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# MONOTONICITY, CONVEXITY, AND INEQUALITIES FOR FUNCTIONS INVOLVING GAMMA FUNCTION

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*Abstract.* In this paper, we study some properties such as the monotonicity, logarithmically complete monotonicity, logarithmic convexity, and geometric convexity, of the combinations of gamma function and power function. The obtained results generalize some related known results for parameters with specific values.

## 1. Introduction

The *gamma function* defined by

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}\mathrm{d}t \quad (\mathrm{Re}\,x>0)$$

is one of the most important functions in analysis and its applications.

The *psi* (*digamma*) *function*, the logarithmic derivative of the gamma function, and the *polygamma functions* can be expressed as

$$\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}=-\gamma+\int_0^\infty\frac{e^{-t}-e^{-xt}}{1-e^{-t}}\mathrm{d}t$$

and

$$\psi^{(n)}(x)=(-1)^{n+1}n!\sum_{k=0}^{\infty}\frac{1}{(x+k)^{n+1}}$$

for $\mathrm{Re}\,x>0$, $n=1,2,\cdots$, where $\gamma=\lim\limits_{n\to\infty}\left(\sum\limits_{k=1}^{n}\frac{1}{k}-\log n\right)=0.57721\cdots$ is the Euler-Mascheroni constant.

Let $I\subset(0,\infty)$ be an interval and $f:I\to(0,\infty)$ be a continuous function. We say that $f$ is *geometrically convex (geometrically concave)* on $I$ if the following is true:

$$f(\sqrt{x_1x_2})\leqslant(\geqslant)\sqrt{f(x_1)f(x_2)}$$

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for all $x_1, x_2 \in I$, see [10, 11].

Let $I \subset \mathbb{R}$ be an interval and $f : I \to (0,\infty)$ be a continuous function. We say that $f$ is *logarithmically convex (logarithmically concave)*, log-convex (log-concave) for abbreviation, if

$$f\left(\frac{x+y}{2}\right) \leqslant (\geqslant) \sqrt{f(x)f(y)}$$

for all $x, y \in I$, see [11].

A function $f$ is called to be *logarithmically completely monotonic* ( LCM ) on an interval $I \subset \mathbb{R}$ if its logarithm $\log f$ satisfies

$$(-1)^n (\log f(x))^{(n)} \geqslant 0 \tag{1.1}$$

for all $x \in I$ and $n = 1, 2, \cdots$. Moreover, the function $f$ is said to be strictly LCM on $I$ if the inequality (1.1) is strict, see [3, 12]. Clearly, the function $f$ is decreasing and log-convex if $f$ is LCM on $I$. The analytical properties for the gamma function and related special functions have been extensively studied recently, see [8, 20, 21, 15, 16, 4, 19].

For $a > 0$, $c \in \mathbb{R}$, let $x \in (-a,\infty) \setminus \{0\}$ and

$$f_{a,c,\pm 1}(x) \equiv \left(\frac{(\Gamma(x+a))^{\frac{1}{x}}}{x^c}\right)^{\pm 1}.$$

The functions $f_{1,0,-1}$ and $f_{1,1,+1}$ are decreasing on $(0,\infty)$. In addition, it is also proved that $f_{1,1-\gamma,-1}$ is decreasing on $(1,\infty)$, see [9]. As a further study, the function $f_{1,0,-1}$ is LCM on $(-1,\infty)$ [14, Theorem 1] and $f_{1,0,+1}$ is geometrically convex on $(0,\infty)$ [5, Theorem 1]. Moreover, the conditions for $f_{1,c,\pm 1}$ to be LCM on $(0,\infty)$ are shown in [14, Theorem 3, Theorem 4]. It is showed that $f_{1,c,-1}$ $(f_{1,c,+1})$ is strictly decreasing on $(0,\infty)$ if and only if $c \leqslant 0$ $(c \geqslant 1)$ [17, Theorem 4(b)].

For $a > 0$, $c \in \mathbb{R}$, let $x \in (-a,\infty) \setminus \{0\}$ and

$$g_{a,c,\pm 1}(x) \equiv \left(\frac{(\Gamma(x+a))^{\frac{1}{x}}}{(x+a)^c}\right)^{\pm 1}.$$

In [13], Theorem 1 shows that $g_{1,1,+1}$ is strictly decreasing and strictly log-convex on $(0,\infty)$, and Theorem 2 shows that $g_{1,\frac{1}{2},-1}$ is strictly decreasing and strictly log-convex on $(0,\infty)$. As a generalization, the conditions for $g_{1,c,\pm 1}$ to be LCM on $(-1,\infty)$ are found [14, Theorem 2]. Theorem 2 in [17] displays that $g_{1,c,+1}$ $(g_{1,c,-1})$ is strictly decreasing on $(0,\infty)$ if and only if $c \geqslant 1$ $\left(c \leqslant \frac{\pi^2}{12}\right)$, and $g_{1,c,+1}$ $(g_{1,c,-1})$ is log-convex on $(0,\infty)$ if and only if $c \geqslant 1$ $(c \leqslant c_0)$, where $0.77797\cdots = \frac{75(28\zeta(3)+\pi^3)}{64} - 75 \leqslant c_0 < 18\left(3 - \gamma - \log\pi - \frac{\pi^2}{8}\right) = 0.79837\cdots$. In addition, the conditions for $g_{a,c,\pm 1}$ to be LCM on $(0,\infty)$ have been discussed [2, Theorem 1.2, Remark 2.1].

The purpose of the present paper is to further study the analytical properties of the gamma function. Specifically, motivated by [14, Theorem 1] which says $f_{1,0,-1}$

is LCM on $(-1,\infty)$, we study the monotonicity property of $f_{a,0,-1}$ for all $a\in\mathbb{R}$ and find the necessary and sufficient conditions for $f_{a,0,-1}$ to be LCM either on $(0,\infty)$ or on $(-a,\infty)$ in Theorem 1. In Theorem 2, we also show the monotonicity, logarithmic convexity and geometric convexity properties of $f_{a,c,+1}$ for certain values $(a,c)$, which is a generalization about the specific parameters in some corresponding results in [5, 14, 17]. Similarly, in Theorem 3 we investigate the monotonicity and geometric convexity properties of $g_{a,c,+1}$ for certain values $(a,c)$ and obtain a generalization of some results in [14, 17] and an improvement of the result in [2].

Before presenting the main results, we give some ranges of parameters, which are needed in describing the corollaries, as follows.

