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NUMERICAL RADIUS OF PRODUCTS OF SPECIAL MATRICES

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Abstract. The purpose of this note is to present upper bounds estimations for the numerical radius of a products and Hadamard products of special matrices, including sectorial and accretive-dissipative matrices.

1. Introduction

Let $\mathbb{M}_n$ be the algebra of all $n\times n$ complex matrices. If $X=[x_{i,j}],Y=[y_{i,j}]\in\mathbb{M}_n$, then their Hadamard product $X\circ Y$ is the matrix $[x_{i,j}y_{i,j}]$. The cartesian decomposition of $X\in\mathbb{M}_n$ is presented as

$$X=A+iB, \tag{1}$$

where A and B are the Hermitian matrices $A=Re(X)=\frac{X+X^*}{2}$ and $B=Im(X)=\frac{X-X^*}{2i}$. A matrix X is said to be accretive (resp. dissipative) if in its cartesian decomposition (1) the matrix A (resp. B) is positive definite. If both A and B, in the decomposition (1), are positive definite, X is called accretive-dissipative.

The numerical range of $X\in\mathbb{M}_n$ is the compact convex subset of the complex plane defined as follows:

$$W(X)=\{\langle Xx,x\rangle : x\in\mathbb{C}^n, ||x||=1\},$$

where $\langle\cdot,\cdot\rangle$ is the standard inner product on $\mathbb{C}^n$ and $||\cdot||$ is the Euclidean norm on $\mathbb{C}^n$. A very important result is that

$$\sigma(X)\subset W(X),$$

where $\sigma(X)$ is the spectrum of X.

For $\alpha\in[0,\pi/2)$, let S_α be the sector defined in the complex plane by

$$S_\alpha=\{z\in\mathbb{C} : Re(z)>0, |Im(z)|\leqslant \tan(\alpha)Re(z)\}.$$

A matrix X is called sectorial if $W(X)\subset S_\alpha$. The smallest possible such α is called the index of sectoriality.

For $\alpha\in[0,\pi/2)$, let $\mathbb{M}^s_{n,\alpha}$ be the class of all $n\times n$ matrices X with $W(zX)\subset S_\alpha$ for some complex number z with $|z|=1$.

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It is clear that X is accretive-dissipative if and only if $W(e^{-\pi i/4}X) \subset S_{\pi/4}$, and hence $X \in \mathbb{M}^s_{n,\alpha}$ with $\alpha = \frac{\pi}{4}$. For more study of sectorial matrices see [1, 2, 3, 4, 10, 11, 12, 13] and the references therein.

A norm N on $\mathbb{M}_n$ is said to be unitarily invariant if it satisfies the property $N(UXV) = N(X)$ for all $X \in \mathbb{M}_n$ and all unitaries $U, V \in \mathbb{M}_n$, and it is said to be multiplicative if $N(AB) \leqslant N(A)N(B)$ for all $A, B \in \mathbb{M}_n$. Examples of such unitarily invariant multiplicative norms are the Schatten p-norm defined by $||X||_p = \left(\sum_{j=1}^n s_j^p(X)\right)^p$, $p \geqslant 1$. When $p = \infty$, this last norm is just the usual operator norm defined by $\|X\| = \sup_{\|x\|=1} \|Xx\|$.

Associated with numerical range, the numerical radius of X is defined by

$$\omega(X) = \sup\{|z| : z \in W(X)\}.$$

It is well known that $\omega(\cdot)$ defines a norm on $\mathbb{M}_n$ which is equivalent to the usual operator norm $\|\cdot\|$. In fact we have

$$\frac{1}{2}\|X\| \leqslant \omega(X) \leqslant \|X\|; \quad \forall X \in \mathbb{M}_n. \tag{2}$$

Moreover, if $X \in \mathbb{M}_n$ is normal then $\omega(X) = \|X\|$. Therefore, the inequalities in (2) are sharp.

Obviously, $\omega(\cdot)$ defines a weakly unitarily invariant norm on $\mathbb{M}_n$; that is it satisfies the property $\omega(UXU) = \omega(X)$ for all $X \in \mathbb{M}_n$ and all unitary $U \in \mathbb{M}_n$.

