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A NEW ISOPERIMETRIC–TYPE INEQUALITY AND ITS STABILITY

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Abstract. In this paper, we discuss the isoperimetric-type inequalities for the closed convex curve in the Euclidean plane $\mathbb{R}^2$. The main result of this paper is that we establish a family of parametric inequalities. By adjusting the parameters, we can derive some well-known isoperimetric inequalities and some new improved isoperimetric-type inequalities. Furthermore, we investigate the stability property of the main inequality.

1. Introduction and main results

The classical isoperimetric inequality in the Euclidean plane $\mathbb{R}^2$ is one of the oldest and the most important geometric inequalities. It states that:

THEOREM 1. (classical isoperimetric inequality) *If γ is a simple colsed curve of length L, enclosing a region of area A, then one gets*

$$L^2 - 4\pi A \geqslant 0, \tag{1.1}$$

and the equality holds if and only if γ is a circle.

This famous fact was known to the ancient Greeks, but the first mathematical proof was only given in the 19th century by Steiner [1]. Since then, there have been many new proofs, sharpened forms, generalizations, and applicationsof this famous inequality.

Recently, in [2], there had established a reverse isoperimetric inequality for convex curves, which states that

THEOREM 2. (Pan-Zhang) *If γ is a closed strictly convex curve in the plane $\mathbb{R}^2$ with length L and enclosing an area A, then*

$$L^2 \leqslant 4\pi(A + |\widetilde{A}|), \tag{1.2}$$

where $\widetilde{A}$ denotes the oriented area of the domain enclosed by the locus of curvature centers of γ, and the equality holds if and only if γ is a circle.

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THEOREM 3. (Gao) [3] *If γ is a closed strictly convex curve in the plane $\mathbb{R}^2$ with length L and enclosing an area A, then*

$$L^2 \leqslant 4\pi A + \pi|\widetilde{A}|, \tag{1.3}$$

where $\widetilde{A}$ denotes the oriented area of the domain enclosed by the locus of curvature centers of γ, and the equality holds if and only if γ is a circle.

THEOREM 4. (Li-Gao) [4] *Let γ be a smooth regular positively oriented and closed strictly convex curve in the Euclidean plane $\mathbb{R}^2$ with length L and enclosing an area A, then*

$$\int_0^{2\pi} \rho(\theta)^2 d\theta \geqslant \frac{L^2}{\pi} - 2A + |\widetilde{A}|, \tag{1.4}$$

where ρ is the radius of curvature and θ is the angle between x-axis and the outward normal vector at the corresponding point p, and the equality in (1.4) holds if and only if γ is a circle.

THEOREM 5. (Bottema) *For the convex domain D in the $\mathbb{E}^2$, if the boundary ∂D of D is an oval curve, the following reverse Bonnesen style inequality (cf. [5], [7])*

$$L^2 - 4\pi A \leqslant \pi^2(\rho_M - \rho_m)^2, \tag{1.5}$$

where ρ_M and ρ_m are the maximum and minimum of the continuous curvature radius ρ of ∂D, respectively. The equality in (1.5) holds if and only if $\rho_M = \rho_m$, that is, ∂D is a circle.

Pleijel had an improvement of the Bottema's result as follows (cf. [6], [7]):

$$L^2 - 4\pi A \leqslant \pi(4-\pi)(\rho_M - \rho_m)^2, \tag{1.6}$$

where the equality holds if and only if ∂D is a circle.

In [4], a family of parametric inequalities have been established, but there exist some unnecessary geomertric quantities. So, in this paper, we establish an improved inequality and gain sharp version of (1.5) and (1.6). We firstly prove the following interesting inequality:

THEOREM 6. *Let γ be a smooth regular positively oriented and closed strictly convex curve in the Euclidean plane $\mathbb{R}^2$ with length L and enclosing an area A. If γ is not circle, then*

$$(\rho_M - \rho_m)^2 > \frac{L^2}{\pi^2}, \tag{1.7}$$

where ρ_M and ρ_m are the maximum and minimum of the continuous curvature radius ρ of γ, respectively.

