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ON p–QUASI–n–HYPONORMAL OPERATORS

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Abstract. An operator $T \in B(H)$ is called p-quasi-n-hyponormal if

$$T^*(T^{*n}T^n)^pT \geqslant T^*(T^nT^{*n})^pT$$

for a positive number $0 < p \leqslant 1$ and a positive integer n, which is a further generalization of normal operator. In this paper we introduce the class of p-quasi-n-hyponormal operators and show its structural properties via Hansen inequality and Löwner-Heinz inequality. As important applications, we obtain that every p-quasi-n-hyponormal operator has a scalar extension. In addition, we prove that if T is a quasiaffine transform of p-quasi-n-hyponormal, then T satisfies Weyl's theorem. Finally some examples are presented.

1. Introduction

Let $B(H)$ denote the C^*-algebra of all bounded linear operators on an infinite dimensional separable Hilbert space H. If $T \in B(H)$, we shall write $N(T)$ and $R(T)$ for the null space and the range space of T, and also, write $\sigma(T)$ and $\omega(T)$ for the spectrum and the Weyl spectrum of T, respectively.

First we define the p-quasi-n-hyponormal operator as follows.

DEFINITION 1. An operator $T \in B(H)$ is called p-quasi-n-hyponormal if

$$T^*(T^{*n}T^n)^pT \geqslant T^*(T^nT^{*n})^pT$$

for a positive number $0 < p \leqslant 1$ and a positive integer n.

A p-quasi-n-hyponormal operator for a positive number $0 < p \leqslant 1$ and a positive integer n is an extension of p-hyponormal operator, i.e., $(T^*T)^p \geqslant (TT^*)^p$, n-th root of p-hyponormal operator, i.e., $(T^{*n}T^n)^p \geqslant (T^nT^{*n})^p$ and p-quasihyponormal operator, i.e., $T^*(T^*T)^pT \geqslant T^*(TT^*)^pT$. A 1-hyponormal operator is called a hyponormal operator and a $\frac{1}{2}$-hyponormal operator is called a semi-hyponormal operator, which has been studied by many authors and it is known that hyponormal operators have many interesting properties similar to those of normal operators (see [22]). A

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1-th root of p-hyponormal operator is called a p-hyponormal operator. A 1-quasi-n-hyponormal operator is called a quasi-n-hyponormal operator (see [25]) and a p-quasi-1-hyponormal operator is called a p-quasihyponormal operator (see [4]). A. Aluthge, E. Ko, A.C. Arora and P. Arora introduced p-hyponormal, n-th root of p-hyponormal and p-quasihyponormal operators, respectively (see [3, 4, 13]), and it is known that these operators have many interesting properties (see [6, 7, 8, 13, 15, 17, 24]). It is well-known that p-hyponormal operators are q-hyponormal if $0 < q \leqslant p$, however, it is not necessarily true that p-quasi-1-hyponormal operators are q-quasi-1-hyponormal even if $0 < q < p$ (see [21]). It is clear that

$$\text{normal} \Rightarrow n\text{-th root of } p\text{-hyponormal} \Rightarrow p\text{-quasi-}n\text{-hyponormal.}$$

An operator $T \in B(H)$ is called scalar of order m if it possesses a spectral distribution of order m, i.e., if there is a continuous unital morphism of topological algebras $\Phi : C_0^m(\mathbb{C}) \to B(H)$ such that $\Phi(z) = T$, where z stands for the identity function on $\mathbb{C}$, and $C_0^m(\mathbb{C})$ stands for the space of compactly supported functions on $\mathbb{C}$, continuously differentiable of order m, $0 \leqslant m \leqslant \infty$. An operator is subscalar if it is similar to the restriction of a scalar operator to an invariant subspace. An operator $T \in B(H)$ is said to have property (β) if for every open subset U of $\mathbb{C}$ and for every sequence $f_n : U \to H$ of H-value analytic functions such that $(T - zI)f_n(z)$ converges uniformly to 0 in norm on compact subsets of U, $f_n(z)$ converges uniformly to 0 in norm on compact subsets of U.