EXAMPLE 1.1. Let $X = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$. Then $\omega(X) = \omega(Y) = 1$ and $\omega(XY) = 4$

Example 1.1 shows that $\omega(\cdot)$ is not a multiplicative norm. However, the inequalities in (2) implies that $4\omega(\cdot)$ is a multiplicative norm. That is for all $X, Y \in \mathbb{M}_n$, $4\omega(XY) \leqslant (4\omega(X))(4\omega(Y))$. Equivalently,

$$\omega(XY) \leqslant 4\omega(X)\omega(Y); \quad \forall X, Y \in \mathbb{M}_n. \tag{3}$$

Obviously, Example 1.1 shows that the inequality (3) is sharp and the constant 4 is the best possible in (3).

The Hadamard product version of (3) can be written as

$$\omega(X \circ Y) \leqslant 2\omega(X)\omega(Y). \tag{4}$$

That is, $2\omega(\cdot)$ is a multiplicative norm over the Hadamard product. See [7, p. 73].

The following example shows that the constant 2 is the best possible in (4).

EXAMPLE 1.2. Let $X = Y = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ Then $\omega(X) = \omega(Y) = 1$ and $\omega(X \circ Y) = 2$.

By considering special matrices X and Y, it is possible to obtain better estimations than those in (3) and (4).

If $XY = YX$, then

$$\omega(XY) \leqslant 2\omega(X)\omega(Y). \tag{5}$$

See [8, Theorem 2.5-2].

If X or Y is normal such that $XY = YX$, then

$$\omega(XY) \leqslant \omega(X)\omega(Y). \tag{6}$$

See [8, Corollary 2.5-6].

If X or Y is normal. Then

$$\omega(X \circ Y) \leqslant \omega(X)\omega(Y). \tag{7}$$

See [7, Corollary 4.2.17].

If $A, B \in \mathbb{M}_n$ and $A = [a_{ij}]$ is positive semidefinite, then

$$\omega(A \circ B) \leqslant \left(\max_j a_{jj}\right) \omega(B). \tag{8}$$

See [5, Corollary 4] and [9, Proposition 4.1].

The purpose of this short note is to add more inequalities to the above list. More precisely, we give estimations of the numerical radius of products or Hadamard products of sectorial matrices and related matrices such as accretive and dissipative matrices.

2. Main results

We start this section by the following two observations.

LEMMA 2.1. *Let $X \in \mathbb{M}_n$. If $W(X) \subset S_\alpha$, then*

$$\|X\| \leqslant \sec(\alpha)\|Re(X)\|.$$

The above Lemma can be found in [1], [3] , [2] and [13].

REMARK 2.1. 1. We recall that $\omega(\cdot)$ defines a self-adjoint norm on $\mathbb{M}_n$, that is it is a norm satisfies the properties $\omega(X^*) = \omega(X)$ for all $X \in \mathbb{M}_n$. Therefore,

$$\begin{aligned}
\omega(Re(X)) &= \omega\left(\frac{X+X^*}{2}\right) \\
&= \frac{1}{2}\left(\omega\left(X+X^*\right)\right) \\
&\leqslant \frac{1}{2}\left(\omega(X)+\omega\left(X^*\right)\right) \\
&= \frac{1}{2}\left(\omega(X)+\omega(X)\right) \\
&= \omega(X).
\end{aligned}$$

2. Let $X \in \mathbb{M}_n$ and let $X = A + iB$ be its cartesian decomposition. If $W(X) \subset S_\alpha$, then for any $v \in \mathbb{C}^n$ with $||v|| = 1$ $\langle Av\ ,\ v\rangle, \langle Bx\ ,\ x\rangle \in \mathbb{R}$ and

$$\langle Xv\ ,\ v\rangle = \langle Av\ ,\ v\rangle + i\,\langle Bv\ ,\ v\rangle \in S_\alpha.$$

Therefore,

$$|\langle Bv\ ,\ v\rangle| \leqslant \tan(\alpha)\langle Av\ ,\ v\rangle.$$

Taking the supremum over all such v's gives

$$\omega(B) \leqslant \tan(\alpha)\omega(A).$$

Now we are ready to state the first result.