Then we consider a family of parametric isoperimetric-type inequalities for closed convex plane curves, which is actually an improved version of the reverse isoperimetric inequalities (1.2), (1.3), (1.4), (1.5) and (1.6), and one of our main results is as follows:

THEOREM 7. (Main Theorem) *Let γ be a smooth regular positively oriented and closed strictly convex curve in the Euclidean plane $\mathbb{R}^2$ with length L and enclosing an area A, let $\widetilde{A}$ denote the area of the domain enclosed by the locus of curvature centers. Then for arbitrary constants λ, δ, σ, ω and ξ satisfying:*

$$\begin{cases} \lambda \geqslant 0 \\ 4\pi^2\delta + \pi\sigma + 4\xi \geqslant 0 \\ \omega - 2\lambda \geqslant 0 \\ 24\lambda - \sigma + 4\omega \geqslant 0 \end{cases} \tag{1.8}$$

then we have:

$$\lambda \int_0^{2\pi} \rho_\beta(\theta)^2 d\theta + \delta L^2 + \sigma A + \omega|\widetilde{A}| + \xi(\rho_M - \rho_m)^2 \geqslant 0, \tag{1.9}$$

where ρ_M and ρ_m are the maximum and minimum of the continuous curvature radius ρ of γ, respectively. ρ_β denote curvature radii of the locus of curvature centers of γ. The equality in (1.9) holds if γ is a circle and the parameters λ, δ, σ, ω and ξ meet

$$4\pi\delta + \sigma = 0.$$

Moreover if the equality in (1.9) holds and the parameters λ, δ, σ, ω and ξ satisfy (1.8), then γ is a circle.

REMARK 1. When $\lambda = \xi = 0$, $\delta = -1$, $\sigma = 4\pi$, $\omega = 4\pi$ and $\lambda = \xi = 0$, $\delta = -1$, $\sigma = 4\pi$, $\omega = \pi$, (1.8) satisfies clearly and the isoperimetric inequality (1.9) respectively turns into (1.2) and (1.3). Hence (1.9) can also be regarded as a reverse isoperimetric-type inequality. Furthermore, if we select other values of the parameters λ, δ, σ, ω and ξ satisfying (1.8), then we can obtain some new reverse isoperimetric inequalities:

COROLLARY 1. *Let γ be a smooth regular positively oriented and closed strictly convex curve in the Euclidean plane $\mathbb{R}^2$ with length L and enclosing an area A, let $\widetilde{A}$ denote the area of the domain enclosed by the locus of curvature centers. Then for arbitrary constants $\alpha, \beta, \lambda, \delta, \sigma, \omega$ and ξ satisfying:*

$$\begin{cases} \xi = 0; \ \ \alpha, \lambda \geqslant 0 \\ 4\pi\delta + \sigma + 2\beta \geqslant 0 \\ \omega + 2\beta - 2\lambda \geqslant 0 \\ 24\lambda - \sigma + 6\beta + 4\omega \geqslant 0 \end{cases} \tag{1.10}$$

then we have:

$$\alpha \int_\gamma k^2 ds + \lambda \int_0^{2\pi} \rho_\beta(\theta)^2 d\theta + \delta L^2 + \sigma A + \omega|\widetilde{A}| \geqslant 0. \tag{1.11}$$

REMARK 2. This inequation can be find in [4]. As in [4], we can obtain a series of results related to (1.11) under condition (1.10).

THEOREM 8. *Let γ be a smooth regular positively oriented and closed strictly convex curve in the Euclidean plane $\mathbb{R}^2$ with length L and enclosing an area A, let $\widetilde{A}$ denote the area of the domain enclosed by the locus of curvature centers. Then we have*

$$L^2 \leqslant 2\pi A + 2\pi|\widetilde{A}| + \pi^2(\rho_M - \rho_m)^2 \tag{1.12}$$

and the equality in (1.12) holds if and only if γ is a circle.

Moreover, when the relationship between A and $|\widetilde{A}|$ satisfies

$$\frac{A}{|\widetilde{A}|} \geqslant \frac{\pi - 2}{8 - 2\pi},$$

we can gain an improvement of the Pleijel's result in (1.6) as follows:

$$L^2 \leqslant (8\pi - 16)A + (2\pi - 4)|\widetilde{A}| + \pi(4 - \pi)(\rho_M - \rho_m)^2 \tag{1.13}$$

and the equality in (1.13) holds if and only if γ is a circle.

THEOREM 9. *Let γ be a smooth regular positively oriented and closed strictly convex curve in the Euclidean plane $\mathbb{R}^2$ with length L and enclosing an area A, let $\widetilde{A}$ denote the area of the domain enclosed by the locus of curvature centers. When γ is not circle, we have*

$$(\rho_M - \rho_m)^2 \leqslant \frac{2}{7\pi}\int_0^{2\pi} \rho_\beta(\theta)^2 d\theta + \frac{4}{7\pi}|\widetilde{A}| + \frac{4A}{\pi} \tag{1.14}$$

where ρ_M and ρ_m are the maximum and minimum of the continuous curvature radius ρ of γ, respectively. ρ_β denote curvature radii of the locus of curvature centers of γ.