In 1984, Putinar [19] proved that every hyponormal operator has a scalar extension, which has been extended from hyponormal operators to p-hyponormal operators [16], to p-quasihyponormal operators [14], to quasi-n-hyponormal operators [20], and to k-th roots of p-hyponormal operators [13]. In Section 2, we show that every p-quasi-n-hyponormal operator is subscalar. As a consequence, we prove that every p-quasi-n-hyponormal operator with rich spectrum has a nontrivial invariant subspace. Finally, we give some examples of p-quasi-n-hyponormal operator in Section 3.

2. Main results

Before we state main theorems, we need several preliminary results.

LEMMA 1. (Hansen inequality [12]) *If $A, B \in B(H)$ satisfy $A \geqslant 0$ and $\| B \| \leqslant 1$, then*

$$(B^*AB)^{\delta} \geqslant B^*A^{\delta}B \quad \textit{for all} \quad \delta \in (0,1].$$

LEMMA 2. (Löwner-Heinz inequality [11]) *$A \geqslant B \geqslant 0$ ensures $A^{\alpha} \geqslant B^{\alpha}$ for any $\alpha \in [0,1]$.*

LEMMA 3. *Suppose that $T \in B(H)$ is a p-quasi-n-hyponormal operator and $R(T)$ is dense. Then T is an n-th root of p-hyponormal operator.*

Proof. Since T has dense range, $\overline{R(T)} = H$. For $\forall y \in H$, there exists a sequence $\{x_k\}_{k=1}^{\infty}$ in H such that $T(x_k) \to y$ as $k \to \infty$. Since T is p-quasi-n-hyponormal, for all positive integers k, a positive number $0 < p \leqslant 1$ and a positive integer n

$$\langle (T^{*n}T^n)^p Tx_k,\, Tx_k\rangle \;\geqslant\; \langle (T^nT^{*n})^p Tx_k,\, Tx_k\rangle.$$

By the continuity of the inner product, for a positive number $0 < p \leqslant 1$ and a positive integer n, we have $\langle (T^{*n}T^n)^p y,\, y\rangle \;\geqslant\; \langle (T^nT^{*n})^p y,\, y\rangle$, and hence $(T^{*n}T^n)^p \geqslant (T^nT^{*n})^p$. Therefore T is n-th root of p-hyponormal. □

THEOREM 1. *Suppose that $T \in B(H)$ is a p-quasi-n-hyponormal operator and $R(T)$ is not dense. Then*

$$T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \quad on \quad H = \overline{R(T)} \oplus N(T^*),$$

where A is an n-th root of p-hyponormal operator and $\sigma(T) = \sigma(A) \cup \{0\}$.

Proof. The spectral inclusion relation is clear and it is sufficient to show that A is n-th root of p-hyponormal. Let E be the orthogonal projection onto $\overline{R(T)}$. Then

$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} = TE = ETE.$$

For $\forall y \in H$, there exists a sequence $\{x_k\}_{k=1}^{\infty}$ in H such that $T(x_k) \to Ey$ as $k \to \infty$. Since T is a p-quasi-n-hyponormal operator, we have

$$\langle ((T^{*n}T^n)^p - (T^nT^{*n})^p)Tx_k,\, Tx_k\rangle = \langle T^*((T^{*n}T^n)^p - (T^nT^{*n})^p)Tx_k,\, x_k\rangle \;\geqslant\; 0.$$