THEOREM 2.1. *Let* $X \in \mathbb{M}^s_{n,\alpha_1}$ *and* $Y \in \mathbb{M}^s_{n,\alpha_2}$, *where* $\alpha_2, \alpha_2 \in [0, \pi/2)$. *Then*

$$\omega(XY) \leqslant \sec(\alpha_1)\sec(\alpha_2)\omega(X)\omega(Y).$$

Proof. Since $X \in \mathbb{M}^s_{n,\alpha_1}$ and $Y \in \mathbb{M}^s_{n,\alpha_2}$, there are two complex numbers $z, w \in \mathbb{C}$ with $|z| = |w| = 1$ such that $W(zX) \subset S_{\alpha_1}$ and $W(wY) \subset S_{\alpha_2}$.
Notice that

$$\begin{aligned}
\omega(XY) &\leqslant ||XY|| \quad (\text{ by (2) })) \\
&\leqslant ||X||\,||Y|| \\
&= ||zX||\,||wY|| \\
&\leqslant \sec(\alpha_1)\sec(\alpha_2)||Re\ (zX)\,|\,|||Re\ (wY)\,|| \quad (\text{by Lemma 2.1}) \\
&= \sec(\alpha_1)\sec(\alpha_2)\omega(Re\ (zX))\ \omega\,(Re\ (wY)) \\
&\qquad (\text{since } Re\ (zX)) \text{ and } Re\ (wY) \text{ are Hermitian }) \\
&\leqslant \sec(\alpha_1)\sec(\alpha_2)\omega\,(zX)\ \ \omega(wY) \quad (\text{by Remark 2.1}) \\
&= \sec(\alpha_1)\sec(\alpha_2)\omega\,(X)\ \ \omega(Y). \qquad \square
\end{aligned}$$

We remark that if $X, Y \in \mathbb{M}^s_{n,\alpha}$, Theorem 2.1 implies that

$$\omega(XY) \leqslant \sec^2(\alpha)\omega(X)\omega(Y). \tag{9}$$

The inequality (9) presents a refinement of the inequality (3) when $0 \leqslant \alpha \leqslant \frac{\pi}{3}$. A particular case is when X and Y are accretive-dissipative as in the following result.

COROLLARY 2.1. *If* $X, Y \in \mathbb{M}_n$ *are accretive-dissipative, then*

$$\omega(XY) \leqslant 2\omega(X)\omega(Y).$$

Proof. The result follows from Theorem 2.1 and the fact that if $X, Y \in \mathbb{M}_n$ are accretive-dissipative then $X, Y \in \mathbb{M}^s_{n,\alpha}$ with $\alpha = \pi/4$. $\square$

REMARK 2.2. 1. If $X \in \mathbb{M}_n$ is accretive, i.e. $Re(X) > 0$, then X is sectorial with sectorial index

$$\alpha_X = \tan^{-1}(|\lambda_1\left((ReX)^{-1} Im(X))|\right) = ||(ReX)^{-1/2}(ImX)(ReX)^{-1/2}||. \quad (10)$$

See [4]. Therefor $X \in \mathbb{M}^s_{n,\alpha}$ with $\alpha = \alpha_X$.

2. If $X \in \mathbb{M}_n$ is dissipative, i.e. $Im(X) > 0$, then iX is accretive and hence $X \in \mathbb{M}^s_{n,\alpha}$ with $\alpha = \alpha_X$.

COROLLARY 2.2. *If $X,Y \in \mathbb{M}_n$ are accretive (or dissipative), then*

$$\omega(XY) \leqslant (1+a^2)\omega(X)\omega(Y),$$

where

$$a = \max\{|\lambda_1\left((ReX)^{-1} ImX\right)|, |\lambda_1\left((ReY)^{-1} ImY\right)|\}.$$

Proof. Since $X,Y \in \mathbb{M}_n$ are accretive (or dissipative), we have $X \in \mathbb{M}^s_{n,\alpha_X}$ and $Y \in \mathbb{M}^s_{n,\alpha_Y}$, where α_X and α_Y are as in (10). By Theorem 2.1, we have

$$\omega(XY) \leqslant \sec(\alpha_X)\sec(\alpha_Y)\omega(X)\omega(Y).$$

Now the result follows by noting that

$$\sec(\alpha_X) = \sec(\tan^{-1}(|\lambda_1\left((Re(X))^{-1} Im(X))|\right)) = \sqrt{1+\left(\lambda_1\left((ReX)^{-1} ImX\right)\right)^2},$$

and

$$\sec(\alpha_Y) = \sec(\tan^{-1}(|\lambda_1\left((ReY)^{-1} ImY)|\right)) = \sqrt{1+\left(\lambda_1\left((ReY)^{-1} ImY\right)\right)^2}. \quad \square$$

Same proof method used to prove Theorem 2.1 can be used to prove the following more general result.