Let

$$D_1=\left\{(a,c)\Big|\frac{1}{2}\leqslant a\leqslant 1, c\in\mathbb{R}\right\},\qquad D_2=\{(a,c)|a\geqslant 2, c\in\mathbb{R}\},$$

$$D_3=\{(a,c)|1<a<2, c\geqslant 0\},\qquad D_4=\{(a,c)|1<a<2, c\leqslant 0\},$$

$$D_5=\left\{(a,c)\Big|\frac{1}{2}\leqslant a\leqslant 1, c\geqslant 1\right\},\qquad D_6=\{(a,c)|a\geqslant 2, c\geqslant 1\},$$

$$D_7=\{(a,c)|a=1, c\leqslant 0\},\qquad D_8=\{(a,c)|a=2, c\leqslant 0\},$$

$$D_9=\left\{(a,c)\Big|\frac{1}{2}\leqslant a\leqslant 1, c\leqslant 0\right\},\qquad D_{10}=\{(a,c)|a\geqslant 2, c\leqslant 0\},$$

$$D_{11}=\left\{(a,c)\Big|a=2, c\leqslant\frac{\pi^2}{6}-1\right\}.$$

Let $a\in\mathbb{R}$, we define the function

$$g_1(x)\equiv f_{a,0,-1}(x)=\frac{1}{(\Gamma(x+a))^{\frac{1}{x}}},$$

where $x\in(-a,\infty)$ for $a\leqslant 0$; $x\in(-a,\infty)\setminus\{0\}$ for $a>0$.

Since

$$g_1(0^-)=\begin{cases}\infty, & 0<a<1 \text{ or } a>2,\\ 0, & 1<a<2,\end{cases}\qquad g_1(0^+)=\begin{cases}0, & 0<a<1 \text{ or } a>2,\\ \infty, & 1<a<2,\end{cases}$$

we only define $g_1(0)$ as follows

$$g_1(0)=\begin{cases}e^{\gamma}, & a=1,\\ e^{\gamma-1}, & a=2.\end{cases}$$

THEOREM 1. *(1) The function $g_1$ is strictly increasing on $(-a,x_0)$ and strictly decreasing on $(x_0,\infty)$ if and only if $a\leqslant 0$; $g_1$ is strictly decreasing on $(-a,x_1)$ and $(x_2,\infty)$, and strictly increasing on $(x_1,0)$ and $(0,x_2)$ if and only if $0<a<1$ or $a>2$; and $g_1$ is strictly decreasing on $(-a,0)$ and $(0,\infty)$ if and only if $1\leqslant a\leqslant 2$, where $x_i$ satisfies $x_i\psi(x_i+a)=\log\Gamma(x_i+a)$, $i=0,1,2$ and $x_1<0<x_2$.*

*(2) The function $g_1$ is strictly LCM on $(0,\infty)$ if and only if $1\leqslant a\leqslant 2$; and $g_1$ is strictly LCM on $(-a,\infty)$ if and only if $a=1$ or $a=2$.*

REMARK 1. The sufficient condition of the LCM property for $g_1$ on $(0,\infty)$ in Theorem 1 (2) can also be obtained by taking $c=0$ in [2, Remark 2.1].

The following inequalities (1.2) and (1.3) can be easily derived from the monotonicity and logarithmic convexity properties of $g_1$ in Theorem 1 (2).

COROLLARY 1. *(1) For $0<x<y$, the inequality*

$$\frac{(\Gamma(x+a))^{\frac{1}{x}}}{(\Gamma(y+a))^{\frac{1}{y}}}<1 \tag{1.2}$$

*holds for $1\leqslant a\leqslant 2$.*

*(2) For $x,y>0$, the inequality*

$$\frac{\left(\Gamma\left(\frac{x+y}{2}+a\right)\right)^{\frac{2}{x+y}}}{\sqrt{(\Gamma(x+a))^{\frac{1}{x}}(\Gamma(y+a))^{\frac{1}{y}}}}\geqslant 1 \tag{1.3}$$

*holds for $1\leqslant a\leqslant 2$. The equality is true if and only if $x=y$.*

THEOREM 2. *For $a>0$, $c\in\mathbb{R}$, let $x\in(0,\infty)$ and*

$$g_2(x)\equiv f_{a,c,+1}(x)=\frac{(\Gamma(x+a))^{\frac{1}{x}}}{x^c}.$$

*(1) The function $g_2$ is strictly decreasing on $(0,\infty)$ if and only if $c\geqslant 1$ for $\frac{1}{2}\leqslant a\leqslant 1$ or $a\geqslant 2$; $g_2$ is strictly increasing on $(0,\infty)$ if and only if $c\leqslant 0$ for $a=1$ or $a=2$; and $g_2$ is strictly increasing on $(0,\infty)$ if and only if $c\leqslant h_2(x_3)$ for $1<a<2$, where $x_3$ satisfies $x_3^2\psi'(x_3+a)+\log\Gamma(x_3+a)=x_3\psi(x_3+a)$ and $h_2(x_3)\equiv x_3\psi'(x_3+a)$.*

*(2) The function $g_2$ is strictly log-convex on $(0,\infty)$ if and only if $c\geqslant 1$ for $a\geqslant 2$; and $g_2$ is strictly log-concave on $(0,\infty)$ if and only if $c\leqslant 0$ for $a=2$.*

*(3) The function $g_2$ is geometrically convex on $(0,\infty)$ if and only if $(a,c)\in D_1\cup D_2$; and $g_2$ is geometrically concave on $(0,x_3)$ and geometrically convex on $(x_3,\infty)$ if and only if $(a,c)\in D_3\cup D_4$.*

REMARK 2. It is clear that Theorem 2 (3) is a generalization of [5, Theorem 1] which says that $f_{1,0,+1}$ is geometrically convex on $(0,\infty)$.

Theorem 2 leads to the following corollary.

COROLLARY 2. *(1) For $0<x<y$, the inequality*

$$\frac{(\Gamma(x+a))^{\frac{1}{x}}}{(\Gamma(y+a))^{\frac{1}{y}}}>\left(\frac{x}{y}\right)^c \tag{1.4}$$

*holds for* $(a,c) \in D_5 \cup D_6$*; and inequality* (1.4) *is reversed for* $(a,c) \in D_7 \cup D_8$.

*(2) For* $x, y > 0$*, the inequality*

$$\frac{\left(\Gamma\left(\frac{x+y}{2}+a\right)\right)^{\frac{2}{x+y}}}{\sqrt{(\Gamma(x+a))^{\frac{1}{x}}\,(\Gamma(y+a))^{\frac{1}{y}}}} \leqslant \left(\frac{x+y}{2\sqrt{xy}}\right)^{c} \tag{1.5}$$

*holds for* $(a,c) \in D_6$*; and inequality* (1.5) *is reversed for* $(a,c) \in D_8$*. The equalities are true if and only if* $x = y$.

*(3) For* $x, y > 0$*, the inequalities*

$$\left(\frac{x}{y}\right)^{\frac{y\psi(y+a)-\log\Gamma(y+a)}{y}} \leqslant \frac{(\Gamma(x+a))^{\frac{1}{x}}}{(\Gamma(y+a))^{\frac{1}{y}}} \leqslant \left(\frac{x}{y}\right)^{\frac{x\psi(x+a)-\log\Gamma(x+a)}{x}} \tag{1.6}$$

*hold for* $(a,c) \in D_1 \cup D_2$*. The equalities are true if and only if* $x = y$.