REMARK 3. Besides those inequalities, we can actually derive more new and interesting geometric isoperimetric type inequalities by selecting the appropriate parameters λ, δ, σ, ω and ξ satisfying (1.8).

The stability problem associated with isoperimetric inequality is also interesting and signficant. In this paper, we will also research the stability properties of our isoperimetric inequality (1.9) with respect to both Hausdorff distance and L^2*-metric*. In section 2, we recall some basic facts about the plane convex geometry and introduce some lemmas. In section 3, we firstly prove the inequalities in Theorem 6, and then provide the proof of Theorem 7 by using Fourier series. Besides, we use Theorem 7 to prove Theorem 8 and Theorem 9. Finally, we discuss stability properties of inequality (1.9). We believe that our trick could be used to derive more interesting isoperimetric inequalities.

2. Geometric quantities and lemmas

In this section, we recall some basic facts about the convex plane curve which will be used later. In this paper we always assume that γ is a closed and convex plane curve which is a smooth regular positively oriented and closed strictly convex curve in the Euclidean plane $\mathbb{R}^2$, such that the curvature radii discussed can be defined and the Fourier series needed in the proof is convergent uniformly. The details can be found in the classical literature [8].

Let $p(\theta)$ denote the Minkowski support function of curve, where θ is the angle between x-axis and the outward normal vector at the corresponding point p. It gives us the parametrization of $\gamma(\theta)$ in terms of θ as follows

$$\gamma(\theta) = (\gamma_1(\theta), \gamma_2(\theta)) = (p(\theta)\cos\theta - p'(\theta)\sin\theta, p(\theta)\sin\theta + p'(\theta)\cos\theta).$$

Therefore the curvature $k(\theta)$ and the radius of curvature $\rho(\theta)$ of $\gamma(\theta)$ can be calculated by

$$k(\theta) = \frac{d\theta}{ds} = \frac{1}{p(\theta) + p''(\theta)} > 0$$

and

$$\rho(\theta) = \frac{ds}{d\theta} = p(\theta) + p''(\theta) > 0,$$

where we use the fact that γ is a strictly convex plane curve. The length L of $\gamma(\theta)$ and the area A it bounds can be also calculated respectively by

$$L = \int_\gamma ds = \int_0^{2\pi} p(\theta)d\theta$$

and

$$A = \frac{1}{2}\int_\gamma p(\theta)ds = \frac{1}{2}\int_0^{2\pi}(p(\theta)^2 - p'(\theta)^2)d\theta.$$

At the same time, we could obtain the locus of centers of curvature of $\gamma(\theta)$ as follows

$$\begin{aligned}\beta(\theta) &= \gamma(\theta) + \rho(\theta)N(\theta)\\ &= (-p'(\theta)\sin\theta - p''(\theta)\cos\theta, p'(\theta)\cos\theta - p''(\theta)\sin\theta)\end{aligned}$$

then

$$\beta'(\theta) = -(p'(\theta) + p'''(\theta))(\cos\theta, \sin\theta).$$

Therefore the radius of curvature $\rho_\beta(\theta)$ of the locus of curvature centers $\beta(\theta)$ can be calculated by

$$\rho_\beta(\theta) = \frac{ds}{d\theta} = p'(\theta) + p'''(\theta)$$

and

$$\int_0^{2\pi}\rho_\beta^2(\theta)d\theta = \int_0^{2\pi}(p'(\theta) + p'''(\theta))^2 d\theta.$$

Moreover, the oriented area of the domain enclosed by $\beta(\theta)$ is given by

$$\widetilde{A}=\frac{1}{2}\int_0^{2\pi}(p'(\theta)^2-p''(\theta)^2)d\theta.$$

Since the Minkowski support function of a given domain K is always continuous, bounded and 2π*-periodic*, it has a Fourier series of the form

$$p(\theta)=a_0+\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta). \tag{2.1}$$

Differentiation of (2.1) with respect to θ gives us

$$p'(\theta)=\sum_{n=1}^{\infty}n(-a_n\sin n\theta+b_n\cos n\theta), \tag{2.2}$$

$$p''(\theta)=-\sum_{n=1}^{\infty}n^2(a_n\cos n\theta+b_n\sin n\theta), \tag{2.3}$$

$$p'''(\theta)=-\sum_{n=1}^{\infty}n^3(-a_n\sin n\theta+b_n\cos n\theta). \tag{2.4}$$