THEOREM 2.2. *Let $X_j \in \mathbb{M}^s_{n,\alpha_j}$, $j = 1,2,3,\ldots,m$. Then*

$$\omega\left(\prod_{j=1}^{m} X_j\right) \leqslant \prod_{j=1}^{m} \sec(\alpha_j)\omega(X_j).$$

Consequently,

COROLLARY 2.3. *If $X_1, X_2, \ldots, X_m \in \mathbb{M}_n$ are accretive-dissipative, then*

$$\omega\left(\prod_{j=1}^{m} X_j\right) \leqslant 2^{m/2}\prod_{j=1}^{m} \omega(X_j).$$

COROLLARY 2.4. *If $X_1, X_2, \ldots, X_m \in \mathbb{M}_n$ are accretive (or dissipative), then*

$$\omega\left(\prod_{j=1}^{m} X_j\right) \leqslant (1+a^2)^{m/2} \prod_{j=1}^{m} \omega(X_j),$$

where

$$a = \max\left\{ |\lambda_1\left((Re\,X_j)^{-1} Im\,X_j\right)|,\ j = 1,2,\ldots,m \right\}.$$

In what follows, we present upper bounds for the numerical ranges of hadamard products of special matrices. The following lemma is important in our analysis. It can be found in [1, 2, 3].

LEMMA 2.2. *Let $T \in \mathbb{M}_n$. If $W(T) \subset S_\alpha$ for some $\alpha \in [0, \pi/2)$. Then*

$$\begin{pmatrix} \sec(\alpha) Re\,(T) & T \\ T^* & \sec(\alpha) Re\,(T) \end{pmatrix} \geqslant 0.$$

Now we estimate the numerical range for a Hadamard product of sectorial matrices.

THEOREM 2.3. *Let $X \in \mathbb{M}^s_{n,\alpha_1}$ and $Y \in \mathbb{M}^s_{n,\alpha_2}$. Then*

$$\omega(X \circ Y) \leqslant \sec(\alpha_1)\sec(\alpha_2)\omega(X)\omega(Y).$$

Proof. Since $X \in \mathbb{M}^s_{n,\alpha_1}$ and $Y \in \mathbb{M}^s_{n,\alpha_2}$, there are two complex numbers $z, w \in \mathbb{C}$ with $|z| = |y| = 1$ such that $W(zX) \subset S_{\alpha_1}$ and $W(wY) \subset S_{\alpha_2}$. Therefore, by Lemma 2.2, the following two block matrices

$$\begin{pmatrix} \sec(\alpha_1) Re\,(zX) & zX \\ \bar{z}X^* & \sec(\alpha_1) Re\,(zX) \end{pmatrix}, \quad \begin{pmatrix} \sec(\alpha_2) Re\,(wY) & wY \\ \overline{w}Y^* & \sec(\alpha_2) Re\,(wY) \end{pmatrix}$$

are positive semidefinite. Hence

$$\begin{pmatrix} \sec(\alpha_1)\sec(\alpha_2) Re\,(zX) \circ Re\,(wY) & zw\,(X \circ Y) \\ \overline{zw}\,(X \circ Y)^* & \sec(\alpha_1)\sec(\alpha_2) Re\,(zX) \circ Re\,(wY) \end{pmatrix}$$

is also positive semidefinite. Since $||\cdot||$ is a Lieb function, we have

$$||X \circ Y|| = ||zw\,(X \circ Y)|| \leqslant \sec(\alpha_1)\sec(\alpha_2) ||Re\,(zX) \circ Re\,(wY)||.$$

Recall that a Lieb function L is a continuous function defined on $\mathbb{M}_n$, which is increasing on the cone of positive matrices. It satisfies the property that for any matrices A and B in $\mathbb{M}_n$, the inequality $|\mathrm{L}(A^*B)|^2 \leqslant \mathrm{L}(A^*A)\mathrm{L}(B^*B)$ holds. For more details, please refer to page 270 in [6].