THEOREM 3. *For* $a > 0$*,* $c \in \mathbb{R}$*, let* $x \in (0,\infty)$ *and*

$$g_3(x) \equiv g_{a,c,+1}(x) = \frac{(\Gamma(x+a))^{\frac{1}{x}}}{(x+a)^c}.$$

*(1) The function* $g_3$ *is strictly decreasing on* $(0,\infty)$ *if and only if* $c \geqslant 1$ *for* $a \geqslant 2$*;* $g_3$ *is strictly increasing on* $(0,\infty)$ *if and only if* $c \leqslant \dfrac{\pi^2}{6} - 1$ *for* $a = 2$*; and* $g_3$ *is strictly increasing on* $(0,\infty)$ *if and only if* $c \leqslant h_4(x_4)$ *for* $\dfrac{3+\sqrt{159}}{12} \leqslant a < 2$*, where* $x_4$ *satisfies* $x_4^2(x_4+a)\psi'(x_4+a) + (x_4+2a)\log\Gamma(x_4+a) = x_4(x_4+2a)\psi(x_4+a)$ *and* $h_4(x_4) \equiv \dfrac{x_4(x_4+a)\psi(x_4+a) - (x_4+a)\log\Gamma(x_4+a)}{x_4^2}$.

*(2) The function* $g_3$ *is geometrically convex on* $(0,\infty)$ *for* $(a,c) \in D_9 \cup D_{10}$*; and* $g_3$ *is geometrically concave on* $(0,x_3)$ *for* $(a,c) \in D_3$ *and geometrically convex on* $(x_3,\infty)$ *for* $(a,c) \in D_4$*, where* $x_3$ *is the same as in Theorem 2 (1).*

REMARK 3. The sufficient condition of the decreasing property for $g_3$ in Theorem 3 (1) can also be obtained by [2, Theorem 1.2].

The following corollary can be directly derived by Theorem 3.

COROLLARY 3. *(1) For* $0 < x < y$*, the inequality*

$$\frac{(\Gamma(x+a))^{\frac{1}{x}}}{(\Gamma(y+a))^{\frac{1}{y}}} > \left(\frac{x+a}{y+a}\right)^{c} \tag{1.7}$$

*holds for* $(a,c) \in D_6$*; and inequality* (1.7) *is reversed for* $(a,c) \in D_{11}$.

*(2) For $x,y>0$, the inequalities*

$$\left(\frac{x}{y}\right)^{\frac{y\psi(y+a)-\log\Gamma(y+a)}{y}-\frac{cy}{y+a}}\left(\frac{x+a}{y+a}\right)^{c}\leqslant\frac{(\Gamma(x+a))^{\frac{1}{x}}}{(\Gamma(y+a))^{\frac{1}{y}}}\leqslant\left(\frac{x}{y}\right)^{\frac{x\psi(x+a)-\log\Gamma(x+a)}{x}-\frac{cx}{x+a}}\left(\frac{x+a}{y+a}\right)^{c} \tag{1.8}$$

*hold for $(a,c)\in D_9\cup D_{10}$. The equalities are true if and only if $x=y$.*

## 2. Lemmas

In this section, we show some lemmas which are needed in the proofs of the main results. The following formulas will be frequently used in the proofs of lemmas [1, 18].

Leibniz's Theorem for differentiation of the product of two functions:

$$(u(x)v(x))^{(n)}=\sum_{k=0}^{n}\binom{n}{k}u^{(k)}(x)v^{(n-k)}(x). \tag{2.1}$$

Recurrence formulas of $\Gamma,\psi$:

$$\Gamma(x+1)=x\Gamma(x),\quad \psi(x+1)=\psi(x)+\frac{1}{x}.$$

Special values of $\Gamma,\psi,\psi'$:

$$\Gamma(1)=1,\quad \psi(1)=-\gamma,\quad \psi'(2)=\frac{\pi^2}{6}-1.$$

Asymptotic formulas of $\log\Gamma$, $\psi$, $\psi'$, $\psi''$: for $x\to\infty$ with $|\arg x|<\pi$,

$$\log\Gamma(x)\sim\left(x-\frac{1}{2}\right)\log x-x+\frac{1}{2}\log(2\pi)+\frac{1}{12x}-\frac{1}{360x^3}+\cdots,$$
$$\psi(x)\sim\log x-\frac{1}{2x}-\frac{1}{12x^2}+\frac{1}{120x^4}-\cdots,$$
$$\psi'(x)\sim\frac{1}{x}+\frac{1}{2x^2}+\frac{1}{6x^3}-\frac{1}{30x^5}+\cdots,$$
$$\psi''(x)\sim-\frac{1}{x^2}-\frac{1}{x^3}-\frac{1}{2x^4}+\cdots.$$

For $x\in(0,\infty)$, the following inequalities of the polygamma functions hold [7, Theorem 3]:

$$\frac{(n-1)!}{x^n}+\frac{n!}{2x^{n+1}}\leqslant(-1)^{n+1}\psi^{(n)}(x)\leqslant\frac{(n-1)!}{x^n}+\frac{n!}{x^{n+1}},\quad n=1,2,\cdots. \tag{2.2}$$

Moreover, there holds the identity for $\psi''$ [6]:

$$\psi''(x)=-\frac{1}{x^2}-\frac{1}{x^3}-\frac{1}{2x^4}+\frac{\theta}{6x^6},\quad 0<\theta<1. \tag{2.3}$$

LEMMA 1. [11, Proposition 4.3] [5, Theorem C] *Let* $I \subset (0,\infty)$ *be an interval. If* $f : I \to (0,\infty)$ *is a differentiable function, then the following assertions are equivalent:*
*(1) The function* $f$ *is geometrically convex (geometrically concave) on* $I$*;*
*(2) The function* $g(x) \equiv \dfrac{xf'(x)}{f(x)}$ *is increasing (decreasing) on* $I$*;*
*(3) The function* $f$ *satisfies the inequalities*

$$\left(\frac{x}{y}\right)^{\frac{yf'(y)}{f(y)}} \leqslant (\geqslant) \frac{f(x)}{f(y)} \leqslant (\geqslant) \left(\frac{x}{y}\right)^{\frac{xf'(x)}{f(x)}}, \quad \forall x,y \in I.$$

LEMMA 2. *For* $n = 1,2,\cdots$*, there hold*

$$(\log g_1(0))^{(n)} = \begin{cases} -\dfrac{\psi^{(n)}(1)}{n+1}, & a=1, \\ -\dfrac{\psi^{(n)}(2)}{n+1}, & a=2. \end{cases} \tag{2.4}$$

*Proof.* We first consider the case for $a=1$.
When $n=1$,

$$\begin{aligned}(\log g_1(0))' &= \lim_{x\to 0} \frac{\log g_1(x) - \log g_1(0)}{x-0} \\ &= \lim_{x\to 0} \frac{-\log\Gamma(x+1) - x\gamma}{x^2} = -\frac{\psi'(1)}{2}.\end{aligned}$$