By the continuity of the inner product, we have

$$\langle E((T^{*n}T^n)^p - (T^nT^{*n})^p)Ey,\, y\rangle = \langle ((T^{*n}T^n)^p - (T^nT^{*n})^p)Ey,\, Ey\rangle \;\geqslant\; 0,$$

and hence $E((T^{*n}T^n)^p - (T^nT^{*n})^p)E \geqslant 0$. Then

$$\begin{aligned} E(T^{*n}T^n)^pE &\leqslant (ET^{*n}T^nE)^p \quad \text{(by Lemma 1)} \\ &= (ET^*\cdots ET^*TE\cdots TE)^p \\ &= \begin{pmatrix} (A^{*n}A^n)^p & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} E(T^nT^{*n})^pE &\geqslant E(T^nPT^{*n})^pE \quad \text{(by Lemma 2)} \\ &= \begin{pmatrix} (A^nA^{*n})^p & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Hence

$$\begin{pmatrix} (A^{*n}A^n)^p & 0 \\ 0 & 0 \end{pmatrix} \geqslant E(T^{*n}T^n)^pE \geqslant E(T^nT^{*n})^pE \geqslant \begin{pmatrix} (A^nA^{*n})^p & 0 \\ 0 & 0 \end{pmatrix},$$

i.e., A is an n-th root of p-hyponormal operator. □

THEOREM 2. *Suppose that $T \in B(H)$ is a p-quasi-n-hyponormal operator and M is its invariant subspace. Then the restriction $T|_M$ of T to M is also a p-quasi-n-hyponormal operator.*

Proof. Let E be the orthogonal projection onto M. Then we can represent T as the following 2×2 operator matrix with respect to the decomposition $M \oplus M^{\perp}$,

$$T = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}.$$

Put $A = T|_M$. Then $TE = ETE$ and $A = (ETE)|_M$. Since T is a p-quasi-n-hyponormal operator, we have

$$ET^*(T^{*n}T^n)^pTE \geqslant ET^*(T^nT^{*n})^pTE.$$

Since

$$\begin{aligned} ET^*(T^{*n}T^n)^pTE &= ET^*E(T^{*n}T^n)^pETE \\ &\leqslant ET^*(ET^{*n}T^nE)^pTE \quad \text{(by Lemma 1)} \\ &= ET^*E(ET^{*n}EET^nE)^pETE \\ &= \begin{pmatrix} A^*(A^{*n}A^n)^pA & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} ET^*(T^nT^{*n})^pTE &= ET^*E(T^nT^{*n})^pETE \\ &\geqslant ET^*E(T^nET^{*n})^pETE \quad \text{(by Lemma 2)} \\ &= ET^*E(ET^nEET^{*n}E)^pETE \\ &= \begin{pmatrix} A^*(A^nA^{*n})^pA & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

we have

$$\begin{pmatrix} A^*(A^{*n}A^n)^pA & 0 \\ 0 & 0 \end{pmatrix} \geqslant \begin{pmatrix} A^*(A^nA^{*n})^pA & 0 \\ 0 & 0 \end{pmatrix}.$$

This implies that A is a p-quasi-n-hyponormal operator. □

For a Banach space X, let $\xi(U,X)$ denote the Fréchet space of all infinite differentiable X-value functions on U (see [10]). T is said to have property $(\beta)_\varepsilon$ if every open subset U in $\mathbb{C}$ the operator

$$T_z: \ \ \xi(U,X) \to \xi(U,X), \quad f \mapsto (T - zI)f$$

is a topological monomorphism, i.e., $T_zf_n \to 0$ implies $f_n \to 0$ for $f_n \in \xi(U,X)$. Now we show that every p-quasi-n-hyponormal operator has a scalar extension, the following lemmas are needed.

LEMMA 4. ([16, Lemma 1]) *For $T \in B(X)$, the following statements are equivalent:*

(i) T is subscalar;

(ii) T has property $(\beta)_\varepsilon$.