Now, observe that

$$\begin{aligned}\omega(X\circ Y) &\leqslant \|X\circ Y\|\\ &\leqslant \sec(\alpha_1)\sec(\alpha_2)\|Re\,(zX)\circ Re\,(wY)\|\\ &= \sec(\alpha_1)\sec(\alpha_2)\omega(Re\,(zX)\circ Re\,(wY))\\ &\leqslant \sec(\alpha_1)\sec(\alpha_2)\omega(Re\,(zX))\omega(Re\,(wY)) \quad \text{(by (7))}\\ &\leqslant \sec(\alpha_1)\sec(\alpha_2)\omega(zX)\omega(wY) \quad \text{(by Remark 2.1)}\\ &= \sec(\alpha_1)\sec(\alpha_2)|z|\omega(X)|w|\omega(Y)\\ &= \sec(\alpha_1)\sec(\alpha_2)\omega(X)\omega(Y). \quad \square\end{aligned} \tag{11}$$

REMARK 2.3. 1. If $X,Y\in\mathbb{M}^s_{n,\alpha}$, then

$$\omega(X\circ Y)\leqslant \sec^2(\alpha)\omega(X)\omega(Y). \tag{12}$$

It is clear that inequality (12) presents a refinement of inequality (4) for $0\leqslant\alpha<\frac{\pi}{4}$. In particular if X and Y are accretive-dissipative, then both (12) and (4) give the same estimation.

2. If $X,Y\in\mathbb{M}_n$ are accretive (or dissipative), then (12) implies that

$$\omega(X\circ Y)\leqslant(1+a^2)\omega(X)\omega(Y),$$

where

$$a=\max\{|\lambda_1\left((Re\,X)^{-1}Im\,X\right)|,|\lambda_1\left((Re\,Y)^{-1}Im\,Y\right)|\}.$$

The argument used to prove Theorem 2.3 can be easily modified to prove the following more general result.

THEOREM 2.4. *Let $X_j\in\mathbb{M}^s_{n,\alpha_1}$, $j=1,2,\ldots,m$. Then*

$$\omega(X_1\circ\ldots\circ X_m)\leqslant\prod_{j=1}^m \sec(\alpha_j)\omega(X_j).$$

Consequently,

COROLLARY 2.5. *If $X_1,X_2,\ldots,X_m\in\mathbb{M}_n$ are accretive-dissipative, then*

$$\omega\left(X_1\circ\ldots\circ X_m\right)\leqslant 2^{m/2}\prod_{j=1}^m\omega(X_j).$$

COROLLARY 2.6. *If $X_1,X_2,\ldots,X_m\in\mathbb{M}_n$ are accretive (or dissipative), then*

$$\omega\left(X_1\circ\ldots\circ X_m\right)\leqslant(1+a^2)^{m/2}\prod_{j=1}^m\omega(X_j),$$

where

$$a=\max\left\{|\lambda_1\left((Re\,X_j)^{-1}Im\,X_j\right)|,\ j=1,2,\ldots,m\right\}.$$

In the following result we give an estimation for $\omega(X\circ Y)$ in terms of the diagonal entries of X and Y.

THEOREM 2.5. *Let* $X=[x_{ij}]\in\mathbb{M}^s_{n,\alpha_1}$ *and* $Y=[y_{ij}]\in\mathbb{M}^s_{n,\alpha_2}$. *Then*

$$\omega(X\circ Y)\leqslant \sec(\alpha_1)\sec(\alpha_2)\min\left\{\max_j|x_{jj}|\ \omega(Y),\max_j|y_{jj}|\ \omega(X)\right\}.$$

Proof. Since $X\in\mathbb{M}^s_{n,\alpha_1}$ and $Y\in\mathbb{M}^s_{n,\alpha_2}$, the inequality before (11) implies that

$$\omega(X\circ Y)\leqslant \sec(\alpha_1)\sec(\alpha_2)\omega(Re\,(zX)\circ Re\,(wY)),$$

for some $z,w\in\mathbb{C}$ with $|z|=|w|=1$. Since $Re\,(zX)$ is positive semidefinite, (8) implies that

$$\omega(Re\,(zX)\circ Re\,(wY))\leqslant \max_j Re\,(zx_{jj})\,\omega\,(Re\,(wY))\,.$$