We assume that (2.4) holds when $n=k$ $(k \in \mathbb{Z}, k > 1)$.
Then by L'Hopital Rule, we get

$$\begin{aligned}(\log g_1(0))^{(k+1)} &= \lim_{x\to 0} \frac{(\log g_1(x))^{(k)} - (\log g_1(0))^{(k)}}{x-0} \\ &= \lim_{x\to 0} \frac{(-1)^k k!\delta_k(x) + \frac{\psi^{(k)}(1)}{k+1}x^{k+1}}{x^{k+2}} \\ &= \lim_{x\to 0} \frac{-\psi^{(k+1)}(x+1)}{k+2} = -\frac{\psi^{(k+1)}(1)}{k+2},\end{aligned}$$

where

$$\delta_k(x) \equiv -\log\Gamma(x+a) - \sum_{j=1}^{k} \frac{(-1)^j x^j}{j!} \psi^{(j-1)}(x+a).$$

By induction, (2.4) holds for $n=1,2,\cdots$ when $a=1$.
In a similar way, we can prove that (2.4) holds for $n=1,2,\cdots$ when $a=2$.
The proof is complete. □

LEMMA 3. *For* $a \in \mathbb{R}$*, let* $x \in (-a,\infty)$ *and*

$$h_1(x) \equiv -x\psi(x+a) + \log\Gamma(x+a).$$

*(1) The function* $h_1$ *is strictly decreasing from* $(-a,\infty)$ *onto* $(-\infty,\infty)$ *if and only if* $a \leqslant 0$.

*(2) The function* $h_1$ *is strictly increasing on* $(-a,0]$ *and strictly decreasing on* $[0,\infty)$ *with* $h_1(x) \in (-\infty, \log\Gamma(a)]$ *if and only if* $a > 0$. *Moreover,* $h_1(x) < 0$ *on* $(-a,x_1)\cup(x_2,\infty)$, $h_1(x) > 0$ *on* $(x_1,x_2)$ *for* $0 < a < 1$ *or* $a > 2$*; and* $h_1(x) < 0$ *on* $(-a,0)\cup(0,\infty)$ *for* $1 \leqslant a \leqslant 2$, *where* $x_1, x_2$ *are the same as in Theorem* 1 (1).

*Proof.* Let $t = x+a$. Then

$$h_1(x) = h_1(t-a) \equiv \widetilde{h}_1(t) = -(t-a)\psi(t) + \log\Gamma(t), \quad t \in (0,\infty).$$

It suffices to study the monotonicity property and the range of $\widetilde{h}_1$.

We first prove the monotonicity property of $\widetilde{h}_1$.

Differentiation yields

$$\widetilde{h}_1'(t) = -(t-a)\psi'(t).$$

Therefore $\widetilde{h}_1$ is strictly decreasing on $(0,\infty)$ if and only if $a \leqslant 0$; and $\widetilde{h}_1$ is strictly increasing on $(0,a]$ and strictly decreasing on $[a,\infty)$ if and only if $a > 0$.

Then we calculate the range of $\widetilde{h}_1$.

By the asymptotic formulas of $\log\Gamma$ and $\psi$, we get

$$\begin{aligned}\lim_{t\to\infty}\widetilde{h}_1(t) &= \lim_{t\to\infty}\left(-(t-a)\left(\log t - \frac{1}{2t} + O\left(\frac{1}{t^2}\right)\right) + \left(t-\frac{1}{2}\right)\log t \right.\\ &\qquad\left. -t + \frac{1}{2}\log(2\pi) + O\left(\frac{1}{t}\right)\right)\\ &= \lim_{t\to\infty}\left(t\left(\left(a-\frac{1}{2}\right)\frac{\log t}{t} - 1\right) + \frac{t-a}{2t} + \frac{1}{2}\log(2\pi) + O\left(\frac{1}{t}\right)\right)\\ &= -\infty, \quad a \in \mathbb{R}.\end{aligned}$$

By the recurrence formulas of $\Gamma$ and $\psi$, we get

$$\lim_{t\to 0^+}\widetilde{h}_1(t) = \lim_{t\to 0^+}\left(-(t-a)\psi(t+1) + \log\Gamma(t) + \frac{t-a}{t}\right) = \infty, \quad a \leqslant 0$$

and

$$\lim_{t\to 0^+}\widetilde{h}_1(t) = \lim_{t\to 0^+}\frac{1}{t}\left(t(-(t-a)\psi(t+1) + \log\Gamma(t+1)) - t\log t + t - a\right) = -\infty, \quad a > 0.$$

The limiting value $\lim\limits_{t\to a}\widetilde{h}_1(t) = \log\Gamma(a)$ is clear for $a > 0$.

Therefore $\widetilde{h}_1(t) \in (-\infty,\infty)$ for $a \leqslant 0$; and $\widetilde{h}_1(t) \in (-\infty, \log\Gamma(a)]$ for $a > 0$. Moreover, for $0 < a < 1$ or $a > 2$, there exist $t_1, t_2$ such that $\widetilde{h}_1(t) < 0$ on $(0,t_1)\cup(t_2,\infty)$ and $\widetilde{h}_1(t) > 0$ on $(t_1,t_2)$; and for $1 \leqslant a \leqslant 2$, $\widetilde{h}_1(t) < 0$ on $(0,a)\cup(a,\infty)$, where $t_i$ satisfies $(t_i - a)\psi(t_i) = \log\Gamma(t_i), i = 1,2$ and $t_1 < a < t_2$.

The proof is complete. $\square$

LEMMA 4. *For $a>0$, let $x\in(0,\infty)$ and*

$$h_2(x)\equiv\frac{x\psi(x+a)-\log\Gamma(x+a)}{x}.$$

*(1) The function $h_2$ is strictly increasing on $(0,\infty)$ if and only if $\frac{1}{2}\leqslant a\leqslant 1$ or $a\geqslant 2$. Moreover, $h_2(x)\in(-\infty,1)$ for $\frac{1}{2}\leqslant a<1$ or $a>2$; and $h_2(x)\in(0,1)$ for $a=1$ or $a=2$.*

*(2) The function $h_2$ is strictly decreasing on $(0,x_3]$ and strictly increasing on $[x_3,\infty)$ with $h_2(x)\in[h_2(x_3),\infty)$ if and only if $1<a<2$, where $x_3$ is the same as in Theorem 2 (1).*

*Proof.* Let $t=x+a$. Then

$$h_2(x)=h_2(t-a)\equiv\widetilde{h}_2(t)=-\frac{\widetilde{h}_1(t)}{t-a},\quad t\in(a,\infty),$$

where $\widetilde{h}_1(t)$ is the same as in the proof of Lemma 3. It suffices to study the monotonicity property and the range of $\widetilde{h}_2$.

We first prove the monotonicity property of $\widetilde{h}_2$.