Thus by (2.1), (2.2), (2.3), (2.4) and the Parseval equality we could express these geometric quantities in terms of the Fourier coefficients of $p(\theta)$ as follows

$$L=2\pi a_0,$$

$$A=\pi a_0^2-\frac{\pi}{2}\sum_{n=2}^{\infty}(n^2-1)(a_n^2+b_n^2),$$

$$|\widetilde{A}|=\frac{\pi}{2}\sum_{n=2}^{\infty}n^2(n^2-1)(a_n^2+b_n^2)$$

and

$$\begin{aligned}
&\int_0^{2\pi}\rho_\beta^2(\theta)d\theta\\
&=\int_0^{2\pi}(p'(\theta)+p'''(\theta))^2d\theta\\
&=\int_0^{2\pi}(\sum_{n=1}^{\infty}n(-a_n\sin n\theta+b_n\cos n\theta)-\sum_{n=1}^{\infty}n^3(-a_n\sin n\theta+b_n\cos n\theta))^2d\theta\\
&=\int_0^{2\pi}(\sum_{0}^{\infty}n(n^2-1)(-a_n\sin n\theta+b_n\cos n\theta))^2d\theta\\
&=\pi\sum_{n=1}^{\infty}n^2(n^2-1)^2(a_n^2+b_n^2).
\end{aligned}$$

The following results are essential to proof of the main resutlt of this note.

LEMMA 1. [2] *Let γ be a C^2 closed and strictly convex curve in the plane, ρ is the radius of curvature of γ, A is the area enclosed by γ and $\widetilde{A}$ the oriented area enclosed by the locus of centers of curvature β. Then we have*

$$\int_0^{2\pi} \rho^2 d\theta = 2(A + |\widetilde{A}|). \tag{2.5}$$

DEFINITION 1. Let γ be a C^2 closed and strictly convex curve in the plane, and $\gamma(\theta,0) \triangleq \gamma_0(\theta)$. If each point on the curve moves along the outward unit normal vector of the curve at that point, the unit-speed outward normal flow is to forms a family of simple closed curves $\gamma(\theta,t)$ on the plane, with original curve $\gamma(\theta,0)$, being equal to $\gamma_0(\theta)$. So that the evolution of curve can be noted by

$$\begin{cases} \frac{\partial \gamma(\theta,t)}{\partial t} = \vec{u}(\theta) \\ \gamma(\theta,0) = \gamma_0(\theta). \end{cases} \tag{2.6}$$

where $\vec{u}(\theta) = (\cos\theta, \sin\theta)$.

This note has shown in [10] that the tangent vector field $\mathbb{T}$ and the unit outward normal vector field $\mathbb{N}$ are independent of the time t. Using the conclusions in [10], we could gain the following results

LEMMA 2. *Let $\gamma(\theta,t)$ be a curve at the time $t \geqslant 0$, then the following equations hold:*

$$\rho(\theta,t) = \rho(\theta,0) + t, \tag{2.7}$$

$$k(\theta,t) = \frac{\rho(\theta,0)}{1 + k(\theta,0)t}, \tag{2.8}$$

$$L(t) = L(0) + 2\pi t, \tag{2.9}$$

$$A(t) = A(0) + L(0)t + \pi t^2, \tag{2.10}$$

where $\rho(\theta,t)$, $k(\theta,t)$, $L(t)$, $A(t)$ are the radius of curvature, curvature, circumference and the area of curve γ enclosing in time t, respectively. The equation (2.10) is usually called the Steiner polynomial.

LEMMA 3. *Let γ be a C^2 closed and strictly convex curve in the plane. $t_1 \geqslant t_2$ are the two roots of the Steiner polynomial $A(t)$, where*

$$A(t) = A(0) + L(0) + \pi t^2.$$

When γ is not circle, then we have [10]

$$-\rho_M < t_2 < -r_e < -\frac{L}{2\pi} < -r_i < t_1 < -\rho_m, \tag{2.11}$$

where r_e and r_i are, respectively, the radius of the minimum circumscribed circle and the maximum inscribed circle of γ, k is the curvature, $\rho = \frac{1}{k}$ is the radius of curvature, and ρ_M and ρ_m are the maximum and minimum of the continuous curvature radius ρ of γ, respectively. The quantities are all equal if the curve γ is a circle.

3. Proof of the main results

In this section, we firstly prove the inequality (1.7).

Proof of Theorem 6. We have defined Steiner polynomial (2.10) at Lemma 2.