LEMMA 5. ([13, Theorem 3.6]) *Suppose that T is an n-th root of p-hyponormal operator. Then T is subscalar of order $4n$.*

THEOREM 3. *Suppose that T is a p-quasi-n-hyponormal operator. Then T is subscalar.*

Proof. Assume that $R(T)$ is dense. Then by Lemma 3 T is an n-th root of p-hyponormal operator, it is subscalar by [13, Theorem 3.6]. So we may assume that T does not have dense range. Then by Theorem 1 the operator T can be decomposed as follow: $T = \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix}$ on $H = \overline{R(T)} \oplus N(T^*)$, where T_1 is an n-th root of p-hyponormal operator. Set $\sigma_{(\beta)_\varepsilon}(S) = \{\mu \in \sigma(S) : S \text{ does not have property } (\beta)_\varepsilon \text{ at } \mu\}$. Recall from [5, Theorem 2.1] that given operators S and R, $\lambda \in \sigma_{(\beta)_\varepsilon}(RS) \Leftrightarrow \lambda \in \sigma_{(\beta)_\varepsilon}(SR)$. Considering $T = \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I_1 & T_2 \\ 0 & I_2 \end{pmatrix} \begin{pmatrix} T_1 & 0 \\ 0 & I_2 \end{pmatrix}$, let $B = \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix}$, $E = \begin{pmatrix} I_1 & T_2 \\ 0 & I_2 \end{pmatrix}$, $A = \begin{pmatrix} T_1 & 0 \\ 0 & I_2 \end{pmatrix}$. Then $T = BEA$. Suppose $\lambda \in \sigma_{(\beta)_\varepsilon}(T) \Leftrightarrow \lambda \in \sigma_{(\beta)_\varepsilon}(BEA)$ $= \sigma_{(\beta)_\varepsilon}(EAB)$. Since E is invertible, $\lambda \in \sigma_{(\beta)_\varepsilon}(AB) = \sigma_{(\beta)_\varepsilon}(T_1 \oplus 0) \Rightarrow \lambda \in \sigma_{(\beta)_\varepsilon}(T_1)$, hence T_1 does not have property $(\beta)_\varepsilon$ and this contradicts the fact that T_1 is subscalar. Thus T has property $(\beta)_\varepsilon$, i.e., T is subscalar. □

COROLLARY 1. *Suppose that T is a p-quasi-n-hyponormal operator. Then T has Bishop's property (β).*

Proof. Since the Bishop's property (β) is transmitted from an operator to its restrictions to closed invariant subspace, we are reduced by Theorem 3 to the case of a scalar operator. Since every scalar operator has Bishop's property (β) [19], T has Bishop's property (β). □

COROLLARY 2. *Suppose that T is a p-quasi-n-hyponormal operator. If $\sigma(T)$ has nonempty interior in $\mathbb{C}$, then T has a nontrivial invariant subspace.*

Proof. It is known from [9, Theorem 1] that if T is subscalar and $\sigma(T)$ has nonempty interior in $\mathbb{C}$, then T has a nontrivial invariant subspace. Since T is a p-quasi-n-hyponormal operator, T is subscalar, hence T has a nontrivial invariant subspace. □

DEFINITION 2. An operator $T \in B(H)$ is said to belong to the class $H(q)$ if there exists a natural number $q := q(\lambda)$ such that

$$H_0(\lambda I - T) = N(\lambda I - T)^q \text{ for all } \lambda \in \mathbb{C},$$

where $H_0(\lambda I-T):=\{x\in H:\lim_{n\to\infty}||(\lambda I-T)^n x||^{\frac{1}{n}}=0\}$.

THEOREM 4. [18] *Every subscalar operator is* $H(q)$.

Classical examples of subscalar operators are hyponormal operators. In this paper we will show that other important classes of operators are $H(q)$.

DEFINITION 3. An operator $T\in B(H)$ is said to be polaroid if every $\lambda\in\operatorname{iso}\sigma(T)$ is a pole of the resolvent of T, where $\operatorname{iso}\sigma(T)$ denotes the isolated points of the spectrum.

The condition of being polaroid may be characterized by means of the quasi-nilpotent part.