Differentiation gives

$$\widetilde{h}_2'(t)\equiv\frac{h_{21}(t)}{(t-a)^2},$$

where

$$h_{21}(t)\equiv(t-a)^2\psi'(t)-(t-a)\psi(t)+\log\Gamma(t).$$

It is easy to obtain

$$h_{21}'(t)=(t-a)((t-a)\psi''(t)+\psi'(t)).$$

By the inequality (2.2) of $\psi'$ and the identity (2.3) of $\psi''$, we get

$$\begin{aligned}(t-a)\psi''(t)+\psi'(t)&>(t-a)\left(-\frac{1}{t^2}-\frac{1}{t^3}-\frac{1}{2t^4}\right)+\frac{1}{t}+\frac{1}{2t^2}\\&=\frac{1}{2t^4}\left((2a-1)t^2+(2a-1)t+a\right).\end{aligned}$$

Since $(2a-1)t^2+(2a-1)t+a>0$ on $(a,\infty)$ if and only if $a\geqslant\frac{1}{2}$, we have that $h_{21}$ is strictly increasing on $(a,\infty)$ for $a\geqslant\frac{1}{2}$ and hence

$$h_{21}(t)>\lim_{t\to a^+}h_{21}(t)=\log\Gamma(a).$$

Thus $h_{21}(t)>0$ on $(a,\infty)$ for $\frac{1}{2}\leqslant a\leqslant 1$ or $a\geqslant 2$.

For $0 < a < \frac{1}{2}$, the limiting value $\lim_{t\to a^+} h_{21}(t) = \log\Gamma(a) > 0$ is clear. By the asymptotic formulas of $\log\Gamma$, $\psi$, and $\psi'$, we get

$$\begin{aligned}
&\lim_{t\to\infty} h_{21}(t)\\
&= \lim_{t\to\infty}\Bigg((t-a)^2\left(\frac{1}{t}+\frac{1}{2t^2}+O\left(\frac{1}{t^3}\right)\right)-(t-a)\left(\log t-\frac{1}{2t}+O\left(\frac{1}{t^2}\right)\right)\\
&\qquad +\left(t-\frac{1}{2}\right)\log t - t + \frac{1}{2}\log(2\pi)+O\left(\frac{1}{t}\right)\Bigg)\\
&= \lim_{t\to\infty}\left(\frac{-2at+a^2}{t}+\left(a-\frac{1}{2}\right)\log t+\frac{(t-a)^2}{2t^2}+\frac{t-a}{2t}+\frac{1}{2}\log(2\pi)+O\left(\frac{1}{t}\right)\right)\\
&= -\infty.
\end{aligned}$$

For $1 < a < 2$, we have $\lim_{t\to a^+} h_{21}(t) = \log\Gamma(a) < 0$ and $\lim_{t\to\infty} h_{21}(t) = \infty$.

Therefore $h_{21}(t) > 0$ on $(a,\infty)$ and hence $\widetilde{h}_2$ is strictly increasing on $(a,\infty)$ if and only if $\frac{1}{2} \leqslant a \leqslant 1$ or $a \geqslant 2$.

Since $h_{21}$ is strictly increasing on $(a,\infty)$ for $1 < a < 2$, there exists $t_3 \in (a,\infty)$ such that $h_{21}(t) < 0$ on $(a,t_3)$ and $h_{21}(t) > 0$ on $(t_3,\infty)$, where $t_3$ satisfies $(t_3-a)^2\psi'(t_3)+\log\Gamma(t_3) = (t_3-a)\psi(t_3)$.

Therefore $\widetilde{h}_2$ is strictly decreasing on $(a,t_3]$ and strictly increasing on $[t_3,\infty)$ if and only if $1 < a < 2$.

Then we calculate the range of $\widetilde{h}_2$.

By the proof of Lemma 3, we have $\lim_{t\to\infty} \widetilde{h}_1(t) = -\infty$. Using L'Hopital Rule and the asymptotic formula of $\psi'$, we obtain

$$\lim_{t\to\infty} \widetilde{h}_2(t) = \lim_{t\to\infty}(t-a)\left(\frac{1}{t}+O\left(\frac{1}{t^2}\right)\right) = 1, \quad a > 0.$$

Calculation yields the limiting value

$$\lim_{t\to a^+} \widetilde{h}_2(t) = \begin{cases} -\infty, & \frac{1}{2} \leqslant a < 1 \text{ or } a > 2,\\ 0, & a = 1 \text{ or } a = 2,\\ \infty, & 1 < a < 2.\end{cases}$$

Therefore $\widetilde{h}_2(t) \in (-\infty,1)$ for $\frac{1}{2} \leqslant a < 1$ or $a > 2$; $\widetilde{h}_2(t) \in (0,1)$ for $a = 1$ or $a = 2$; and $\widetilde{h}_2(t) \in [\widetilde{h}_2(t_3),\infty)$ for $1 < a < 2$.

The proof is complete. □

OPEN PROBLEM 1. *What is the monotonicity property of $h_2$ on $(0,\infty)$ for $0 < a < \frac{1}{2}$?*

LEMMA 5. *For $a > 0$, let $x \in (0,\infty)$ and*

$$h_3(x) \equiv \frac{-x^2\psi'(x+a)+2x\psi(x+a)-2\log\Gamma(x+a)}{x}.$$

*Then the function $h_3$ is strictly increasing on $(0,\infty)$ for $a \geqslant 2$. Moreover, $h_3(x) \in (-\infty,1)$ for $a > 2$; and $h_3(x) \in (0,1)$ for $a = 2$.*

*Proof.* Let $t = x+a$. Then

$$\begin{aligned} h_3(x) &= h_3(t-a) \equiv \widetilde{h}_3(t) \\ &= \frac{-(t-a)^2\psi'(t)+2(t-a)\psi(t)-2\log\Gamma(t)}{t-a}, \quad t \in (a,\infty). \end{aligned}$$

It suffices to study the monotonicity property and the range of $\widetilde{h}_3$.

We first prove the monotonicity property of $\widetilde{h}_3$.

Differentiation gives

$$\widetilde{h}_3'(t) \equiv \frac{h_{31}(t)}{(t-a)^2},$$

where

$$h_{31}(t) \equiv -(t-a)^3\psi''(t)+(t-a)^2\psi'(t)-2(t-a)\psi(t)+2\log\Gamma(t).$$

It is easy to obtain

$$h_{31}'(t) = -(t-a)^2\left((t-a)\psi'''(t)+2\psi''(t)\right).$$

By the inequalities (2.2) of $\psi''$ and $\psi'''$, we have

$$\begin{aligned} (t-a)\psi'''(t)+2\psi''(t) &\leqslant (t-a)\left(\frac{2}{t^3}+\frac{6}{t^4}\right)+2\left(-\frac{1}{t^2}-\frac{1}{t^3}\right) \\ &= \frac{1}{t^4}\left(2(2-a)t-6a\right). \end{aligned}$$

Since $2(2-a)t-6a<0$ on $(a,\infty)$ if and only if $a \geqslant 2$, we have that $h_{31}$ is strictly increasing on $(a,\infty)$ for $a \geqslant 2$ and hence

$$h_{31}(t) > \lim_{t\to a^+} h_{31}(t) = 2\log\Gamma(a).$$

Therefore $h_{31}(t) > 0$ on $(a,\infty)$ and hence $\widetilde{h}_3$ is strictly increasing on $(a,\infty)$ for $a \geqslant 2$.

Then we calculate the range of $\widetilde{h}_3$.