Firstly, let the closed convex curve γ be $A(t)$, that is to say, $A(t) \triangleq A(\gamma) \triangleq A$. Then let the closed convex curve be γ_0 when $t=0$, where $A(0)$ and $L(0)$ are, respectively, the area enclosed by γ_0 and the length of γ_0. We have

$$A(t) = A(0) + L(0)t + \pi t^2,$$

equally,

$$A_0 + L_0 t + \pi t^2 - A = 0. \tag{3.1}$$

This means that the equation (3.1) has two roots:

$$t_1 = \frac{-L_0 + \sqrt{{L_0}^2 - 4\pi(A_0 - A)}}{2\pi},$$
$$t_2 = \frac{-L_0 - \sqrt{{L_0}^2 - 4\pi(A_0 - A)}}{2\pi}.$$

Thus we have

$$(t_1 - t_2)^2 = \frac{L_0^2 - 4\pi(A_0 - A)}{\pi^2} \tag{3.2}$$

then applying Lamma 3, we gain

$$(\rho_M - \rho_m)^2 > (t_1 - t_2)^2 = \frac{{L_0}^2 - 4\pi A_0 + 4\pi A}{\pi^2}. \tag{3.3}$$

By (2.9) and (2.10), the following equation holds,

$$\begin{aligned} L(t)^2 - 4\pi A(t) &= (L(0) + 2\pi t)^2 - 4\pi(A(0) + L(0)t + \pi t^2) \\ &= L(0)^2 + 4\pi t L(0) + 4\pi^2 t^2 - 4\pi A(0) - 4\pi t L(0) - 4\pi^2 t^2 \\ &= L(0)^2 - 4\pi A(0). \end{aligned}$$

Applying the above equation to (3.3), we have

$$(\rho_M - \rho_m)^2 > \frac{L^2 - 4\pi A + 4\pi A}{\pi^2} = \frac{L^2}{\pi^2},$$

where γ must not be a circle. If γ is a circle, $\rho_M = \rho_m$. □

Now we turn to prove our main result Theorem 7.

Proof of Theorem 7. To prove (1.9), it suffices to prove the inequality below under the condition (1.8)

$$\begin{aligned} &\lambda \int_0^{2\pi} \rho_\beta(\theta)^2 d\theta + \delta L^2 + \sigma A + \omega|\widetilde{A}| + \xi(\rho_M - \rho_m)^2 \\ \geqslant\; &\lambda \int_0^{2\pi} \rho_\beta(\theta)^2 d\theta + \left(\delta + \frac{\xi}{\pi^2}\right) L^2 + \sigma A + \omega|\widetilde{A}| \\ \geqslant\; &0. \end{aligned} \tag{3.4}$$

Then by using the expression of the geometric quantities in terms of the Fourier coefficients of $p(\theta)$ in section 2. The equation (3.4) can be written as

$$\begin{aligned}
&\lambda\int_0^{2\pi}\rho_\beta(\theta)^2d\theta+\left(\delta+\frac{\xi}{\pi^2}\right)L^2+\sigma A+\omega|\widetilde{A}|\\
=&\lambda\pi\sum_{n=1}^{\infty}n^2(n^2-1)^2(a_n^2+b_n^2)+\left(\delta+\frac{\xi}{\pi^2}\right)(2\pi a_0)^2\\
&+\sigma\left(\pi a_0^2-\frac{\pi}{2}\sum_{n=2}^{\infty}(n^2-1)(a_n^2+b_n^2)\right)+\frac{\pi}{2}\omega\sum_{n=2}^{\infty}n^2(n^2-1)(a_n^2+b_n^2)\\
=&(4\pi^2\delta+4\xi+\pi\sigma)a_0^2+\pi\sum_{n=2}^{\infty}\left(\lambda n^2(n^2-1)-\frac{\sigma}{2}+\frac{\omega n^2}{2}\right)(n^2-1)(a_n^2+b_n^2).
\end{aligned}$$

Thus it follows from (1.8) that

$$4\pi^2\delta+4\xi+\pi\sigma>0$$

and

$$\begin{aligned}
&\pi\sum_{n=2}^{\infty}\left(\lambda n^2(n^2-1)-\frac{\sigma}{2}+\frac{\omega n^2}{2}\right)(n^2-1)(a_n^2+b_n^2)\\
\geqslant&3\pi\sum_{n=2}^{\infty}(12\lambda-\frac{\sigma}{2}+2\omega)(a_n^2+b_n^2)\\
=&\frac{3}{2}\pi\sum_{n=2}^{\infty}(24\lambda-\sigma+4\omega)(a_n^2+b_n^2)\\
\geqslant&0,
\end{aligned}$$

which implies that

$$\lambda\int_0^{2\pi}\rho_\beta(\theta)^2d\theta+\left(\delta+\frac{\xi}{\pi^2}\right)L^2+\sigma A+\omega|\widetilde{A}|\geqslant 0.$$