THEOREM 5. [2] *An operator* $T\in B(H)$ *is polaroid if and only if there exists a natural number* $q:=q(\lambda)$ *such that*

$$H_0(\lambda I-T)=N(\lambda I-T)^q \textit{ for all } \lambda\in\operatorname{iso}\sigma(T).$$

Note that every $H(q)$ operator is polaroid. By using Theorem 3 and Theorem 4, we deduce the following corollaries.

COROLLARY 3. *Every* p*-quasi-*n*-hyponormal operator is* $H(q)$.

COROLLARY 4. *Every* p*-quasi-*n*-hyponormal operator is polaroid.*

Recall that an operator $Y\in B(H_1,H_2)$ is called a quasiaffinity if it has trivial kernel and dense range. An operator $S\in B(H_1)$ is said to be a quasiaffine transform of $T\in B(H_2)$ if there is a quasiaffinity $Y\in B(H_1,H_2)$ such that $YS=TY$.

COROLLARY 5. *Suppose that* T *is a* p*-quasi-*n*-hyponormal operator. If* S *is a quasiaffine transform of* T, *then* S *satisfies Weyl's theorem* (i.e., $\sigma(S)-\omega(S)=\pi_{00}(S)$, *where* $\pi_{00}(S)=\{\lambda\in\operatorname{iso}\sigma(S):0<N(S-\lambda I)<\infty\}$).

Proof. If T is a p-quasi-n-hyponormal operator, then $H_0(\lambda I-T)=N(\lambda I-T)^q$ for some integer $q:=q(\lambda)\geqslant 0$ and all $\lambda\in\mathbb{C}$. Suppose $US=TU$ with U injective and $x\in H_0(\lambda I-S)$. Then

$$||(\lambda I-T)^n Ux||^{\frac{1}{n}}=||U(\lambda I-S)^n x||^{\frac{1}{n}}\leqslant||U||^{\frac{1}{n}}||(\lambda I-S)^n x||^{\frac{1}{n}},$$

for which we obtain that $Ux\in H_0(\lambda I-T)=N(\lambda I-T)^q$. Hence

$$U(\lambda I-S)^q x=(\lambda I-T)^q Ux=0,$$

and since U injective this implies that $(\lambda I-S)^q x=0$. Consequently $H_0(\lambda I-S)=N(\lambda I-S)^q$ for some integer $q:=q(\lambda)\geqslant 0$ and all $\lambda\in\mathbb{C}$. By [1, Theorem 3.10] Weyl's theorem holds for S. $\square$

3. Examples

Consider unilateral weighted shift operator as an infinite dimensional Hilbert space operator. Recall that given a bounded sequence of positive numbers $\alpha : \alpha_1, \alpha_2, \ldots$ (called weights), the unilateral weighted shift W_α associated with α is the operator on $H = l_2$ defined by $W_\alpha e_n := \alpha_n e_{n+1}$ for all $n \geqslant 1$, where $\{e_n\}_{n=1}^{\infty}$ is the canonical orthonormal basis for l_2. It is well known that W_α is p-quasihyponormal if and only if α is monotonically increasing (see [23, Example 2.3]).

LEMMA 6. *W_α belongs to p-quasi-n-hyponormal if and only if*

$$W_\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \cdots \\ \alpha_1 & 0 & 0 & 0 & 0 \cdots \\ 0 & \alpha_2 & 0 & 0 & 0 \cdots \\ 0 & 0 & \alpha_3 & 0 & 0 \cdots \\ 0 & 0 & 0 & \alpha_4 & 0 \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \ddots \end{pmatrix},$$

where

$$\alpha_k \alpha_{k+1} \ldots \alpha_{k+n-1} \leqslant \alpha_{k+n} \alpha_{k+n+1} \ldots \alpha_{k+2n-1} \quad (k = 1, 2, 3, \ldots).$$