By the asymptotic formulas of $\log\Gamma$, $\psi$, and $\psi'$, we get

$$\begin{aligned}
&\lim_{t\to\infty}\left(-(t-a)^2\psi'(t)+2(t-a)\psi(t)-2\log\Gamma(t)\right)\\
=&\lim_{t\to\infty}\left(-(t-a)^2\left(\frac{1}{t}+\frac{1}{2t^2}+O\left(\frac{1}{t^3}\right)\right)+2(t-a)\left(\log t-\frac{1}{2t}+O\left(\frac{1}{t^2}\right)\right)\right.\\
&\qquad\left.-2\left(\left(t-\frac{1}{2}\right)\log t-t+\frac{1}{2}\log(2\pi)+O\left(\frac{1}{t}\right)\right)\right)\\
=&\lim_{t\to\infty}\left(t\left((1-2a)\frac{\log t}{t}-\frac{(t-a)^2}{t^2}+2\right)-\frac{(t-a)^2}{2t^2}-\frac{t-a}{t}-\log(2\pi)+O\left(\frac{1}{t}\right)\right)\\
=&\infty.
\end{aligned}$$

L'Hopital Rule and the asymptotic formula of $\psi''$ yield

$$\lim_{t\to\infty}\widetilde{h}_3(t)=\lim_{t\to\infty}\left(-(t-a)^2\left(-\frac{1}{t^2}+O\left(\frac{1}{t^3}\right)\right)\right)=1.$$

It is easy to obtain

$$\lim_{t\to a^+}\widetilde{h}_3(t)=\begin{cases}-\infty, & a>2,\\ 0, & a=2.\end{cases}$$

Therefore $\widetilde{h}_3(t)\in(-\infty,1)$ for $a>2$; and $\widetilde{h}_3(t)\in(0,1)$ for $a=2$. $\square$

OPEN PROBLEM 2. *What is the monotonicity property of $h_3$ on $(0,\infty)$ for $0<a<2$?*

LEMMA 6. *For $a>0$, let $x\in(0,\infty)$ and*

$$h_4(x)\equiv\frac{x(x+a)\psi(x+a)-(x+a)\log\Gamma(x+a)}{x^2}.$$

*(1) The function $h_4$ is strictly increasing on $(0,\infty)$ for $a\geqslant 2$. Moreover, $h_4(x)\in(-\infty,1)$ for $a>2$; and $h_4(x)\in\left(\frac{\pi^2}{6}-1,1\right)$ for $a=2$.*

*(2) The function $h_4$ is strictly decreasing on $(0,x_4]$ and strictly increasing on $[x_4,\infty)$ with $h_4(x)\in[h_4(x_4),\infty)$ for $\frac{3+\sqrt{159}}{12}\leqslant a<2$, where $x_4$ is the same as in Theorem 3 (1).*

*Proof.* Let $t=x+a$. Then

$$h_4(x)=h_4(t-a)\equiv\widetilde{h}_4(t)=\frac{t(t-a)\psi(t)-t\log\Gamma(t)}{(t-a)^2},\quad t\in(a,\infty).$$

It suffices to study the monotonicity property and the range of $\widetilde{h}_4$.

We first prove the monotonicity property of $\widetilde{h}_4$.

Differentiation gives

$$\widetilde{h}_4'(t) \equiv \frac{h_{41}(t)}{(t-a)^3},$$

where

$$h_{41}(t) \equiv t(t-a)^2\psi'(t) - (t^2-a^2)\psi(t) + (t+a)\log\Gamma(t).$$

It is easy to obtain

$$h_{41}'(t) = t(t-a)^2\psi''(t) + 2(t-a)^2\psi'(t) - (t-a)\psi(t) + \log\Gamma(t)$$

and

$$h_{41}''(t) = (t-a)\left(t(t-a)\psi'''(t) + (5t-3a)\psi''(t) + 3\psi'(t)\right).$$

By the inequalities (2.2) of $\psi'$ and $\psi'''$ and the identity (2.3) of $\psi''$, we get

$$\begin{aligned}
& t(t-a)\psi'''(t) + (5t-3a)\psi''(t) + 3\psi'(t) \\
> & \, t(t-a)\left(\frac{2}{t^3} + \frac{3}{t^4}\right) + (5t-3a)\left(-\frac{1}{t^2} - \frac{1}{t^3} - \frac{1}{2t^4}\right) + 3\left(\frac{1}{t} + \frac{1}{2t^2}\right) \\
= & \, \frac{1}{2t^4}\left((2a-1)t^2 - 5t + 3a\right).
\end{aligned}$$

Since $(2a-1)t^2 - 5t + 3a \geqslant 0$ on $(a,\infty)$ if and only if $a \geqslant \dfrac{3+\sqrt{159}}{12} \approx 1.3$, we have that $h_{41}'$ is strictly increasing on $(a,\infty)$ for $a \geqslant \dfrac{3+\sqrt{159}}{12}$ and hence

$$h_{41}'(t) > \lim_{t\to a^+} h_{41}'(t) = \log\Gamma(a).$$

Moreover, $h_{41}'(t) > 0$ and hence $h_{41}$ is strictly increasing on $(a,\infty)$ for $a \geqslant 2$.

Then for $a \geqslant 2$, we have

$$h_{41}(t) > \lim_{t\to a^+} h_{41}(t) = 2a\log\Gamma(a).$$

Therefore $h_{41}(t) > 0$ and hence $\widetilde{h}_4$ is strictly increasing on $(a,\infty)$ for $a \geqslant 2$.

We consider the case for $\dfrac{3+\sqrt{159}}{12} \leqslant a < 2$ in the following.

The limiting value $\lim_{t\to a^+} h_{41}'(t) = \log\Gamma(a) < 0$ is clear.

By the asymptotic formulas of $\log\Gamma$, $\psi$, $\psi'$, and $\psi''$, we get

$$\begin{aligned}
&\lim_{t\to\infty} h_{41}'(t)\\
&=\lim_{t\to\infty}\left(t(t-a)^2\left(-\frac{1}{t^2}-\frac{1}{t^3}+O\left(\frac{1}{t^4}\right)\right)+2(t-a)^2\left(\frac{1}{t}+\frac{1}{2t^2}+O\left(\frac{1}{t^3}\right)\right)\right.\\
&\qquad\left.-(t-a)\left(\log t-\frac{1}{2t}+O\left(\frac{1}{t^2}\right)\right)+\left(t-\frac{1}{2}\right)\log t-t+\frac{1}{2}\log(2\pi)+O\left(\frac{1}{t}\right)\right)\\
&=\lim_{t\to\infty}\left(\frac{-2at+a^2}{t}+\left(a-\frac{1}{2}\right)\log t+\frac{t-a}{2t}+\frac{1}{2}\log(2\pi)+O\left(\frac{1}{t}\right)\right)\\
&=\infty.
\end{aligned}$$

Since $h_{41}'$ is strictly increasing on $(a,\infty)$ for $a\geqslant\dfrac{3+\sqrt{159}}{12}$, there exists $\tilde t_4\in(a,\infty)$ such that $h_{41}'(t)<0$ on $(a,\tilde t_4)$ and $h_{41}'(t)>0$ on $(\tilde t_4,\infty)$, where $\tilde t_4$ satisfies $\tilde t_4(\tilde t_4-a)^2\psi''(\tilde t_4)+2(\tilde t_4-a)^2\psi'(\tilde t_4)+\log\Gamma(\tilde t_4)=(\tilde t_4-a)\psi(\tilde t_4)$.