Then the main inequality in Theorem 7 is proved. □

Furthermore, if γ is a circle, then $\rho_M=\rho_m$ and together with the equality condition in (1.1) we have

$$\lambda\int_0^{2\pi}\rho_\beta(\theta)^2d\theta+\delta L^2+\sigma A+\omega|\widetilde{A}|+\xi(\rho_M-\rho_m)^2=\delta L^2+\sigma A,$$

then for the parameters $\lambda,\delta,\sigma,\omega,\xi$ satisfying

$$4\pi\delta+\sigma=0$$

we have

$$\lambda\int_0^{2\pi}\rho_\beta(\theta)^2d\theta+\delta L^2+\sigma A+\omega|\widetilde{A}|+\xi(\rho_M-\rho_m)^2=0.$$

Proof of Theorem 8. In fact, we can proof the inequality (1.12) and (1.13) by selecting the appropriate parameters $\lambda, \delta, \sigma, \omega, \xi$ satisfying (1.8).

Firstly, let $\lambda = 0,\ \delta = -1,\ \xi = \pi^2, \sigma = \omega = 2\pi$, we obtain (1.12) which, in fact, is a improved version of (1.5).

Then, let $\lambda = 0,\ \delta = -1,\ \xi = \pi(4-\pi)$, thus it follows from (1.8) that

$$\begin{cases} \sigma \geqslant 8\pi - 16 \\ \sigma \leqslant 4\omega. \end{cases}$$

In order to get an improved and sharp version of (1.6), the following inequality must be satisfied

$$\sigma A + \omega|\widetilde{A}| \leqslant 4\pi A$$

Let $\sigma = 8\pi - 16$, then we can obtain

$$2\pi - 4 \leqslant \omega \leqslant \frac{(16-4\pi)A}{|\widetilde{A}|}.$$

Because A and $|\widetilde{A}|$ satisfy the following relationship at this time

$$\frac{A}{|\widetilde{A}|} \geqslant \frac{\pi - 2}{8 - 2\pi},$$

ω must exist. Take $\omega = 2\pi - 4$, then

$$L^2 \leqslant (8\pi - 16)A + (2\pi - 4)|\widetilde{A}| + \pi(4-\pi)(\rho_M - \rho_m)^2. \quad \square$$

REMARK 4. Moreover, let $\lambda = 0,\ \delta = -1,\ \sigma = 4\pi,\ \omega = 2\pi,\ \xi = 0$ and combine with Lemma 1, then we have

$$\int_0^{2\pi} \rho(\theta)^2 d\theta \geqslant \frac{L^2 - 2\pi A}{\pi}. \tag{3.5}$$

On the other hand, let $\lambda = 0,\ \delta = -1,\ \sigma = 4\pi,\ \omega = \pi,\ \xi = 0$. Using the same method as above, we can prove the inequality (1.4) which Li-Gao have proved in [4].

Proof of Theorem 9. If we select $\lambda = \frac{2\pi}{7},\ \delta = 0,\ \sigma = 4\pi,\ \omega = \frac{4\pi}{7},\ \xi = -\pi^2$, then (1.8) also satisfies and we obtain (1.14). $\square$

4. The stability properties of the isoperimetric inequality

Let K and M be two convex domains with respective Minkowski support functions p_K and p_M. The most frequently used function to measure the deviation between K and M is the Hausdorff distance

$$h_1(K,M) = \max_u |p_K(u) - p_M(u)|. \tag{4.1}$$

Another such measure which appears to be of particular value with respect to stability problems is the measure that corresponds to the L^2*-metric* in the function space, which is defined by

$$h_2(K,M) = \left(\int_0^{2\pi} |p_K(\theta) - p_M(\theta)|^2 d\theta \right)^{\frac{1}{2}}. \tag{4.2}$$

It is abvious that $h_1(K,M) = 0$ or $h_2(K,M) = 0$ if and only if $K = M$.

Now, we consider the stability properties of the reverse isoperimetric inequality (1.9) with respect to the deviation measures h_1 and h_2.