Proof. By simple calculations,

$$W_\alpha^{*n} W_\alpha^n = (\alpha_1^2 \alpha_2^2 \ldots \alpha_n^2) \oplus (\alpha_2^2 \alpha_3^2 \ldots \alpha_{n+1}^2) \oplus (\alpha_3^2 \alpha_4^2 \ldots \alpha_{n+2}^2) \oplus \ldots$$

and

$$W_\alpha^n W_\alpha^{*n} = \overbrace{0 \oplus \cdots \oplus 0}^{n \text{ items}} \oplus (\alpha_1^2 \alpha_2^2 \ldots \alpha_n^2) \oplus (\alpha_2^2 \alpha_3^2 \ldots \alpha_{n+1}^2) \oplus \ldots.$$

Hence

$$\begin{aligned} W_\alpha^* (W_\alpha^{*n} W_\alpha^n)^p W_\alpha = {} & \alpha_1^2 (\alpha_2^{2p} \alpha_3^{2p} \ldots \alpha_{n+1}^{2p}) \oplus \alpha_2^2 (\alpha_3^{2p} \alpha_4^{2p} \ldots \alpha_{n+2}^{2p}) \oplus \ldots \\ & \oplus \alpha_n^2 (\alpha_{n+1}^{2p} \alpha_{n+2}^{2p} \ldots \alpha_{2n}^{2p}) \oplus \alpha_{n+1}^2 (\alpha_{n+2}^{2p} \alpha_{n+3}^{2p} \ldots \alpha_{2n+1}^{2p}) \oplus \ldots \end{aligned}$$

and

$$\begin{aligned} W_\alpha^* (W_\alpha^n W_\alpha^{*n})^p W_\alpha = {} & \overbrace{0 \oplus \cdots \oplus 0}^{n-1 \text{ items}} \oplus \alpha_n^2 (\alpha_1^{2p} \alpha_2^{2p} \ldots \alpha_n^{2p}) \\ & \oplus \alpha_{n+1}^2 (\alpha_2^{2p} \alpha_3^{2p} \ldots \alpha_{n+1}^{2p}) \oplus \ldots. \end{aligned}$$

Thus W_α belongs to p-quasi-n-hyponormal if and only if

$$\alpha_k \alpha_{k+1} \ldots \alpha_{k+n-1} \leqslant \alpha_{k+n} \alpha_{k+n+1} \ldots \alpha_{k+2n-1} \quad (k = 1, 2, 3, \ldots). \qquad \square$$

The following example provides an operator which is p-quasi-n-hyponormal for all $n \geqslant 2$ but not p-quasihyponormal.

EXAMPLE 1. A p-quasi-n-hyponormal operator which is not p-quasihyponormal.

Proof. Let W_α be a unilateral weighted shift operator with weights $\alpha_n = 2$ $(n \neq 2)$ and $\alpha_2 = 1$. Simple calculations show that W_α is p-quasi-n-hyponormal, but W_α is non-p-quasihyponormal. □

Finally we give an example to show that the class of n-th root of p-hyponormal operators is properly contained in the class of p-quasi-n-hyponormal operators. The following lemma is needed.

LEMMA 7. *Let $K = \bigoplus_{m=1}^{+\infty} H_m$, and $H_m \cong H$. For given positive operators A and B on H, and any fixed positive integer n, define the operator $T = T_{A,B,n}$ on K as*

$$T(x_1,x_2,x_3,\ldots) = (0,Ax_1,Ax_2,\ldots,Ax_n,Bx_{n+1},Bx_{n+2},\ldots).$$

Then the following assertions hold

(i) *T belongs to n-th root of p-hyponormal if and only if*

$$\begin{cases} B^{2np} \geqslant A^{2np}, \\ B^{2np} \geqslant (BA^{2n-2}B)^p, \\ B^{2np} \geqslant (B^2A^{2n-4}B^2)^p, \\ \cdots\cdots \\ B^{2np} \geqslant (B^{n-1}A^2B^{n-1})^p. \end{cases} \tag{3.1}$$