Hence $h_{41}$ is strictly decreasing on $(a,\tilde t_4]$ and strictly increasing on $[\tilde t_4,\infty)$ for $\dfrac{3+\sqrt{159}}{12}\leqslant a<2$.

The limit values $\lim_{t\to a^+} h_{41}(t)=2a\log\Gamma(a)<0$ is clear.

By the asymptotic formulas of $\log\Gamma$, $\psi$, and $\psi'$, we get

$$\begin{aligned}
&\lim_{t\to\infty} h_{41}(t)\\
&=\lim_{t\to\infty}\left(t(t-a)^2\left(\frac{1}{t}+\frac{1}{2t^2}+\frac{1}{6t^3}+O\left(\frac{1}{t^5}\right)\right)-(t^2-a^2)\left(\log t-\frac{1}{2t}-\frac{1}{12t^2}\right.\right.\\
&\qquad\left.\left.+O\left(\frac{1}{t^4}\right)\right)+(t+a)\left(\left(t-\frac{1}{2}\right)\log t-t+\frac{1}{2}\log 2\pi+\frac{1}{12t}+O\left(\frac{1}{t^3}\right)\right)\right)\\
&=\lim_{t\to\infty}\left(t\left(\left(a-\frac{1}{2}\right)\log t+\left(a^2-\frac{a}{2}\right)\frac{\log t}{t}+1+\frac{1}{2}\log(2\pi)-3a\right)\right.\\
&\qquad\left.+a\left(a+\frac{1}{2}\log(2\pi)-1\right)+\frac{4t^2-3at+a^2}{12t^2}+O\left(\frac{1}{t^2}\right)\right)\\
&=\infty.
\end{aligned}$$

By the monotonicity property of $h_{41}$, there exists $t_4\,(>\tilde t_4)$ such that $h_{41}(t)<0$ on $(a,t_4)$ and $h_{41}(t)>0$ on $(t_4,\infty)$, where $t_4$ satisfies $t_4(t_4-a)^2\psi'(t_4)+(t_4+a)\log\Gamma(t_4)=(t_4^2-a^2)\psi(t_4)$.

Therefore $\widetilde{h}_4$ is strictly decreasing on $(a,t_4]$ and strictly increasing on $[t_4,\infty)$ for $\dfrac{3+\sqrt{159}}{12}\leqslant a<2$.

Then we calculate the range of $\widetilde{h}_4$.

The following limiting values

$$\lim_{t\to\infty}\widetilde{h}_4(t)=\lim_{t\to\infty}\frac{t}{t-a}\widetilde{h}_2(t)=1,\quad a>0,$$

and

$$\lim_{t\to a^+}\widetilde{h}_4(t)=\begin{cases}\infty, & \dfrac{3+\sqrt{159}}{12}\leqslant a<2,\\ -\infty, & a>2\end{cases}$$

are clear.

For $a=2$, by L'Hopital Rule, we get

$$\lim_{t\to 2^+}\widetilde{h}_4(t)=\psi'(2)=\frac{\pi^2}{6}-1.$$

Therefore $\widetilde{h}_4(t)\in(-\infty,1)$ for $a>2$; $\widetilde{h}_4(t)\in\left(\frac{\pi^2}{6}-1,1\right)$ for $a=2$; and $\widetilde{h}_4(t)\in[\widetilde{h}_4(t_4),\infty)$ for $\frac{3+\sqrt{159}}{12}\leqslant a<2$.

The proof is complete. □

OPEN PROBLEM 3. *What is the monotonicity property of* $h_4$ *on* $(0,\infty)$ *for* $0<a<\frac{3+\sqrt{159}}{12}$*?*

### 3. Proofs of main results

*Proof of Theorem* 1. (1) Logarithmic differentiation gives

$$\frac{g_1'(x)}{g_1(x)}\equiv\frac{h_1(x)}{x^2},$$

where $h_1(x)$ is the same as in Lemma 3.

By Lemma 3, we have that there exists $x_0\in(-a,\infty)$ such that $g_1$ is strictly increasing on $(-a,x_0)$ and strictly decreasing on $(x_0,\infty)$ if and only if $a\leqslant 0$, where $x_0$ satisfies $x_0\psi(x_0+a)=\log\Gamma(x_0+a)$; $g_1$ is strictly decreasing on $(-a,x_1)$, $(x_2,\infty)$, and strictly increasing on $(x_1,0)$, $(0,x_2)$ if and only if $0<a<1$ or $a>2$, where $x_i$ satisfies $x_i\psi(x_i+a)=\log\Gamma(x_i+a)$, $i=1,2$; and $g_1$ is strictly decreasing on $(-a,0)$ and $(0,\infty)$ if and only if $1\leqslant a\leqslant 2$.

(2) By (1), we have that $g_1$ is not LCM on $(-a,\infty)$ or $(0,\infty)$ for $a\leqslant 0$, $0<a<1$ or $a>2$. Therefore we only need to consider the LCM property of $g_1$ for $1\leqslant a\leqslant 2$.

For $x\in(-a,\infty)\setminus\{0\}$, by the formula (2.1), we get

$$\begin{aligned}(-1)^n(\log g_1(x))^{(n)}&=(-1)^{n+1}\left(\frac{(-1)^n n!}{x^{n+1}}\log\Gamma(x+a)+\sum_{k=1}^{n}\frac{(-1)^{n-k}n!}{k!x^{n-k+1}}\psi^{(k-1)}(x+a)\right)\\&\equiv\frac{n!}{x^{n+1}}\delta_n(x),\end{aligned}$$

where $\delta_n(x)\equiv-\log\Gamma(x+a)-\sum_{k=1}^{n}\frac{(-1)^k x^k}{k!}\psi^{(k-1)}(x+a)$ is the same as in the proof of Lemma 2.

By differentiation, we get

$$\delta_n'(x) = \frac{(-1)^{n+1}x^n}{n!}\psi^{(n)}(x+a) = \sum_{k=0}^{\infty} \frac{x^n}{(k+x+a)^{n+1}}$$

and hence

$$\delta_n'(x) \begin{cases} < 0, & x \in (-a,0), & \text{if } n \text{ is odd}, \\ > 0, & x \in (0,\infty), & \text{if } n \text{ is odd}, \\ > 0, & x \in (-a,\infty) \setminus \{0\}, & \text{if } n \text{ is even}. \end{cases}$$

For $n$ is odd and $x \in (-a,\infty) \setminus \{0\}$, we have

$$\delta_n(x) > \lim_{x \to 0} \delta_n(x) = -\log\Gamma(a).$$

Then $\delta_n(x) > 0$ and hence $(-1)^n(\log g_1(x))^{(n)} > 0$ on $(-a,\infty) \setminus \{0\}$ if and only if $1 \leqslant a \leqslant 2$.