THEOREM 10. *Let K be a domain enclosed by a C^2 closed and strictly convex plane curve γ with area $A(K)$ and perimeter $L(K)$, and let $\widetilde{A}(K)$ denote the oriented area of the domain enclosed by the locus of curvature centers of γ, $S(K)$ denotes the Steiner disc associated with K. Then for arbitrary constants $\lambda, \delta, \sigma, \omega, \xi$ satisfying*

$$\begin{cases} \lambda \geqslant 0 \\ 4\pi^2\delta + \pi\sigma + 4\xi \geqslant 0 \\ \omega - 2\lambda \geqslant 0 \\ 24\lambda - \sigma + 4\omega \geqslant 0 \end{cases} \tag{4.3}$$

then we have

$$\begin{aligned} h_1(K,S(K))^2 &\leqslant C(\lambda,\sigma,\omega) \\ &\quad \times \left(\lambda \int_0^{2\pi} \rho_\beta^2(\theta) d\theta + \delta L^2 + \sigma A + \omega |\widetilde{A}| + \xi (\rho_M - \rho_m)^2 \right) \end{aligned} \tag{4.4}$$

where k is the curvature of γ, ρ and ρ_β respectively denote curvature radii of the curve γ and the locus of curvature centers,

$$C(\lambda,\sigma,\omega) = \max\left\{ 1, \sum_{n=2}^{\infty} \frac{1}{\pi(\lambda n^2(n^2-1) - \frac{\sigma}{2} + \frac{\omega n^2}{2})(n^2-1)} \right\}.$$

The equality holds if γ is a circle and the parameters $\lambda, \delta, \sigma, \omega, \xi$ satisfy

$$4\pi\delta + \sigma = 0.$$

Proof. We assume $\vec{S}(K) = (0,0)$, then the support functions P_K and $P_{S(K)}$ can be written as

$$P_K(\theta) = \frac{L(K)}{2\pi} + \sum_{n=2}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)$$

and

$$P_{S(K)}(\theta) = \frac{L(K)}{2\pi}$$

it follows that

$$\begin{aligned}|P_K(\theta)-P_{S(K)}(\theta)| &= \Big|\frac{L(K)}{2\pi}+\sum_{n=2}^{\infty}(a_n\cos n\theta+b_n\sin n\theta)-\frac{L(K)}{2\pi}\Big|\\ &\leqslant \sum_{n=2}^{\infty}|a_n\cos n\theta+b_n\sin n\theta|\\ &\leqslant \sum_{n=2}^{\infty}\sqrt{a_n^2+b_n^2}.\end{aligned}$$

We have proved the following inequality in section 3

$$\begin{aligned}&\lambda\int_0^{2\pi}\rho_\beta(\theta)^2d\theta+\delta L^2+\sigma A+\omega|\widetilde{A}|+\xi(\rho_M-\rho_m)^2\\ &\geqslant \lambda\int_0^{2\pi}\rho_\beta(\theta)^2d\theta+\left(\delta+\frac{\xi}{\pi^2}\right)L^2+\sigma A+\omega|\widetilde{A}|\\ &=(4\pi^2\delta+4\xi+\pi\sigma)a_0^2\\ &\quad+\pi\sum_{n=2}^{\infty}\left(\lambda n^2(n^2-1)-\frac{\sigma}{2}+\frac{\omega n^2}{2}\right)(n^2-1)(a_n^2+b_n^2).\end{aligned}\tag{4.5}$$

Using Hölder's inequality, together with (4.5) we have

$$\begin{aligned}&h_1(K,S(K))^2\\ &\leqslant (\sum_{n=2}^{\infty}\sqrt{a_n^2+b_n^2})^2\\ &\leqslant (4\pi^2\delta+4\xi+\pi\sigma)a_0^2\\ &\quad+\left(\pi\sum_{n=2}^{\infty}\left(\lambda n^2(n^2-1)-\frac{\sigma}{2}+\frac{\omega n^2}{2}\right)(n^2-1)(a_n^2+b_n^2)\right)\\ &\quad\times\left(\sum_{n=2}^{\infty}\frac{1}{\pi(\lambda n^2(n^2-1)-\frac{\sigma}{2}+\frac{\omega n^2}{2})(n^2-1)}\right)\\ &\leqslant \max\left\{1,\sum_{n=2}^{\infty}\frac{1}{\pi(\lambda n^2(n^2-1)-\frac{\sigma}{2}+\frac{\omega n^2}{2})(n^2-1)}\right\}\\ &\quad\times\left(\lambda\int_0^{2\pi}\rho_\beta^2(\theta)d\theta+\left(\delta+\frac{\xi}{\pi^2}\right)L^2+\sigma A+\omega|\widetilde{A}|\right)\\ &\leqslant C(\lambda,\sigma,\omega)(\lambda\int_0^{2\pi}\rho_\beta^2d\theta+\delta L^2+\sigma A+\omega|\widetilde{A}|+\xi(\rho_M-\rho_m)^2)\end{aligned}$$

for arbitrary constants $\lambda,\delta,\sigma,\omega,\xi$ satisfying (4.3). Besides, if γ is a circle,

$$\lambda\int_0^{2\pi}\rho_\beta^2(\theta)d\theta+\delta L^2+\sigma A+\xi(\rho_M-\rho_m)^2=0.$$