(ii) *T belongs to p-quasi-n-hyponormal if and only if*

$$\begin{cases} AB^{2np}A \geqslant AA^{2np}A, \\ BB^{2np}B \geqslant B(BA^{2n-2}B)^pB, \\ BB^{2np}B \geqslant B(B^2A^{2n-4}B^2)^pB, \\ \cdots\cdots \\ BB^{2np}B \geqslant B(B^{n-1}A^2B^{n-1})^pB. \end{cases} \tag{3.2}$$

Proof. Since

$$T(x_1,x_2,x_3,\ldots) = (0,Ax_1,Ax_2,\ldots,Ax_n,Bx_{n+1},Bx_{n+2},\ldots),$$

we obtain

$$T^*(x_1,x_2,x_3,\ldots) = (Ax_2,Ax_3,\ldots,Ax_{n+1},Bx_{n+2},Bx_{n+3},\ldots).$$

By simple calculations, the following equalities hold.

$$\begin{aligned} T^n(x_1,x_2,x_3,\ldots) = (&\overbrace{0,\ldots,0}^{n \text{ items}},A^nx_1,BA^{n-1}x_2,B^2A^{n-2}x_3, \\ &\ldots,B^{n-1}Ax_n,B^nx_{n+1},B^nx_{n+2},\ldots); \end{aligned}$$

$$T^{*n}(x_1,x_2,x_3,\ldots)=(A^n x_{n+1},A^{n-1}Bx_{n+2},A^{n-2}B^2x_{n+3},\\ \ldots,AB^{n-1}x_{2n},B^n x_{2n+1},B^n x_{2n+2},\ldots).$$

Hence

$$(T^{*n}T^n)^p(x_1,x_2,x_3,\ldots)=\{A^{2np}x_1,(A^{n-1}B^2A^{n-1})^p x_2,(A^{n-2}B^4A^{n-2})^p x_3,\\ \ldots,(AB^{2n-2}A)^p x_n,B^{2np}x_{n+1},B^{2np}x_{n+2},\ldots\};$$

$$(T^nT^{*n})^p(x_1,x_2,x_3,\ldots)=\{\overbrace{0,\ldots,0}^{n\text{ items}},A^{2np}x_{n+1},(BA^{2n-2}B)^p x_{n+2},\\ (B^2A^{2n-4}B^2)^p x_{n+3},\ldots,(B^{n-1}A^2B^{n-1})^p x_{2n},\\ B^{2np}x_{2n+1},B^{2np}x_{2n+2},\ldots\};$$

$$T^*(T^{*n}T^n)^pT(x_1,x_2,x_3,\ldots)=\{A(A^{n-1}B^2A^{n-1})^pAx_1,A(A^{n-2}B^4A^{n-2})^pAx_2,\\ \ldots,AB^{2np}Ax_n,B^{2np+2}x_{n+1},B^{2np+2}x_{n+2},\ldots\};$$

$$T^*(T^nT^{*n})^pT(x_1,x_2,x_3,\ldots)=\{\overbrace{0,\ldots,0}^{n-1\text{ items}},AA^{2np}Ax_n,B(BA^{2n-2}B)^pBx_{n+1},\\ \ldots,B(B^{n-1}A^2B^{n-1})^pBx_{2n-1},BB^{2np}Bx_{2n},BB^{2np}Bx_{2n+1},\ldots\}.$$

Therefore, T is n-th root of p-hyponormal if and only if

$$\begin{cases} B^{2np}\geqslant A^{2np},\\ B^{2np}\geqslant (BA^{2n-2}B)^p,\\ B^{2np}\geqslant (B^2A^{2n-4}B^2)^p,\\ \cdots\cdots\\ B^{2np}\geqslant (B^{n-1}A^2B^{n-1})^p. \end{cases}$$

Similarly, T is p-quasi-n-hyponormal if and only if

$$\begin{cases} AB^{2np}A\geqslant AA^{2np}A,\\ BB^{2np}B\geqslant B(BA^{2n-2}B)^pB,\\ BB^{2np}B\geqslant B(B^2A^{2n-4}B^2)^pB,\\ \cdots\cdots\\ BB^{2np}B\geqslant B(B^{n-1}A^2B^{n-1})^pB. \quad\square \end{cases}$$

EXAMPLE 2. A $\frac{1}{2}$-quasi-n-hyponormal operator which is not n-th root of $\frac{1}{2}$-hyponormal.