For $n$ is even and $x \in (-a,0)$, we have

$$\delta_n(x) < \lim_{x \to 0} \delta_n(x) = -\log\Gamma(a).$$

Then $\delta_n(x) < 0$ and hence $(-1)^n(\log g_1(x))^{(n)} > 0$ on $(-a,0)$ if and only if $0 < a \leqslant 1$ or $a \geqslant 2$.

For $n$ is even and $x \in (0,\infty)$, we have

$$\delta_n(x) > \lim_{x \to 0} \delta_n(x) = -\log\Gamma(a).$$

Then $\delta_n(x) > 0$ and hence $(-1)^n(\log g_1(x))^{(n)} > 0$ on $(0,\infty)$ if and only if $1 \leqslant a \leqslant 2$.

Therefore $g_1$ is strictly LCM on $(0,\infty)$ if and only if $1 \leqslant a \leqslant 2$.

(2) By Lemma 2, we get

$$(-1)^n(\log g_1(0))^{(n)} = \begin{cases} \dfrac{(-1)^{n+1}\psi^{(n)}(1)}{n+1}, & a = 1, \\ \dfrac{(-1)^{n+1}\psi^{(n)}(2)}{n+1}, & a = 2, \end{cases}$$

which are clearly positive.

Together with the proof in (1), we have that $g_1$ is strictly LCM on $(-a,\infty)$ if and only if $a = 1$ or $a = 2$.

The proof is complete. □

*Proof of Theorem 2.* (1) Logarithmic differentiation leads to

$$\frac{g_2'(x)}{g_2(x)} \equiv \frac{h_2(x) - c}{x}, \tag{3.1}$$

where $h_2(x)$ is the same as in Lemma 4.

By Lemma 4, we have that $g_2$ is strictly decreasing on $(0,\infty)$ if and only if $c \geqslant 1$ for $\frac{1}{2} \leqslant a \leqslant 1$ or $a \geqslant 2$; $g_2$ is strictly increasing on $(0,\infty)$ if and only if $c \leqslant 0$ for $a=1$ or $a=2$; and $g_2$ is strictly increasing on $(0,\infty)$ if and only if $c \leqslant h_2(x_3)$ for $1<a<2$, where $x_3$ satisfies $x_3^2\psi'(x_3+a)+\log\Gamma(x_3+a)=x_3\psi(x_3+a)$.

(2) Differentiation gives

$$(\log g_2(x))'' \equiv \frac{c-h_3(x)}{x^2},$$

where $h_3(x)$ is the same as in Lemma 5.

By Lemma 5, we have that $g_2$ is strictly log-convex on $(0,\infty)$ if and only if $c \geqslant 1$ for $a \geqslant 2$; and $g_2$ is strictly log-concave on $(0,\infty)$ if and only if $c \leqslant 0$ for $a=2$.

(3) By (3.1), it is easy to obtain

$$x\frac{g_2'(x)}{g_2(x)} \equiv h_2(x)-c.$$

By Lemma 1 and Lemma 4, we have that $g_2$ is geometrically convex on $(0,\infty)$ if and only if $(a,c)\in D_1\cup D_2$; and $g_2$ is geometrically concave on $(0,x_3)$ and geometrically convex on $(x_3,\infty)$ if and only if $(a,c)\in D_3\cup D_4$.

The proof is complete. □

*Proof of Theorem 3.* (1) Logarithmic differentiation gives

$$\frac{g_3'(x)}{g_3(x)} \equiv \frac{h_4(x)-c}{x+a}, \tag{3.2}$$

where $h_4(x)$ is the same as in Lemma 6.

By Lemma 6, we have that $g_3$ is strictly decreasing on $(0,\infty)$ if and only if $c \geqslant 1$ for $a \geqslant 2$; $g_3$ is strictly increasing on $(0,\infty)$ if and only if $c \leqslant \frac{\pi^2}{6}-1$ for $a=2$; and $g_3$ is strictly increasing on $(0,\infty)$ if and only if $c \leqslant h_4(x_4)$ for $\frac{3+\sqrt{159}}{12} \leqslant a < 2$, where $x_4$ satisfies $x_4^2(x_4+a)\psi'(x_4+a)+(x_4+2a)\log\Gamma(x_4+a)=x_4(x_4+2a)\psi(x_4+a)$.

(2) By (3.2), it is easy to obtain

$$x\frac{g_3'(x)}{g_3(x)} \equiv h_2(x)+\frac{ac}{x+a}-c, \tag{3.3}$$

where $h_2(x)$ is the same as in Lemma 4.

By Lemma 1 and Lemma 4, we have that $g_3$ is geometrically convex on $(0,\infty)$ for $(a,c)\in D_9\cup D_{10}$; and $g_3$ is geometrically concave on $(0,x_3)$ for $(a,c)\in D_3$ and geometrically convex on $(x_3,\infty)$ for $(a,c)\in D_4$.

The proof is complete. □

## 4. Comparison of inequalities

In this section, we compare the inequalities appeared in the corollaries in Section 1.

REMARK 4. For $c \leqslant 0$ and $x,y > 0$, there holds

$$\left(\frac{x+y}{2\sqrt{xy}}\right)^c \leqslant 1.$$

Thus the inequality (1.3) is better than the reversed one of inequality (1.5) for $(a,c) \in D_8$.

REMARK 5. By Lemma 4 (1), for $0 < x < y$, we have

$$\left(\frac{x}{y}\right)^{\frac{x\psi(x+a)-\log\Gamma(x+a)}{x}} < 1 < \left(\frac{x}{y}\right)^c$$

for $(a,c) \in D_7 \cup D_8$; and

$$\left(\frac{x}{y}\right)^c < \left(\frac{x}{y}\right)^{\frac{y\psi(y+a)-\log\Gamma(y+a)}{y}}$$

for $(a,c) \in D_5 \cup D_6$.

Thus the right side of the inequalities (1.6) is better than the inequality (1.2) and the reversed one of inequality (1.4) for $(a,c) \in D_7 \cup D_8$; and the left side of the inequalities (1.6) is better than the inequality (1.4) for $(a,c) \in D_5 \cup D_6$.

REMARK 6. By (3.3) and Lemma 4 (1), it is easy to obtain

$$\lim_{x\to 0^+} x\frac{g_3'(x)}{g_3(x)} = 0 \qquad \text{and} \qquad \lim_{x\to\infty} x\frac{g_3'(x)}{g_3(x)} = 1-c,$$

and hence for $0 < x < y$, there holds

$$\left(\frac{x}{y}\right)^{\frac{x\psi(x+a)-\log\Gamma(x+a)}{x}-\frac{cx}{x+a}} < 1$$

for $(a,c) \in D_8$.

Thus the right side of the inequalities (1.8) is better than the reversed one of inequality (1.7) for $(a,c) \in D_8$.

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