It is obvious that $h_1(K,S(K))=0$. $\square$

THEOREM 11. *Under the same assumptions of Theorem 10, Then for arbitrary constants* $\lambda,\delta,\sigma,\omega,\xi$

$$\begin{cases}\lambda \geqslant 0\\ 4\pi^2\delta+\pi\sigma+4\xi \geqslant 0\\ \omega-2\lambda \geqslant 0\\ 72\lambda-3\sigma+12\omega-2 \geqslant 0\end{cases} \tag{4.6}$$

then we have

$$h_2(K,S(K))^2 \leqslant \lambda\int_0^{2\pi}\rho_\beta^2(\theta)d\theta+\delta L^2+\sigma A+\omega|\widetilde{A}|+\xi(\rho_M-\rho_m)^2. \tag{4.7}$$

The equality holds if γ *is a circle and the parameters* $\lambda,\delta,\sigma,\omega,\xi$ *satisfy*

$$4\pi\delta+\sigma=0.$$

Proof. Using Parseval equality, we have

$$\begin{aligned}h_2(K,S(K))^2 &= \int_0^{2\pi}|P_K(\theta)-P_{S(K)}(\theta)|^2d\theta\\ &= \sum_{n=2}^{\infty}((\sqrt{\pi}a_n)^2+(\sqrt{\pi}b_n)^2)\\ &= \pi\sum_{n=2}^{\infty}(a_n^2+b_n^2).\end{aligned}$$

Since it is easily seen that

$$\begin{aligned}&\lambda\int_0^{2\pi}\rho_\beta^2(\theta)d\theta+\left(\sigma+\frac{\xi}{\pi^2}\right)L^2+\sigma A+\omega|\widetilde{A}|-h_2(K,S(K))^2\\ &=(4\pi^2\delta+4\xi+\pi\sigma)a_0^2\\ &\quad+\pi\sum_{n=2}^{\infty}\left(\left(\lambda n^2(n^2-1)-\frac{\sigma}{2}+\frac{\omega n^2}{2}\right)(n^2-1)-1\right)(a_n^2+b_n^2).\end{aligned}$$

Becasue the parameters $\lambda,\delta,\sigma,\omega,\xi$ satisfy (4.6), we have

$$\begin{aligned}&\pi\sum_{n=2}^{\infty}\left(\left(\lambda n^2(n^2-1)-\frac{\sigma}{2}+\frac{\omega n^2}{2}\right)(n^2-1)-1\right)(a_n^2+b_n^2)\\ &\geqslant \pi\sum_{n=2}^{\infty}\left(3\left(12\lambda-\frac{\sigma}{2}+2\omega\right)-1\right)(a_n^2+b_n^2)\\ &=\frac{\pi}{2}\sum_{n=2}^{\infty}(72\lambda-3\sigma+12\omega-2)(a_n^2+b_n^2)\\ &\geqslant 0.\end{aligned}$$

Thus

$$\begin{aligned} & h_2(K,S(K))^2 \\ \leqslant & \lambda \int_0^{2\pi} \rho_\beta^2(\theta)d\theta + \left(\delta + \frac{\xi}{\pi^2}\right)L^2 + \sigma A + \omega|\widetilde{A}| \\ \leqslant & \lambda \int_0^{2\pi} \rho_\beta^2(\theta)d\theta + \delta L^2 + \sigma A + \omega|\widetilde{A}| + \xi(\rho_M - \rho_m)^2. \quad \square \end{aligned}$$

REMARK 5. Combining Theorem 10 and Theorem 11, we can obtain:

$$\begin{aligned} & \max\{h_1(K,S(K))^2, h_2(K,S(K))^2\} \\ \leqslant & C(\lambda,\sigma,\omega)(\lambda \int_0^{2\pi} \rho_\beta^2(\theta)d\theta + \delta L^2 + \sigma A + \omega|\widetilde{A}| + \xi(\rho_M - \rho_m)^2) \end{aligned}$$

where

$$C(\lambda,\sigma,\omega) = \max\left\{1, \sum_{n=2}^{\infty} \frac{1}{\pi(\lambda n^2(n^2-1) - \frac{\sigma}{2} + \frac{\omega n^2}{2})(n^2-1)}\right\}$$

which states that the isoperimetric inequality (1.9) does have well stability properties with respect to both Hausdorff distance and L^2*-metric*.

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