Now, we have

$$\begin{aligned}\omega(X\circ Y)&=\sec(\alpha_1)\sec(\alpha_2)\omega(Re\,(zX)\circ Re\,(wY))\\&\leqslant \sec(\alpha_1)\sec(\alpha_2)\max_j Re\,(zx_{jj})\,\omega\,(Re\,(wY))\\&\leqslant \sec(\alpha_1)\sec(\alpha_2)\max_j |zx_{jj}|\,\omega\,(wY)\\&= \sec(\alpha_1)\sec(\alpha_2)\max_j |x_{jj}|\,\omega\,(Y)\,.\end{aligned}$$

Hence,

$$\omega(X\circ Y)\leqslant \sec(\alpha_1)\sec(\alpha_2)\max_j |x_{jj}|\,\omega\,(Y)\,. \tag{13}$$

Similarly, we have

$$\omega(X\circ Y)\leqslant \sec(\alpha_1)\sec(\alpha_2)\max_j |y_{jj}|\,\omega\,(X)\,. \tag{14}$$

The result follows by combining (13) and (14). □

COROLLARY 2.7. *If* $X=[x_{ij}],Y=[y_{ij}]\in\mathbb{M}_n$ *are accretive-dissipative, then*

$$\omega(X\circ Y)\leqslant 2\min\{\max_j|x_{jj}|\ \omega(Y),\max_j|y_{jj}|\ \omega(X)\}.$$

COROLLARY 2.8. *If* $X,Y\in\mathbb{M}_n$ *are accretive (or dissipative), then*

$$\omega(X\circ Y)\leqslant (1+a^2)\min\left\{\max_j|x_{jj}|\ \omega(Y),\max_j|y_{jj}|\ \omega(X)\right\},$$

where

$$a=\max\{|\lambda_1\left((Re\,X)^{-1}\,Im\,X\right)|,|\lambda_1\left((Re\,Y)^{-1}\,Im\,Y\right)|\}.$$

Another upper bound for the numerical radius of a Hadamard product of two sectorial matrices can be obtained as follows.

THEOREM 2.6. *Let* $X \in \mathbb{M}^s_{n,\alpha_1}$ *and* $Y \in \mathbb{M}^s_{n,\alpha_2}$. *Then*

$$\omega(X \circ Y) \leqslant \min\{(1+\tan\alpha_1)\omega(Re\,X)\omega(Y), (1+\tan\alpha_2)\omega(X)\omega(Re\,Y)\}.$$

Consequently,

$$\omega(X \circ Y) \leqslant (1+\tan\alpha)\omega(X)\omega(Y),$$

where $\alpha = \max\{\alpha_1, \alpha_1\}$.

Proof. The second inequality follows from the the first one and fact that $\omega(Re\,X) \leqslant \omega(X)\, \forall X \in \mathbb{M}_n$. To prove the first inequality, let $X = A + iB$ be the cartesian decomposition of X. Then

$$\begin{aligned}
\omega(X \circ Y) &= \omega((A+iB)\circ Y)\\
&= \omega(A\circ Y + iB\circ Y)\\
&\leqslant \omega(A\circ Y) + \omega(B\circ Y)\\
&\leqslant \omega(A)\omega(Y) + \omega(B)\omega(Y) \quad \text{by (7)}\\
&\leqslant \omega(A)\omega(Y) + \tan(\alpha_1)\,\omega(A)\omega(Y) \quad \text{(by part 2 of Remark 2.1)}\\
&\leqslant (1+\tan\alpha_1)\omega(A)\omega(Y)\\
&= (1+\tan\alpha_1)\omega(Re\,(X))\omega(Y).
\end{aligned}$$

Hence,

$$\omega(X \circ Y) \leqslant (1+\tan\alpha_1)\omega(Re\,X)\omega(Y). \tag{15}$$

Similarly, one can show that

$$\omega(X \circ Y) \leqslant (1+\tan\alpha_2)\omega(Re\,Y)\omega(X). \tag{16}$$

The result follows by combining (15) and (16). □

COROLLARY 2.9. *Let* $X, Y \in \mathbb{M}^s_{n,\alpha}$. *Then*

$$\omega(X \circ Y) \leqslant (1+\tan\alpha)\min\{(\omega(Re\,X)\omega(Y), \omega(X)\omega(Re\,Y)\}.$$

Consequently,

$$\omega(X \circ Y) \leqslant (1+\tan\alpha)\omega(X)\omega(Y). \tag{17}$$

Finally, we remark that inequality (17) presents an improvement to (4) for $0 \leqslant \alpha < \frac{\pi}{4}$.

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Data availability. No datasets were generated or analyzed during the current study.

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