Proof. Let H be a two dimensional Hilbert space and $p=\frac{1}{2}$. Take A and B as

$$A=\begin{pmatrix}\frac{1}{2} & 0\\ 0 & 0\end{pmatrix},\ \ B=\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}.$$

Then

$$A^n=\begin{pmatrix}\frac{1}{2^n} & 0\\ 0 & 0\end{pmatrix},\ \ B^n=B=\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix},$$

we have

$$B^n-A^n=\begin{pmatrix}\frac{2^n-2}{2^{n+1}} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}\ngeqslant 0.$$

Hence T is not an n-th root of $\frac{1}{2}$-hyponormal operator.

On the other hand,

$$\begin{aligned}A(B^n-A^n)A&=\begin{pmatrix}\frac{1}{2} & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\frac{2^n-2}{2^{n+1}} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}\frac{1}{2} & 0\\ 0 & 0\end{pmatrix}\\ &=\begin{pmatrix}\frac{2^n-2}{2^{n+3}} & 0\\ 0 & 0\end{pmatrix}\geqslant 0,\end{aligned}$$

$$\begin{aligned}B(B^n-(BA^{2n-2}B)^{\frac{1}{2}})B&=\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}\frac{2^n-\sqrt{2}}{2^{n+1}} & \frac{2^n-\sqrt{2}}{2^{n+1}}\\ \frac{2^n-\sqrt{2}}{2^{n+1}} & \frac{2^n-\sqrt{2}}{2^{n+1}}\end{pmatrix}\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}\\ &=\begin{pmatrix}\frac{2^n-\sqrt{2}}{2^{n+1}} & \frac{2^n-\sqrt{2}}{2^{n+1}}\\ \frac{2^n-\sqrt{2}}{2^{n+1}} & \frac{2^n-\sqrt{2}}{2^{n+1}}\end{pmatrix}\geqslant 0,\end{aligned}$$

$$\begin{aligned}B(B^n-(B^2A^{2n-4}B^2)^{\frac{1}{2}})B&=\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}\frac{2^{n-1}-\sqrt{2}}{2^n} & \frac{2^{n-1}-\sqrt{2}}{2^n}\\ \frac{2^{n-1}-\sqrt{2}}{2^n} & \frac{2^{n-1}-\sqrt{2}}{2^n}\end{pmatrix}\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}\\ &=\begin{pmatrix}\frac{2^{n-1}-\sqrt{2}}{2^n} & \frac{2^{n-1}-\sqrt{2}}{2^n}\\ \frac{2^{n-1}-\sqrt{2}}{2^n} & \frac{2^{n-1}-\sqrt{2}}{2^n}\end{pmatrix}\geqslant 0,\\ &\cdots\cdots\end{aligned}$$

$$\begin{aligned}B(B^n-(B^{n-1}A^2B^{n-1})^{\frac{1}{2}})B&=\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}\frac{4-\sqrt{2}}{8} & \frac{4-\sqrt{2}}{8}\\ \frac{4-\sqrt{2}}{8} & \frac{4-\sqrt{2}}{8}\end{pmatrix}\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}\\ &=\begin{pmatrix}\frac{4-\sqrt{2}}{8} & \frac{4-\sqrt{2}}{8}\\ \frac{4-\sqrt{2}}{8} & \frac{4-\sqrt{2}}{8}\end{pmatrix}\geqslant 0.\end{aligned}$$

Thus T is a $\frac{1}{2}$-quasi-n-hyponormal operator. $\square$

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