if and only if $f(t)$ equals to the probability density function of a continuous uniform distribution on $\mathbb{R}_{\geqslant 0}$.

These results are a generalization of the results of [5]. The paper [5] made the assumptions (I') and (II):

(I') The prediction error $z := \hat{y} - y$ is the realized value of a random variable Z, whose probability density function is a generalized Gaussian distribution function (see, e.g., [1], [2], and [3]) with mean zero

$$f(z) := \frac{1}{2ab\Gamma(a)} \exp\left(-\left|\frac{z}{b}\right|^{\frac{1}{a}}\right),$$

where $\Gamma(a)$ is the gamma function and $a, b > 0$.

Assumption (I) is weaker than (I'). Thus, we assume a more general situation than in [5]. In [5], under the assumptions (I') and (II), the minimization problem for the expected value of $L(Z+c)$ is solved and the inequality $\mathrm{V}[L(Z+C)] \leqslant \mathrm{V}[L(Z)]$ is obtained. This inequality is derived from the following inequality: *For $a, x > 0$, we have*

$$x^a\gamma(a,x)^2 - x^a\Gamma(a)^2 + 2\gamma(a,x)\Gamma(2a,x) > 0, \tag{1}$$

where

$$\Gamma(a) := \int_0^{+\infty} t^{a-1}e^{-t}dt, \quad \Gamma(a,x) := \int_x^{+\infty} t^{a-1}e^{-t}dt, \quad \gamma(a,x) := \int_0^x t^{a-1}e^{-t}dt.$$

Inequality (1) is the special case of Lemma 2 that $f(z)$ is a generalized Gaussian distribution function.

Assumptions (I) and (II) have a background in the procurement from an electricity market. Suppose that we purchase electricity $\hat{y}$ from an market, based on a forecast of the electricity y that will be needed. This situation makes the assumption (I). If $\hat{y} - y > 0$, then there is a waste of procurement fee proportional to $\hat{y} - y$. If $y - \hat{y} > 0$, then we are charged with a penalty proportional to $y - \hat{y}$. This situation makes the assumption (II). For details, see [4].

2. Proof of results

For $c\in\mathbb{R}$, let $\operatorname{sgn}(c):=1\,(c\geqslant 0);\ -1\,(c<0)$. From $\int_0^\infty f(z)dz=\frac{1}{2}$, the expected value of $L(Z+c)$ and $L(Z+c)^2$ are as follows: For any $c\in\mathbb{R}$,

$$\begin{aligned}
\mathrm{E}[L(Z+c)] &= (k_1+k_2)\int_{|c|}^{\infty} zf(z)dz+\frac{c(k_1-k_2)}{2}+|c|(k_1+k_2)\int_0^{|c|} f(z)dz,\\
\mathrm{E}[L(Z+c)^2] &= (k_1^2+k_2^2)\int_0^{\infty} z^2 f(z)dz+\operatorname{sgn}(c)(k_1^2-k_2^2)\int_0^{|c|} z^2 f(z)dz\\
&\quad +2c(k_1^2-k_2^2)\int_{|c|}^{\infty} zf(z)dz+\frac{c^2(k_1^2+k_2^2)}{2}+c|c|(k_1^2-k_2^2)\int_0^{|c|} f(z)dz.
\end{aligned}$$

Therefore, the expected value and the variance of $L(Z)$ are as follows:

$$\begin{aligned}
\mathrm{E}[L(Z)] &= (k_1+k_2)\int_0^{\infty} zf(z)dz,\\
\mathrm{V}[L(Z)] &= (k_1^2+k_2^2)\int_0^{\infty} z^2 f(z)dz-(k_1+k_2)^2\left(\int_0^{\infty} zf(z)dz\right)^2.
\end{aligned}$$

We determine the value c that gives the minimum value of $\mathrm{E}[L(Z+c)]$. From

$$\begin{aligned}
\frac{d}{dc}\mathrm{E}[L(Z+c)] &= \frac{k_1-k_2}{2}+\operatorname{sgn}(c)(k_1+k_2)\int_0^{|c|} f(z)dz,\\
\frac{d^2}{dc^2}\mathrm{E}[L(Z+c)] &= (k_1+k_2)f(c)\geqslant 0,
\end{aligned}$$

we can see that $\mathrm{E}[L(Z+c)]$ has the minimum value at the zero point of $\frac{d}{dc}\mathrm{E}[L(Z+c)]$. The zero point C satisfies the following equation:

$$\frac{k_1-k_2}{2}+\operatorname{sgn}(C)(k_1+k_2)\int_0^{|C|} f(z)dz=0.$$

From this, $C=0$ if and only if $k_1=k_2$. Also, we have

$$\begin{aligned}
\mathrm{E}[L(Z+C)] &= (k_1+k_2)\int_{|C|}^{\infty} zf(z)dz,\\
\mathrm{V}[L(Z+C)] &= (k_1^2+k_2^2)\int_0^{\infty} z^2 f(z)dz-2(k_1+k_2)^2\int_0^{|C|} f(z)dz\int_0^{|C|} z^2 f(z)dz\\
&\quad -4|C|(k_1+k_2)^2\int_0^{|C|} f(z)dz\int_{|C|}^{\infty} zf(z)dz+\frac{C^2(k_1+k_2)^2}{4}\\
&\quad -(k_1+k_2)^2\left(\int_{|C|}^{\infty} zf(z)dz\right)^2-C^2(k_1+k_2)^2\left(\int_0^{|C|} f(z)dz\right)^2.
\end{aligned}$$

Let

$$\beta(x) := -\left(\int_0^\infty zf(z)dz\right)^2 + 2\int_0^x f(z)dz\int_0^x z^2 f(z)dz + 4x\int_0^x f(z)dz\int_x^\infty zf(z)dz \\ -\frac{x^2}{4} + \left(\int_x^\infty zf(z)dz\right)^2 + x^2\left(\int_0^x f(z)dz\right)^2.$$

Then, $\mathrm{V}[L(Z)] - \mathrm{V}[L(Z+C)] = (k_1+k_2)^2\beta(C)$ holds. From $\beta(0) = 0$ and

$$\frac{d}{dx}\beta(x) = 4\int_0^x f(z)dz\int_x^\infty zf(z)dz - \frac{x}{2} + 2x\left(\int_0^x f(z)dz\right)^2 \\ + 2f(x)\int_0^x z^2 f(z)dz + 2xf(x)\int_x^\infty zf(z)dz,$$

if Lemma 2 is proved, then Theorem 1 is immediately obtained. We prove Lemma 2.

Proof of Lemma 2. Take any $x \geqslant 0$. If $f(x) = 0$, then $\alpha(x) = 0 - \frac{x}{2} + 2x\cdot\frac{1}{4} = 0$. Below, we consider the case that $f(x) > 0$. Let $\gamma := \int_0^x f(t)dt$. For a function $g = g(t)$ satisfying $f(x) \geqslant g(t) \geqslant 0$ for $x \leqslant t$ and $\gamma + \int_x^\infty g(t)dt = \frac{1}{2}$, we define a functional $S(g)$ by

$$S(g) := \int_x^\infty tg(t)dt.$$

Regarding $S(g)$ as a solid with the bottom surface area $\int_x^\infty g(t)dt = \frac{1}{2} - \gamma$ (constant), we find that if we make $g(t)$ as large as possible within the range where t is small, then $S(g)$ become smaller. Thus, the function g that minimizes $S(g)$ is $g(t) = u(t)$ defined by

$$u(t) := \begin{cases} f(x), & x \leqslant t \leqslant x + \frac{1}{f(x)}\left(\frac{1}{2} - \gamma\right), \\ 0, & \text{otherwise.} \end{cases}$$

From

$$S(u) = \int_x^\infty tu(t)dt = x\left(\frac{1}{2} - \gamma\right) + \frac{1}{2f(x)}\left(\gamma^2 - \gamma + \frac{1}{4}\right)$$

and $\gamma \geqslant xf(x)$, we have

$$\begin{aligned} \alpha(x) &\geqslant 4\gamma S(u) - \frac{x}{2} + 2x\gamma^2 \\ &= 4\gamma\left\{x\left(\frac{1}{2} - \gamma\right) + \frac{1}{2f(x)}\left(\gamma^2 - \gamma + \frac{1}{4}\right)\right\} - \frac{x}{2} + 2x\gamma^2 \\ &\geqslant 2x\gamma - 4x\gamma^2 + 2x\left(\gamma^2 - \gamma + \frac{1}{4}\right) - \frac{x}{2} + 2x\gamma^2 \\ &= 0. \end{aligned}$$

Also, from this, if $f(t)$ is strictly decreasing, then $\alpha(x) > 0$ holds for $x > 0$. In addition, $f(t)$ is the function of the form

$$f(t) = \begin{cases} \frac{1}{2a}, & 0 \leqslant t \leqslant a, \\ 0, & t > a \end{cases}$$

if and only if $\alpha(x) = 0$ holds for $x \geqslant 0$. $\square$

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